Kumamoto J. Math.
Vol. 8, 101~145 March (1995)

Monodromy Groups for Certain Hypergeometric Systems

Mitsuhiko KOHNO, Kazuhiro TSURUMARU and Hidenori HARUKI

(Received August 20, 1994)

Dedicated to Professor Kenjiro Okubo on his 60th birthday

Abstract

Professor K. Okubo verified a very interesting theorem that if a hypergeometric system, the form of which denotes a general Fuchsian differential equation, has no accessory parameters, then the monodromy group of the system can be evaluated entirely by algebraic procedures.

Following his method, several authors tried to calculate explicitly monodromy groups of certain hypergeometric systems. In those cases, they had to solve nonlinear indeterminate equations. So, K. Okubo has also attempted to solve such complicated indeterminate equations by means of algebraic manipulations based on Gröbner basis.

In this paper, we shall deal with exact calculations of monodromy groups of hypergeometric systems, which may have logarithmic solutions. In stead of treating nonlinear equations, we solve merely systems of linear equations.

1. Generators of monodromy group

We shall consider the monodromy group of the hypergeometric system

$$(t-B)\frac{dX}{dt} = AX,$$

where $B = \mathrm{diag}\,(\lambda_1, \lambda_1, \lambda_2, \lambda_2)$, which are positions of finite regular singular points, and A is a constant matrix. In this case, the linear transformation $X = DY$, where D is a block-diagonal constant matrix

$$D = \left(\begin{array}{c|c} D_1 & 0 \\ \hline 0 & D_2 \end{array}\right),$$

the D_j being 2 by 2 matrices, does not change the form of the hypergeometric system. And hence the constant matrix A may be considered to be of the form

$$A = \left(\begin{array}{cc|cc} \alpha_1 & 0 & a_{13} & a_{14} \\ \varepsilon_1 & \alpha_2 & a_{23} & a_{24} \\ \hline a_{31} & a_{32} & \alpha_3 & 0 \\ a_{41} & a_{42} & \varepsilon_2 & \alpha_4 \end{array}\right),$$

where matrices of the form $\begin{pmatrix} \alpha & 0 \\ \varepsilon & \alpha' \end{pmatrix}$ are Jordan canonical matrices, i.e., $\varepsilon = 1$ or 0.

Here A is assumed to be similar to a diagonal matrix, i.e.,

$$A \sim \operatorname{diag}(\nu_1, \nu_2, \nu_3, \nu_4), \qquad (\nu_1 \equiv \nu_2 \pmod{\mathbf{Z}}),$$

and

$$\begin{cases} \alpha_k \not\equiv 0, \quad \alpha_1 \not\equiv \alpha_2, \quad \alpha_3 \not\equiv \alpha_4 \pmod{\mathbf{Z}} \qquad (k = 1, 2, 3, 4), \\ \alpha_k \not\equiv \nu_i, \quad \nu_i \not\equiv 0 \pmod{\mathbf{Z}} \qquad (k, i = 1, 2, 3, 4), \\ \nu_i \not\equiv \nu_j, \quad \nu_3 \not\equiv \nu_4 \pmod{\mathbf{Z}} \qquad (i = 1, 2;\ j = 3, 4). \end{cases}$$

It is remarked that from the trace relation, there holds

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = \nu_1 + \nu_2 + \nu_3 + \nu_4,$$

which is called the *Fuchs relation.*

Under the above assumptions, near each finite regular singular point $t = \lambda_i$ $(i = 1, 2)$, one can find non-holomorphic solutions corresponding to non-zero characteristic exponent as follows : for example, in case $\varepsilon_1 = 0$,

$$\begin{cases} X_1(t) = (x_1(t), x_2(t)), \\ x_i(t) = (t - \lambda_1)^{\alpha_i} \sum_{m=0}^{\infty} g_i(m) (t - \lambda_1)^m \qquad (i = 1, 2), \end{cases}$$

and in case $\varepsilon_1 = 1$,

$$\begin{cases} X_1(t) = \widehat{X}_1(t) (t - \lambda_1)^J, \\ \widehat{X}_1(t) = (t - \lambda_1)^{\alpha_1} \sum_{m=0}^{\infty} G_1(m) (t - \lambda_1)^m, \end{cases}$$

J being the 2 by 2 shifting matrix

$$J = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Then it is verified that $\mathcal{X}(t) = (X_1(t), X_2(t))$ forms a fundamental matrix solution of the hypergeometric system and plays an important role in the calculation of its monodromy group.

Now, we shall explain how to calculate the monodromy group. First, we have to seek holomorphic solutions corresponding to zero characteristic exponent near regular singular points. By the connection formulas :

$$X_i(t) = X_k(t)C_{k\,i} + Y_{k\,i}(t) \qquad (i \neq k \,:\, i,k = 1,2),$$

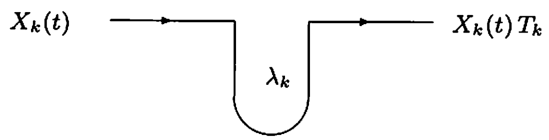
where the $C_{k\,i}$ are 2 by 2 constant matrices, one can derive holomorphic solutions near regular singular points, that is,

$$\begin{aligned} \mathcal{X}(t:\lambda_1) &= (X_1(t),\ Y_{12}(t)), \\ \mathcal{X}(t:\lambda_2) &= (Y_{21}(t),\ X_2(t)) \end{aligned}$$

form fundamental sets of solutions near $t = \lambda_1$ and $t = \lambda_2$, respectively. This fact is easily seen from the relations

$$\begin{cases} \mathcal{X}(t) = \mathcal{X}(t:\lambda_1)\,L_1 \equiv (X_1(t),\ Y_{12}(t)) \begin{pmatrix} I & C_{12} \\ 0 & I \end{pmatrix}, \\[2ex] \mathcal{X}(t) = \mathcal{X}(t:\lambda_2)\,L_2 \equiv (Y_{21}(t),\ X_2(t)) \begin{pmatrix} I & 0 \\ C_{21} & I \end{pmatrix}. \end{cases}$$

We can then calculate the circuit matrices of $X_k(t)$ and $\mathcal{X}(t:\lambda_k)$ $(k = 1,2)$ as follows :



where

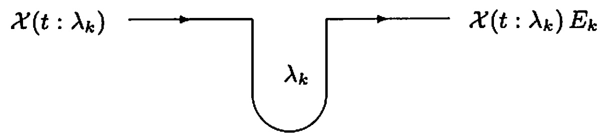
$$T_1 = \begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix}, \qquad T_2 = \begin{pmatrix} e_3 & 0 \\ 0 & e_4 \end{pmatrix} \qquad (\,e_k = \exp(2\pi i \alpha_k)\,)$$

for non-logarithmic solutions and

$$\begin{aligned} T_1 &= \exp\{2\pi i(\alpha_1 + J)\} = \exp(2\pi i \alpha_1)\exp(2\pi i J) \\ &= e_1 \left\{ I + \frac{1}{1!} 2\pi i J + \frac{1}{2!}(2\pi i J)^2 + \cdots \right\} \\ &= e_1 \begin{pmatrix} 1 & 2\pi i \\ 0 & 1 \end{pmatrix}, \end{aligned}$$

$$T_2 = e_3 \begin{pmatrix} 1 & 2\pi i \\ 0 & 1 \end{pmatrix}$$

for logarithmic solutions, and hence



with

$$E_1 = \begin{pmatrix} T_1 & 0 \\ 0 & I \end{pmatrix}, \quad E_2 = \begin{pmatrix} I & 0 \\ 0 & T_2 \end{pmatrix}.$$

We therefore obtain generators of the monodromy group with respect to the fundamental matrix solution $\mathcal{X}(t)$

$$M_1 = L_1^{-1} E_1 L_1 = \begin{pmatrix} I & C_{12} \\ 0 & I \end{pmatrix}^{-1} \begin{pmatrix} T_1 & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & C_{12} \\ 0 & I \end{pmatrix},$$

$$M_2 = L_2^{-1} E_2 L_2 = \begin{pmatrix} I & 0 \\ C_{21} & I \end{pmatrix}^{-1} \begin{pmatrix} I & 0 \\ 0 & T_2 \end{pmatrix} \begin{pmatrix} I & 0 \\ C_{21} & I \end{pmatrix}.$$

Since

$$\begin{pmatrix} I & C \\ 0 & I \end{pmatrix} \begin{pmatrix} I & -C \\ 0 & I \end{pmatrix} = I, \quad \begin{pmatrix} I & 0 \\ C & I \end{pmatrix} \begin{pmatrix} I & 0 \\ -C & I \end{pmatrix} = I,$$

we consequently obtain

$$M_1 = \begin{pmatrix} I & -C_{12} \\ 0 & I \end{pmatrix} \begin{pmatrix} T_1 & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & C_{12} \\ 0 & I \end{pmatrix} = \begin{pmatrix} T_1 & (T_1 - I)C_{12} \\ 0 & I \end{pmatrix},$$

$$M_2 = \begin{pmatrix} I & 0 \\ -C_{21} & I \end{pmatrix}^{-1} \begin{pmatrix} I & 0 \\ 0 & T_2 \end{pmatrix} \begin{pmatrix} I & 0 \\ C_{21} & I \end{pmatrix} = \begin{pmatrix} I & 0 \\ (T_2 - I)C_{21} & T_2 \end{pmatrix}.$$

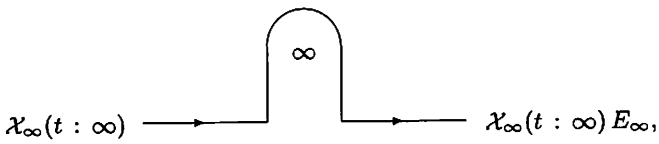
On the other hand, one can find a fundamental set of solutions near $t = \infty$

$$\mathcal{X}(t:\infty) = (x^1(t), x^2(t), x^3(t), x^4(t)),$$

where

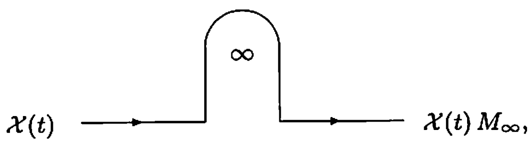
$$x^k(t) = t^{\nu_k} \sum_{s=0}^{\infty} h_k(s)\, t^{-s} \qquad (k = 1, 2, 3, 4).$$

Let E_∞ be a circuit matrix with respect to $\mathcal{X}(t:\infty)$ in the negative direction around the infinity :



$$E_\infty \;=\; \operatorname{diag}(f_1, f_2, f_3, f_4), \qquad f_k \;=\; \exp(2\pi i \nu_k) \quad (k = 1,2,3,4).$$

Denoting the connection coefficient between $\mathcal{X}(t)$ and $\mathcal{X}(t:\infty)$ by L_∞, we then have



$$M_\infty \;=\; L_\infty^{-1}\,E_\infty\,L_\infty\,.$$

Moreover, it is not difficult to see that there holds

$$M_1\,M_2 \;=\; M_\infty\,,$$

and hence that eigenvalues of $M_1\,M_2$ are equal to f_k $(k = 1,2,3,4)$.

Taking account of the above fact, we shall calculate the generators M_k $(k = 1,2)$. To this end, from the relation

$$M_1\,M_2 - f = (M_1 - f\,M_2^{-1})M_2,$$

where

$$M_2^{-1} = \begin{pmatrix} I & 0 \\ -T_2^{-1}(T_2 - I)C_{21} & T_2^{-1} \end{pmatrix},$$

we have only to consider the matrix $M_1 - f\,M_2^{-1}$.

Non-logarithmic case

We put

$$M_1 = \begin{pmatrix} e_1 & 0 & c_1 & c_2 \\ 0 & e_2 & c_3 & c_4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad M_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ d_1 & d_2 & e_3 & 0 \\ d_3 & d_4 & 0 & e_4 \end{pmatrix},$$

obtaining

$$M_2^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -e_3^{-1}d_1 & -e_3^{-1}d_2 & e_3^{-1} & 0 \\ -e_4^{-1}d_3 & -e_4^{-1}d_4 & 0 & e_4^{-1} \end{pmatrix},$$

$$\text{(1)} \qquad M_1 - f\,M_2^{-1} = \begin{pmatrix} e_1 - f & 0 & c_1 & c_2 \\ 0 & e_2 - f & c_3 & c_4 \\ gd_1 & gd_2 & 1-g & 0 \\ hd_3 & hd_4 & 0 & 1-h \end{pmatrix} \equiv (A_1, A_2, A_3, A_4),$$

where $g = f/e_3$, $h = f/e_4$ and the A_k denote column vectors.

From the form of E_∞ and the assumption that $f_1 \equiv f_2$, we can see that the rank of the matrix (1) is equal to 3 for $f = f_3, f_4$ and 2 for $f = f_1$. Only taking these facts into consideration, we shall determine all the constants c_k and d_k.

Since it is easy to see that the column vectors A_1 and A_2 are linearly independent, for $f = f_1$ we can take

$$\text{(2)} \qquad A_3 = \delta\,A_1 + \gamma\,A_2, \qquad A_4 = \delta_1\,A_1 + \gamma_1\,A_2,$$

that is,

$$\begin{cases} c_1 &= \delta(e_1 - f_1), \\ c_3 &= \gamma(e_2 - f_1), \\ 1 - g_1 &= \delta g_1 d_1 + \gamma g_1 d_2, \\ 0 &= \delta h_1 d_3 + \gamma h_1 d_4 \end{cases} \qquad \begin{cases} c_2 &= \delta_1(e_1 - f_1), \\ c_4 &= \gamma_1(e_2 - f_1), \\ 0 &= \delta_1 g_1 d_1 + \gamma_1 g_1 d_2, \\ 1 - h_1 &= \delta_1 h_1 d_3 + \gamma_1 h_1 d_4 \end{cases}$$

obtaining, by solving linear equations of the above last two lines,

$$c_1 = \delta(e_1 - f_1), \qquad c_2 = \delta_1(e_1 - f_1),$$

$$c_3 = \gamma(e_2 - f_1), \qquad c_4 = \gamma_1(e_2 - f_1),$$

$$d_1 = \frac{\gamma_1}{\Delta}\left(\frac{1-g_1}{g_1}\right) = \frac{\gamma_1}{\Delta}\left(\frac{e_3-f_1}{f_1}\right),$$

$$d_2 = -\frac{\delta_1}{\Delta}\left(\frac{1-g_1}{g_1}\right) = -\frac{\delta_1}{\Delta}\left(\frac{e_3-f_1}{f_1}\right),$$

$$d_3 = -\frac{\gamma}{\Delta}\left(\frac{1-h_1}{h_1}\right) = -\frac{\gamma}{\Delta}\left(\frac{e_4-f_1}{f_1}\right),$$

$$d_4 = \frac{\delta}{\Delta}\left(\frac{1-h_1}{h_1}\right) = \frac{\delta}{\Delta}\left(\frac{e_4-f_1}{f_1}\right)$$

where

$$\Delta = \delta\gamma_1 - \gamma\delta_1.$$

Since the rank is 3 for $f = f_3$ and A_1, A_2, A_3 are considered to be linearly independent, we have

$$A_4 = \beta_1 A_1 + \beta_2 A_2 + \beta_3 A_3, \tag{3}$$

that is,

$$\begin{cases} \beta_1(e_1 - f_3) + \beta_3 c_1 = c_2, \\ \beta_2(e_2 - f_3) + \beta_3 c_3 = c_4, \\ \beta_1 g_3 d_1 + \beta_2 g_3 d_2 + \beta_3(1 - g_3) = 0, \\ \beta_1 h_3 d_3 + \beta_2 h_3 d_4 = 1 - h_3. \end{cases}$$

From the last two formulas, we solve β_1, β_2 in terms of β_3 as follows :

$$\beta_1 = \delta_1 H - \beta_3\,\delta G, \qquad \beta_2 = \gamma_1 H - \beta_3\,\gamma G, \tag{4}$$

where

$$H = \left(\frac{1-h_3}{h_3}\right)\left(\frac{h_1}{1-h_1}\right) = \left(\frac{e_4-f_3}{f_3}\right)\left(\frac{f_1}{e_4-f_1}\right),$$

$$G = \left(\frac{1-g_3}{g_3}\right)\left(\frac{g_1}{1-g_1}\right) = \left(\frac{e_3-f_3}{f_3}\right)\left(\frac{f_1}{e_3-f_1}\right).$$

The substitution of these into the first two above formulas leads to

$$\begin{cases} \beta_3\{(e_1-f_1)-G(e_1-f_3)\}\delta = \{(e_1-f_1)-H(e_1-f_3)\}\delta_1, \\ \\ \beta_3\{(e_2-f_1)-G(e_2-f_3)\}\gamma = \{(e_2-f_1)-H(e_2-f_3)\}\gamma_1, \end{cases}$$

whence we cosequently obtain

$$\frac{\delta}{\gamma}\left(\frac{f_1f_3-e_1e_3}{f_1f_3-e_2e_3}\right) = \frac{\delta_1}{\gamma_1}\left(\frac{f_1f_3-e_1e_4}{f_1f_3-e_2e_4}\right),$$

which yields

$$\Delta = \delta\gamma_1\left(\frac{(e_4-e_3)(e_2-e_1)}{f_1f_3+f_1f_4-e_1e_4-e_2e_3}\right) = \delta_1\gamma\left(\frac{(e_4-e_3)(e_2-e_1)}{f_1f_3+f_1f_4-e_2e_4-e_1e_3}\right).$$

In this calculation we have used the Fuchs relation $e_1e_2e_3e_4 = f_1^2f_3f_4$. Moreover, the above determination justifies that the rank of the matrix (1) is also 3 for $f = f_4$.

We have thus proved that 8 constants c_i, d_i $(i = 1,2,3,4)$ are expressed in terms of 4 parameters δ, δ_1, γ, γ_1 among which there holds one relation. So it is easy to see that if three parameters are given, then all the constants included in generators of the monodromy group are completely determined. This is accomplished by the change of fundamental matrix solutions :

$$\mathcal{Y}(t) = \mathcal{X}(t)\,\mathrm{diag}(b_1,\, b_2,\, b_3,\, b_4) = (\,b_1x_1(t),\, b_2x_2(t),\, b_3x_3(t),\, b_4x_4(t))$$

for which, denoting the diagonal matrix by D, we have generators $D^{-1}M_kD$ $(k = 1,2)$. Hence, one can always assign any non-zero values to three parameters.

Logarithmic case

First we shall consider a case in which one of $X_k(t)$ $(k = 1,2)$, for example, $X_1(t)$ is a logarithmic matrix solution. In this case, of course, $\alpha_1 = \alpha_2$, $\varepsilon_1 = 1$. And we have

$$M_1 = \begin{pmatrix} e_1 & 2\pi ie_1 & c_1 & c_2 \\ 0 & e_1 & c_3 & c_4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

When we follow the calculation stated above, putting $e_2 = e_1$, we have only to replace two formulas in (2) by

$$c_1 = \delta(e_1-f_1)+\gamma z, \qquad c_2 = \delta_1(e_1-f_1)+\gamma_1 z$$

and one formula in (3) by

$$\beta_1(e_1-f_3)+\beta_2 z+\beta_3 c_1 = c_2 \qquad (\,z = 2\pi ie_1\,).$$

Substituting c_1, c_2, together with formulas derived for β_1, β_2 and

$$\beta_3\{(e_1 - f_1) - G(e_1 - f_3)\}\gamma = \{(e_1 - f_1) - H(e_1 - f_3)\}\gamma_1,$$

into the last formula, we obtain

$$\frac{\delta}{\gamma} - \frac{\delta_1}{\gamma_1} = \left(\frac{e_3 - e_4}{f_1 f_3 + f_1 f_4 - e_1 e_4 - e_1 e_3}\right) z,$$

whence in this case we have

$$\Delta = \gamma\gamma_1 \left(\frac{e_3 - e_4}{f_1 f_3 + f_1 f_4 - e_1 e_4 - e_1 e_3}\right) z.$$

Therefore, by assigning arbitrarily any values to three parameters, we can determine the monodromy group.

Lastly, we shall consider a case in which both $X_k(t)$ $(k = 1, 2)$ are logarithmic matrix solutions. In this case, $\alpha_1 = \alpha_2$, $\varepsilon_1 = 1$ and $\alpha_3 = \alpha_4$, $\varepsilon_2 = 1$.

In order to simplify the calculation, we put

$$M_1 = \begin{pmatrix} T_1 & D_1 \\ 0 & I \end{pmatrix}, \qquad M_2 = \begin{pmatrix} I & 0 \\ D_2 & T_2 \end{pmatrix},$$

where

$$\begin{aligned} T_1 &= e_1 \begin{pmatrix} 1 & 2\pi i \\ 0 & 1 \end{pmatrix}, \qquad T_2 = e_3 \begin{pmatrix} 1 & 2\pi i \\ 0 & 1 \end{pmatrix}, \\ D_1 &= e_1 \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix}, \qquad D_2 = \begin{pmatrix} 1 & 2\pi i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} d_1 & d_2 \\ d_3 & d_4 \end{pmatrix} \end{aligned}$$

and we have

$$M_1 - f M_2^{-1} = \begin{pmatrix} e_1 - f & 2\pi i e_1 & e_1 c_1 & e_1 c_2 \\ 0 & e_1 - f & e_1 c_3 & e_1 c_4 \\ g d_1 & g d_2 & 1 - g & 2\pi i g \\ g d_3 & g d_4 & 0 & 1 - g \end{pmatrix} \equiv (A_1, A_2, A_3, A_4),$$

where again $g = f/e_3$.

From the above form, we may assume that A_2 and A_4 are linearly dependent for $f = f_3$ because the rank of the matrix is 3. Putting $A_4 = \beta A_2$, we immediately obtain

$$\begin{aligned} c_2 &= 2\pi i \beta, \qquad c_4 = \beta\left(\frac{e_1 - f_3}{e_1}\right), \\ d_2 &= \frac{2\pi i}{\beta}, \qquad d_4 = \frac{1}{\beta}\left(\frac{e_3 - f_3}{f_3}\right). \end{aligned}$$

Since the rank is 2 for $f = f_1$, from the relations (2), we again obtain

$$\begin{aligned} c_1 &= \delta\left(\frac{e_1 - f_1}{e_1}\right) + 2\pi i\,\gamma, \\ c_3 &= \gamma\left(\frac{e_1 - f_1}{e_1}\right), \\ d_1 &= \frac{1}{\delta}\left\{\frac{e_3 - f_1}{f_1} - \gamma\, d_2\right\}, \\ d_3 &= -\frac{\gamma}{\delta}\, d_4 \end{aligned}$$

and

$$\begin{aligned} c_2 &= \delta_1\left(\frac{e_1 - f_1}{e_1}\right) + 2\pi i\,\gamma_1, \\ c_4 &= \gamma_1\left(\frac{e_1 - f_1}{e_1}\right), \\ d_1 &= \frac{1}{\delta_1}\left\{2\pi i - \gamma_1\, d_2\right\}, \\ d_3 &= \frac{1}{\delta_1}\left\{\frac{e_3 - f_1}{f_1} - \gamma_1\, d_4\right\}. \end{aligned}$$

Combining these relations, we can express all the constants c_k and d_k only in terms of one parameter β. In fact, from the relations for c_4 and c_2

$$\beta\left(\frac{e_1 - f_3}{e_1}\right) = \gamma_1\left(\frac{e_1 - f_1}{e_1}\right),$$

$$2\pi i\,\beta = \delta_1\left(\frac{e_1 - f_1}{e_1}\right) + 2\pi i\,\gamma_1,$$

respectively, we have

$$\gamma_1 = \beta\left(\frac{e_1 - f_3}{e_1 - f_1}\right), \qquad \delta_1 = 2\pi i\,\beta\frac{(f_3 - f_1)e_1}{(e_1 - f_1)^2},$$

which then determine the values of d_1 and d_3 as follows :

$$\begin{aligned} d_1 &= \frac{1}{\beta}\left(\frac{e_1 - f_1}{e_1}\right), \\ d_3 &= \frac{1}{2\pi i\,\beta}\frac{(e_1 - f_1)(e_1 e_3 - f_1 f_3)}{e_1 f_1 f_3}. \end{aligned}$$

Combining these values with the relations for d_1 and d_3, we can determine δ and γ, and then we finally obtain c_1 and c_3 as follows :

$$\begin{aligned} \delta &= \beta \frac{-e_1(e_3-f_1)(e_3-f_3)}{e_3(e_1-f_1)^2}, \\ \gamma &= \frac{\beta}{2\pi i}\frac{(e_3-f_1)(e_1e_3-f_1f_3)}{e_3f_1(e_1-f_1)}, \\ c_1 &= \beta\frac{(e_3-f_1)}{f_1}, \\ c_3 &= \frac{\beta}{2\pi i}\frac{(e_3-f_1)(e_1e_3-f_1f_3)}{e_1e_3f_1}. \end{aligned}$$

We have thus determined all elements of generators in terms of one parameter β. It is not difficult to verify that the rank of the matrix considered is also equal to 3 for $f = f_4$ by means of the Fuchs relation $e_1^2\, e_3^2 = f_1^2\, f_3\, f_4$.

Any value can be assigned to the parameter β by the transformation

$$\mathcal{Y}(t) = \mathcal{X}(t)\begin{pmatrix} b_1 I & 0 \\ 0 & b_2 I \end{pmatrix} = (\, b_1X_1(t),\, b_2X_2(t)\,),$$

which does not change the form of the matrix A.

We have thus derived the elements of generators of the monodromy group :

$$\begin{aligned} D_1 &= \begin{pmatrix} \beta\,\frac{e_1(e_3-f_1)}{f_1} & 2\pi i\,\beta\, e_1 \\ \frac{\beta}{2\pi i}\frac{(e_3-f_1)(e_1e_3-f_1f_3)}{e_3f_1} & \beta\,(e_1-f_3) \end{pmatrix}, \\ D_2 &= \begin{pmatrix} 1 & 2\pi i \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{\beta}\left(\frac{e_1-f_1}{e_1}\right) & \frac{2\pi i}{\beta} \\ \frac{1}{2\pi i\,\beta}\frac{(e_1-f_1)(e_1e_3-f_1f_3)}{e_1f_1f_3} & \frac{1}{\beta}\left(\frac{e_3-f_3}{f_3}\right) \end{pmatrix}. \end{aligned}$$

Symbolic manipulation

In order to calculate the monodromy group, we have used conditions concerning ranks of eigenvalues. Such conditions of rank of a matrix can be analyzed by its minors, though it was done by linear dependence of vectors in this paper. Whichever methods are taken, one can determine all elements of generators merely by several conditions among many conditions, the rest of which become identical formulas. For example, the constants c_k and d_k are determined only by f_1 and f_3. Then the conditions for f_4 must be identities under the Fuchs relation. On the other hand, generators of the monodromy group depends on the choice of fundamental matrix solutions. We have many expressions of generators. We then want to know whether one is similar to the other. Such an evaluation is not difficult, but tedious. To this end, the symbolic manipulation is very effective. Here we shall show one example.

```
procedure minor(i,j);
begin
n:=first length m$
for k:=1:n do for l:=1:n do mm(k,l):=m(k,l)$
for k:=1:n do mm(i,k):=0$
mm(i,j):=1$
s:=det(mm);
return s$
end;

matrix m(4,4),mm(4,4)$

m(1,1):=e1-f$m(1,2):=z$m(1,3):=c1$m(1,4):=c2$
m(2,1):=0$m(2,2):=e1-f$m(2,3):=c3$m(2,4):=c4$
m(3,1):=g*d1$m(3,2):=g*d2$m(3,3):=1-g$m(3,4):=0$
m(4,1):=h*d3$m(4,2):=h*d4$m(4,3):=0$m(4,4):=1-h$

c1:=a*(e1-f1)+c*z$
c2:=b*(e1-f1)+d*z$
c3:=c*(e1-f1)$
c4:=d*(e1-f1)$

d1:=(d*(e3-f1))/(delta*f1)$
d2:=-(b*(e3-f1))/(delta*f1)$
d3:=-(c*(e4-f1))/(delta*f1)$
d4:=(a*(e4-f1))/(delta*f1)$

a:=(delta+c*b)/d$
delta:=(c*d*(e3-e4)*z)/(f1*f3+f1*f4-e4*e1-e3*e1)$
f4:=(e1**2*e3*e4)/(f1**2*f3);
g:=f/e3$
h:=f/e4$

f:=f3;
```

```
w:=det(m);<-------- This shows that the rank is 3 for f3.

f:=f1;<------------ This part shows that the rank is 2 for f1.
y:=det(m);
for i:=1:4 do
for j:=1:4 do <<r:=minor(i,j);write "Minor", "(",i,",",j,")=",r>>;

f:=f4;<------------ This certainly shows the rank is 3 for f4 !!
z:=det(m);

end;
```

2. 5-th order hypergeometric system

For the hypergeometric system

$$(t-B)\,\frac{dX}{dt} \;=\; A\,X,$$

let us assume that B has multiple eigenvalues :

$$B = \mathrm{diag}\,(\overbrace{\lambda_1,\ldots,\lambda_1}^{n_1},\ \overbrace{\lambda_2,\ldots,\lambda_2}^{n_2},\ \ldots,\overbrace{\lambda_p,\ldots,\lambda_p}^{n_p}),$$

where

$$\lambda_i \neq \lambda_j \qquad (i \neq j) \qquad n_i \geq 1 \qquad (i,j = 1,2,...,p)$$

and

$$n_1 + n_2 + \cdots + n_p = n.$$

As explained before, applying the transformation $Y = DX$, where D is a block-diagonal matrix, we may assume from the outset that $A = (A_{ij})$ is blockwise decomposed according to the multiplicities of B and the diagonal block A_{ii} are Jordan canonical matrices. Hereafter it will be assumed that the eigenvalues of A_{ii} are not congruent to zero and to eigenvalues of A modulo integers.

Moreover, assume that A is similar to a diagonal matrix of the form

$$A \sim \mathrm{diag}\,(\overbrace{\nu_{1,1},\ldots,\nu_{1,\theta_1}}^{\theta_1},\ \overbrace{\nu_{2,1},\ldots,\nu_{2,\theta_2}}^{\theta_2},\ \ldots,\overbrace{\nu_{q,1},\ldots,\nu_{q,\theta_q}}^{\theta_q}),$$

where

$$\nu_{i,1} \equiv \nu_{i,2} \equiv \cdots \equiv \nu_{i,\theta_i} \pmod{\mathbf{Z}} \qquad (i = 1,2,...,q)$$

and

$$\theta_1 + \theta_2 + \cdots + \theta_q = n.$$

Labels $L(B) = n_1 \cdot n_2 \cdots n_p$ and $L(A) = \theta_1 \cdot \theta_2 \cdots \theta_q$ are given to such a hypergeometric system in which

$$\mathcal{N} = n^2 - n + 2 - \sum_{i=1}^{p} n_i^2 - \sum_{k=1}^{q} \theta_k^2$$

accessary parameters are included. It is well-known (see [3] and also [2],[1]) that if $\mathcal{N} \leq 0$, that is, a hypergeometric system is free of accessary parameters, then its monodromy group is computable. The typical examples are Jordan-Pochhammer equation with labels $L(B) = 1 \cdot 1 \cdots 1$, $L(A) = (n-1) \cdot 1$ and Generalized Hypergeometric Equation with labels $L(B) = (n-1) \cdot 1$, $L(A) = 1 \cdot 1 \cdots 1$. As a matter of course, the hypergeometric system treated in the preceding section has the labels $L(B) = 2 \cdot 2$, $L(A) = 2 \cdot 1 \cdot 1$ and is also free of accessary parameters.

In this section we shall deal with the evaluation of monodromy groups for hypergeometric systems with labels $L(B) = 3 \cdot 2$, $L(A) = 2 \cdot 2 \cdot 1$ and $L(B) = 2 \cdot 2 \cdot 1$, $L(A) = 3 \cdot 2$.

2.1 $L(B) = 3 \cdot 2$, $L(A) = 2 \cdot 2 \cdot 1$

According to Jordan canonical forms of A_{11} :

$$\begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}, \quad \begin{pmatrix} a_1 & 0 & 0 \\ 1 & a_1 & 0 \\ 0 & 0 & a_3 \end{pmatrix}, \quad \begin{pmatrix} a_1 & 0 & 0 \\ 1 & a_1 & 0 \\ 0 & 1 & a_1 \end{pmatrix},$$

the respective circuit matrices of non-holomorphic or logarithmic solutions near $t = \lambda_1$ become

$$\begin{pmatrix} e_1 & 0 & 0 \\ 0 & e_2 & 0 \\ 0 & 0 & e_3 \end{pmatrix}, \quad \begin{pmatrix} e_1 & 2\pi i e_1 & 0 \\ 0 & e_1 & 0 \\ 0 & 0 & e_3 \end{pmatrix}, \quad \begin{pmatrix} e_1 & 2\pi i e_1 & (2\pi i)^2 e_1/2! \\ 0 & e_1 & 2\pi i e_1 \\ 0 & 0 & e_1 \end{pmatrix},$$

where $e_k = \exp(2\pi i\, a_k)$. Just like this, the circuit matrix around $t = \lambda_2$ behaves according to the form of A_{22}.

So we shall now attempt to calculate generators of the monodromy group by putting

$$M_1 = \begin{pmatrix} e_1 & p_1 & p_2 & c_1 & c_2 \\ 0 & e_2 & p_3 & c_3 & c_4 \\ 0 & 0 & e_3 & c_5 & c_6 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ d_1 & d_2 & d_3 & e_3 & q_1 \\ d_4 & d_5 & d_6 & 0 & e_4 \end{pmatrix},$$

where p_i $(i = 1, 2, 3)$ and q_1 are vanishing in a generic case, and p_1, p_3 are equal to $2\pi i\, e_1$ or zero, p_2 is equal to $(2\pi i)^2 e_1/2!$ or zero and q_1 is equal to $2\pi i e_4$ or zero in logarithmic cases.

Now, in order to evaluate all constants c_i, d_i $(i = 1, 2, 3, 4, 5, 6)$, we again consider the matrix $M_1 - f\, M_2^{-1}$, the rank of which is equal to 3 for $f = f_1$, f_2 and 4 for $f = f_3$, the f_j being $\exp(2\pi i \nu_{j,*})$. The rank condition can be analyzed by the vanishing of minors. But this method is very complicated. So, as explained in the preceding section, we introduce the same number of parameters, which are coefficients of linear combinations of vectors, and then we express the c_i, d_i explicitly in terms of such parameters.

First, we shall consider a case in which $p_2 = p_3 = 0$, since a more simple method can be applied to a case in which p_2 and p_3 are not vanishing. In stead of the matrix $M_1 - f\, M_2^{-1}$, we may treat its equivalent matrix as follows :

$$\begin{pmatrix} e_1 - f & p & 0 & c_1 & c_2 \\ 0 & e_2 - f & 0 & c_3 & c_4 \\ 0 & 0 & e_3 - f & c_5 & c_6 \\ d_1 & d_2 & d_3 & (e_4 - f)/f & q \\ d_4 & d_5 & d_6 & 0 & (e_5 - f)/f \end{pmatrix} \equiv (A_1, A_2, A_3, A_4, A_5).$$

Here it should be remarked that if $q \neq 0$, i.e., $q = 2\pi i$, then the d_i are different from those of M_2, which are, in fact, derived by

$$\begin{pmatrix} 1 & 2\pi i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} d_1 & d_2 & d_3 \\ d_4 & d_5 & d_6 \end{pmatrix}.$$

Now, since the rank of the matrix is 3 for $f = f_1$, we take

$$A_4 = \gamma_1\, A_1 + \gamma_2\, A_2 + \gamma_3\, A_3, \qquad A_5 = \delta_1\, A_1 + \delta_2\, A_2 + \delta_3\, A_3,$$

whence

$$\left\{ \begin{array}{ll} c_1 & = \gamma_1(e_1 - f_1) + \gamma_2 p, \\ c_3 & = \gamma_2(e_2 - f_1), \\ c_5 & = \gamma_3(e_3 - f_1), \\ \dfrac{e_4 - f_1}{f_1} & = \gamma_1 d_1 + \gamma_2 d_2 + \gamma_3 d_3, \\ 0 & = \gamma_1 d_4 + \gamma_2 d_5 + \gamma_3 d_6, \end{array} \right. \qquad \left\{ \begin{array}{ll} c_2 & = \delta_1(e_1 - f_1) + \delta_2 p, \\ c_4 & = \delta_2(e_2 - f_1), \\ c_6 & = \delta_3(e_3 - f_1), \\ q & = \delta_1 d_1 + \delta_2 d_2 + \delta_3 d_3, \\ \dfrac{e_5 - f_1}{f_1} & = \delta_1 d_4 + \delta_2 d_5 + \delta_3 d_6. \end{array} \right.$$

Putting here and hereafter

$$\varphi_i = e_i - f_1, \qquad \psi_i = e_i - f_2 \quad (i = 1, 2, 3, 4, 5),$$

we solve linear equations of the above last two lines to express d_2, d_3 and d_5, d_6 in terms of d_1 and d_4, respectively. We then have

$$\text{(5)} \qquad \begin{cases} d_2 = -\frac{\Delta_1}{\Delta_2} d_1 + \frac{1}{\Delta_2}\left\{\gamma_3\, q - \delta_3 \left(\frac{\varphi_4}{f_1}\right)\right\}, \\ d_3 = \frac{\Delta}{\Delta_2} d_1 - \frac{1}{\Delta_2}\left\{\gamma_2\, q - \delta_2 \left(\frac{\varphi_4}{f_1}\right)\right\}, \end{cases}$$

$$\text{(6)} \qquad \begin{cases} d_5 = -\frac{\Delta_1}{\Delta_2} d_4 + \frac{\gamma_3}{\Delta_2}\left(\frac{\varphi_5}{f_1}\right), \\ d_6 = \frac{\Delta}{\Delta_2} d_4 - \frac{\gamma_2}{\Delta_2}\left(\frac{\varphi_5}{f_1}\right), \end{cases}$$

where we have put

$$\text{(7)} \qquad \Delta_2 = \delta_2\,\gamma_3 - \gamma_2\,\delta_3, \qquad \Delta_1 = \delta_1\,\gamma_3 - \gamma_1\,\delta_3, \qquad \Delta = \delta_1\,\gamma_2 - \gamma_1\,\delta_2.$$

On the other hand, since the rank is also 3 for $f = f_2$, we can take

$$A_4 = \gamma_1'\, A_1 + \gamma_2'\, A_2 + \gamma_3'\, A_3, \qquad A_5 = \delta_1'\, A_1 + \delta_2'\, A_2 + \gamma_3'\, A_3,$$

whence we have the same formulas as (5) and (6), where the γ_i, δ_i and φ_4, φ_5 are replaced by γ_i', δ_i' and ψ_4, ψ_5. From formulas for the c_i, it is immediate to see that there hold

$$\gamma_1\,\varphi_1 + \gamma_2\, p = \gamma_1'\,\psi_1 + \gamma_2'\, p, \qquad \gamma_2\,\varphi_2 = \gamma_2'\,\psi_2, \qquad \gamma_3\,\varphi_3 = \gamma_3'\,\psi_3,$$

$$\delta_1\,\varphi_1 + \delta_2\, p = \delta_1'\,\psi_1 + \delta_2'\, p, \qquad \delta_2\,\varphi_2 = \delta_2'\,\psi_2, \qquad \delta_3\,\varphi_3 = \delta_3'\,\psi_3,$$

which yield

$$\Delta_2' = \delta_2'\,\gamma_3' - \gamma_2'\,\delta_3' = \frac{\varphi_2\varphi_3}{\psi_2\psi_3}\,\Delta_2,$$

$$\Delta_1' = \delta_1'\,\gamma_3' - \gamma_1'\,\delta_3' = \frac{\varphi_1\varphi_3}{\psi_1\psi_3}\,\Delta_1 + \frac{(\psi_2 - \varphi_2)\varphi_3}{\psi_1\psi_2\psi_3}\, p\,\Delta_2,$$

$$\Delta' = \delta_1'\,\gamma_2' - \gamma_1'\,\delta_2' = \frac{\varphi_1\varphi_2}{\psi_1\psi_2}\,\Delta.$$

Taking account of these relations, we consequently obtain

$$d_2 \;=\; -\frac{\Delta_1'}{\Delta_2'} d_1 + \frac{1}{\Delta_2'}\left\{\gamma_3'\, q - \delta_3'\left(\frac{\psi_4}{f_2}\right)\right\}$$

$$
\begin{aligned}
&= -\left(\frac{\psi_2\varphi_1}{\varphi_2\psi_1}\frac{\Delta_1}{\Delta_2}+\frac{(\psi_2-\varphi_2)}{\psi_1\varphi_2}p\right)d_1+\frac{1}{\Delta_2}\frac{\psi_2}{\varphi_2}\left\{\gamma_3\,q-\delta_3\left(\frac{\psi_4}{f_2}\right)\right\},\\
d_3 &= \frac{\Delta'}{\Delta_2'}d_1-\frac{1}{\Delta_2'}\left\{\gamma_2'\,q-\delta_2'\left(\frac{\psi_4}{f_2}\right)\right\}\\
&= \left(\frac{\psi_3\varphi_1}{\varphi_3\psi_1}\right)\frac{\Delta}{\Delta_2}d_1-\frac{1}{\Delta_2}\frac{\psi_3}{\varphi_3}\left\{\gamma_2\,q-\delta_2\left(\frac{\psi_4}{f_2}\right)\right\},\\
d_5 &= -\frac{\Delta_1'}{\Delta_2'}d_4+\frac{\gamma_3'}{\Delta_2'}\left(\frac{\psi_5}{f_2}\right)\\
&= -\left(\frac{\psi_2\varphi_1}{\varphi_2\psi_1}\frac{\Delta_1}{\Delta_2}+\frac{(\psi_2-\varphi_2)}{\psi_1\varphi_2}p\right)d_4+\frac{\gamma_3}{\Delta_2}\frac{\psi_2}{\varphi_2}\frac{\psi_5}{f_2},\\
d_6 &= \frac{\Delta'}{\Delta_2'}d_4-\frac{\gamma_2'}{\Delta_2'}\left(\frac{\psi_5}{f_2}\right)\\
&= \left(\frac{\psi_3\varphi_1}{\varphi_3\psi_1}\right)\frac{\Delta}{\Delta_2}d_4-\frac{\gamma_2}{\Delta_2}\frac{\psi_3}{\varphi_3}\frac{\psi_5}{f_2}.
\end{aligned}
$$

Combining these relations with (5) and (6), we now seek d_1 and d_4. For simplicity of notations, we first put

$$
\begin{aligned}
\Phi_1 &= \left(\frac{\varphi_4}{f_1}-\left(\frac{\psi_2}{\varphi_2}\right)\frac{\psi_4}{f_2}\right)\Big/\left(\left(\frac{\psi_2}{\varphi_2}\right)\left(\frac{\varphi_1}{\psi_1}\right)-1\right),\\
\Phi_2 &= \left(\frac{\varphi_4}{f_1}-\left(\frac{\psi_3}{\varphi_3}\right)\frac{\psi_4}{f_2}\right)\Big/\left(\left(\frac{\psi_3}{\varphi_3}\right)\left(\frac{\varphi_1}{\psi_1}\right)-1\right),\\
\Psi_1 &= \left(\frac{\varphi_5}{f_1}-\left(\frac{\psi_2}{\varphi_2}\right)\frac{\psi_5}{f_2}\right)\Big/\left(\left(\frac{\psi_2}{\varphi_2}\right)\left(\frac{\varphi_1}{\psi_1}\right)-1\right),\\
\Psi_2 &= \left(\frac{\varphi_5}{f_1}-\left(\frac{\psi_3}{\varphi_3}\right)\frac{\psi_5}{f_2}\right)\Big/\left(\left(\frac{\psi_3}{\varphi_3}\right)\left(\frac{\varphi_1}{\psi_1}\right)-1\right),\\
P &= \left(\frac{(\psi_2-\varphi_2)}{\psi_1\varphi_2}\right)p\Big/\left(\left(\frac{\psi_2}{\varphi_2}\right)\left(\frac{\varphi_1}{\psi_1}\right)-1\right),\\
Q_1 &= \left(\frac{(\psi_2-\varphi_2)}{\varphi_2}\right)q\Big/\left(\left(\frac{\psi_2}{\varphi_2}\right)\left(\frac{\varphi_1}{\psi_1}\right)-1\right),\\
Q_2 &= \left(\frac{(\psi_3-\varphi_3)}{\varphi_3}\right)q\Big/\left(\left(\frac{\psi_3}{\varphi_3}\right)\left(\frac{\varphi_1}{\psi_1}\right)-1\right).
\end{aligned}
$$

Then we have

$$(8)\qquad \begin{cases} d_1 = \dfrac{Q_1\gamma_3 + \Phi_1\delta_3}{\Delta_1 + P\Delta_2} = \dfrac{Q_2\gamma_2 + \Phi_2\delta_2}{\Delta}, \\[2ex] d_4 = \dfrac{-\gamma_3\Psi_1}{\Delta_1 + P\Delta_2} = \dfrac{-\gamma_2\Psi_2}{\Delta}. \end{cases}$$

From relations thus derived, we can determine all the constants of generators by assigning any values to 4 parameters among 6 parameters γ_i, δ_i $(i = 1,2,3)$. To see this fact in more precise form, we here put

$$\frac{\delta_1}{\gamma_1} = X, \qquad \frac{\delta_2}{\gamma_2} = \beta, \qquad \frac{\delta_3}{\gamma_3} = \alpha.$$

Then, from (8) we obtain

$$\begin{cases} \dfrac{Q_1 + \Phi_1\alpha}{\gamma_1(X-\alpha) + P\gamma_2(\beta-\alpha)} = \dfrac{Q_2 + \Phi_2\beta}{\gamma_1(X-\beta)}, \\[2ex] \dfrac{-\Psi_1}{\gamma_1(X-\alpha) + P\gamma_2(\beta-\alpha)} = \dfrac{-\Psi_2}{\gamma_1(X-\beta)}, \end{cases}$$

the solution of which leads directly to

$$\begin{aligned} \beta &= \frac{\Phi_1\Psi_2}{\Psi_1\Phi_2}\alpha + \frac{Q_1\Psi_2 - Q_2\Psi_1}{\Psi_1\Phi_2}, \\ X &= \frac{\Psi_2}{\Phi_2}\left\{\frac{\Phi_1 - \Phi_2}{\Psi_1 - \Psi_2} + P\frac{\Phi_1\Psi_2 - \Phi_2\Psi_1}{\Psi_1(\Psi_1 - \Psi_2)}\frac{\gamma_2}{\gamma_1}\right\}\alpha \\ &\quad + \left(\frac{Q_1\Psi_2 - Q_2\Psi_1}{\Psi_1\Phi_2(\Psi_1 - \Psi_2)}\right)\left(\Psi_1 + P\Psi_2\frac{\gamma_2}{\gamma_1}\right). \end{aligned}$$

Together with the above formulas, all the c_i and d_i can be considered to be exactly determined. For example, let $\underline{\gamma_1, \gamma_2, \gamma_3}$ and $\underline{\delta_3}$ be given. Then α is a known value, and hence β, X are known too. It is clear that the c_i are expressed in terms of those values. As for the d_i, we have

$$\begin{aligned} d_1 &= \frac{Q_2 + \Phi_2\beta}{\gamma_1(X-\beta)}, \\ d_4 &= -\frac{\Psi_2}{\gamma_1(X-\beta)}, \\ d_2 &= -\frac{1}{\gamma_2(\beta-\alpha)}\left\{\left(\frac{X-\alpha}{X-\beta}\right)(Q_2 + \Phi_2\beta) + \alpha\left(\frac{\varphi_4}{f_1}\right)\right\}, \\ d_3 &= \frac{1}{\gamma_3(\beta-\alpha)}\left\{Q_2 + \left(\Phi_2 + \left(\frac{\varphi_4}{f_1}\right)\right)\beta\right\}, \end{aligned}$$

$$\begin{aligned} d_5 &= \frac{1}{\gamma_2(\beta-\alpha)}\left\{\left(\frac{X-\alpha}{X-\beta}\right)\Psi_2+\left(\frac{\varphi_5}{f_1}\right)\right\}, \\ d_6 &= -\frac{1}{\gamma_3(\beta-\alpha)}\left\{\Psi_2+\left(\frac{\varphi_5}{f_1}\right)\right\}. \end{aligned}$$

Non-logarithmic case

In this generic case, we have only to put $p=q=0$ and hence $P=Q_1=Q_2=0$ in the above formulas, obtaining

$$\begin{aligned} c_1 &= \gamma_1(e_1-f_1), \qquad c_2 = \frac{(e_1e_4-f_1f_2)(e_3e_5-f_1f_2)(e_1-f_1)}{(e_1e_5-f_1f_2)(e_3e_4-f_1f_2)}\alpha\gamma_1, \\ c_3 &= \gamma_2(e_2-f_1), \qquad c_4 = \frac{(e_2e_4-f_1f_2)(e_3e_5-f_1f_2)(e_2-f_1)}{(e_2e_5-f_1f_2)(e_3e_4-f_1f_2)}\alpha\gamma_2, \\ c_5 &= \gamma_3(e_3-f_1), \qquad c_6 = (e_3-f_1)\alpha\gamma_3, \\ d_1 &= \frac{(e_1e_5-f_1f_2)(e_2e_4-f_1f_2)(e_3e_4-f_1f_2)(e_1-f_2)}{(e_1-e_2)(e_1-e_3)(e_4-e_5)f_1^2f_2^2}\frac{1}{\gamma_1}, \\ d_2 &= \frac{-(e_1e_4-f_1f_2)(e_2e_5-f_1f_2)(e_3e_4-f_1f_2)(e_2-f_2)}{(e_1-e_2)(e_2-e_3)(e_4-e_5)f_1^2f_2^2}\frac{1}{\gamma_2}, \\ d_3 &= \frac{(e_1e_4-f_1f_2)(e_2e_4-f_1f_2)(e_3e_5-f_1f_2)(e_3-f_2)}{(e_1-e_3)(e_2-e_3)(e_4-e_5)f_1^2f_2^2}\frac{1}{\gamma_3}, \\ d_4 &= \frac{-(e_1e_5-f_1f_2)(e_2e_5-f_1f_2)(e_3e_4-f_1f_2)(e_1-f_2)}{(e_1-e_2)(e_1-e_3)(e_4-e_5)f_1^2f_2^2}\frac{1}{\alpha\gamma_1}, \\ d_5 &= \frac{(e_1e_5-f_1f_2)(e_2e_5-f_1f_2)(e_3e_4-f_1f_2)(e_2-f_2)}{(e_1-e_2)(e_2-e_3)(e_4-e_5)f_1^2f_2^2}\frac{1}{\alpha\gamma_2}, \\ d_6 &= \frac{-(e_1e_5-f_1f_2)(e_2e_5-f_1f_2)(e_3e_4-f_1f_2)(e_3-f_2)}{(e_1-e_3)(e_2-e_3)(e_4-e_5)f_1^2f_2^2}\frac{1}{\alpha\gamma_3}. \end{aligned}$$

Full-logarithmic case

Now we consider a case, where there exist 5 by 3 logarithmic matrix solution near $t=\lambda_1$

and 5 by 2 logarithmic matrix solution near $t = \lambda_2$, dealing with the matrix

$$\begin{pmatrix} e_1 - f & p_1 & p_2 & c_1 & c_2 \\ 0 & e_1 - f & p_1 & c_3 & c_4 \\ 0 & 0 & e_1 - f & c_5 & c_6 \\ d_1 & d_2 & d_3 & (e_4 - f)/f & q \\ d_4 & d_5 & d_6 & 0 & (e_4 - f)/f \end{pmatrix} \equiv (A_1, A_2, A_3, A_4, A_5).$$

In this case, one can assume that A_3 and A_5 are linearly dependent for $f = f_3$, for which the rank of the matrix is 4. Putting $A_5 = \beta A_3$, we have

$$c_2 = p_2 \beta, \qquad c_4 = p_1 \beta, \qquad c_6 = \eta_1 \beta,$$

$$d_3 = \frac{q}{\beta}, \qquad d_6 = \frac{\eta_4}{f_3 \beta},$$

where $\eta_i = e_i - f_3 \, (i = 1, 4)$.

Again, from the rank condition for $f = f_1$ stated above, we put

$$A_4 = \gamma_1 A_1 + \gamma_2 A_2 + \gamma_3 A_3, \qquad A_5 = \delta_1 A_1 + \delta_2 A_2 + \delta_3 A_3,$$

whence

$$\begin{cases} c_1 &= \gamma_1 \varphi_1 + \gamma_2 p_1 + \gamma_3 p_2, \\ c_3 &= \gamma_2 \varphi_1 + \gamma_3 p_1, \\ c_5 &= \gamma_3 \varphi_1, \\ \frac{\varphi_4}{f_1} &= \gamma_1 d_1 + \gamma_2 d_2 + \gamma_3 d_3, \\ 0 &= \gamma_1 d_4 + \gamma_2 d_5 + \gamma_3 d_6, \end{cases} \qquad \begin{cases} c_2 &= \delta_1 \varphi_1 + \delta_2 p_1 + \delta_3 p_2, \\ c_4 &= \delta_2 \varphi_1 + \delta_3 p_1, \\ c_6 &= \delta_3 \varphi_1, \\ q &= \delta_1 d_1 + \delta_2 d_2 + \delta_3 d_3, \\ \frac{\varphi_4}{f_1} &= \delta_1 d_4 + \delta_2 d_5 + \delta_3 d_6. \end{cases}$$

Now, combining relations for c_i $(i = 2, 4, 6)$, we can express the δ_i in terms of one parameter β :

(9) $$\delta_3 = \frac{\eta_1}{\varphi_1} \beta, \qquad \delta_2 = \frac{\varphi_1 - \eta_1}{\varphi_1^2} p_1 \beta, \qquad \delta_1 = \frac{\varphi_1 - \eta_1}{\varphi_1^2} \left\{ p_2 - \left(\frac{p_1^2}{\varphi_1} \right) \right\} \beta.$$

On the other hand, solving d_1, d_2 and d_4, d_5 in terms of d_3 and d_6, respectively, we have

(10) $$\begin{cases} d_1 = \frac{\Delta_2}{\Delta} d_3 + \frac{1}{\Delta} \left\{ \gamma_2 \, q - \delta_2 \left(\frac{\varphi_4}{f_1} \right) \right\}, \\ d_2 = -\frac{\Delta_1}{\Delta} d_3 - \frac{1}{\Delta} \left\{ \gamma_1 \, q - \delta_1 \left(\frac{\varphi_4}{f_1} \right) \right\}, \end{cases}$$

$$(11)\qquad \begin{cases} d_4 = \frac{\Delta_2}{\Delta} d_6 + \frac{\gamma_2}{\Delta}\left(\frac{\varphi_4}{f_1}\right), \\[2ex] d_5 = -\frac{\Delta_1}{\Delta} d_6 - \frac{\gamma_1}{\Delta}\left(\frac{\varphi_4}{f_1}\right), \end{cases}$$

where Δ, Δ_1, Δ_2 are given by (7).

Consequently, taking account of values of δ_i $(i = 1, 2, 3)$ in (9), one can easily see that if 4 parameters $\underline{\gamma_1, \gamma_2, \gamma_3}$ and $\underline{\beta}$ are given, then all constants c_i and d_i are determined explicitly.

2.2 $L(B) = 2 \cdot 2 \cdot 1,\ L(A) = 3 \cdot 2$

In this case, general forms of generators are given by

$$M_1 = \left(\begin{array}{cc|ccc} e_1 & p_1 & b_1' & b_2' & b_3' \\ 0 & e_2 & b_4' & b_5' & b_6' \\ \hline 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right),$$

$$M_2 = \left(\begin{array}{cc|cc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline c_1 & c_2 & e_3 & q & c_3 \\ c_4 & c_5 & 0 & e_4 & c_6 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right), \qquad M_3 = \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline d_1 & d_2 & d_3 & d_4 & e_5 \end{array}\right),$$

where $p_1 = 2\pi i e_1$ or zero in case $e_1 = e_2$ and $q = 2\pi i e_3$ or zero in case $e_3 = e_4$.

In order to evaluate 16 constants in the generators, we consider the matrix $M_2 M_3 - f\, M_1^{-1}$, the rank of which is equal to 2 for $f = f_1$ and 3 for $f = f_2$, as follows :

$$M_2 M_3 - f\, M_1^{-1} = \begin{pmatrix} 1 - f/e_1 & * & * & * & *, \\ 0 & 1 - f/e_2 & * & * & * \\ c_1 + c_3 d_1 & c_2 + c_3 d_2 & e_3 - f + c_3 d_3 & q + c_3 d_4 & c_3 e_5 \\ c_4 + c_6 d_1 & c_5 + c_6 d_2 & c_6 d_3 & e_4 - f + c_6 d_4 & c_6 e_5 \\ d_1 & d_2 & d_3 & d_4 & e_5 - f \end{pmatrix}.$$

So, we may deal with its equivalent matrix

$$\begin{pmatrix} (e_1 - f)/f & p & b_1 & b_2 & b_3 \\ 0 & (e_2 - f)/f & b_4 & b_5 & b_6 \\ c_1 & c_2 & e_3 - f & q & c_3 f \\ c_4 & c_5 & 0 & e_4 - f & c_6 f \\ d_1 & d_2 & d_3 & d_4 & e_5 - f \end{pmatrix} = \begin{pmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \\ B_5 \end{pmatrix},$$

where $p = 2\pi i$ or zero, and the B_i denote the corresponding row vectors. The relation between the b'_i and the b_i is expressed in the form

$$\begin{pmatrix} b'_1 & b'_2 & b'_3 \\ b'_4 & b'_5 & b'_6 \end{pmatrix} = \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix} \begin{pmatrix} b_1 & b_2 & b_3 \\ b_4 & b_5 & b_6 \end{pmatrix}.$$

Now, since B_1, B_2 and B_4 are linearly independent row vectors, we put

$$(12) \qquad \begin{cases} B_3 = \delta_1 B_1 + \delta_2 B_2, \\ B_4 = \gamma_1 B_1 + \gamma_2 B_2, \\ B_5 = \beta_1 B_1 + \beta_2 B_2 \end{cases}$$

for $f = f_1$ and

$$(13) \qquad \begin{cases} B_3 = \alpha'_1 B_1 + \alpha'_2 B_2 + \alpha'_3 B_4, \\ B_5 = \alpha_1 B_1 + \alpha_2 B_2 + \alpha_3 B_4 \end{cases}$$

for $f = f_2$.

Using the notations as for φ_i, ψ_i in the preceding subsection, from the following relations of B_3, B_4 in (12) :

$$\begin{cases} \delta_1 b_1 + \delta_2 b_4 = \varphi_3, \\ \gamma_1 b_1 + \gamma_2 b_4 = 0, \end{cases} \qquad \begin{cases} \delta_1 b_2 + \delta_2 b_5 = q, \\ \gamma_1 b_2 + \gamma_2 b_5 = \varphi_4, \end{cases}$$

we first seek b_1, b_4 and b_2, b_5, obtaining

$$b_1 = \frac{\varphi_3\gamma_2}{\Delta}, \qquad b_4 = -\frac{\varphi_3\gamma_1}{\Delta},$$

$$b_2 = \frac{q\gamma_2 - \varphi_4\delta_2}{\Delta}, \qquad b_5 = -\frac{q\gamma_1 - \varphi_4\delta_1}{\Delta},$$

where Δ is just the same value as in (7).

Next, solving the formulas for d_i $(i = 1, 2, 3, 4)$ in B_5 of (12) and (13) :

$$d_1 : \quad \beta_1 \left(\frac{\varphi_1}{f_1}\right) - \alpha_1 \left(\frac{\psi_1}{f_2}\right) = \alpha_3 c_4,$$

$$d_2 : \quad \beta_2 \left(\frac{\varphi_2}{f_1}\right) - \alpha_2 \left(\frac{\psi_2}{f_2}\right) + (\beta_1 - \alpha_1)p = \alpha_3 c_5,$$

$$d_3 : \quad (\beta_1 - \alpha_1)b_1 + (\beta_2 - \alpha_2)b_4 = 0,$$

$$d_4 \;:\qquad (\beta_1-\alpha_1)b_2+(\beta_2-\alpha_2)b_5 \;=\; \alpha_3\,\psi_4,$$

we have

$$\begin{aligned}
\frac{\alpha_1}{\alpha_3} &= \frac{\xi_4}{\eta_1}\gamma_1,\\
\frac{\beta_1}{\alpha_3} &= \frac{\Phi_1}{\eta_1}\gamma_1,\\
\frac{\alpha_2}{\alpha_3} &= \frac{\xi_4}{\eta_2}\gamma_2+\left(\frac{f_1}{\varphi_2}\right)\left(\frac{\xi_4}{\eta_2}\right)\gamma_1 p,\\
\frac{\beta_2}{\alpha_3} &= \frac{\Phi_2}{\eta_2}\gamma_2+\left(\frac{f_1}{\varphi_2}\right)\left(\frac{\xi_4}{\eta_2}\right)\gamma_1 p.
\end{aligned}$$

Here and hereafter we introduce the following notations :

$$\begin{aligned}
\eta_1 &= \left(\frac{\psi_1}{\varphi_1}\right)\frac{f_1}{f_2}-1, & \eta_2 &= \left(\frac{\psi_2}{\varphi_2}\right)\frac{f_1}{f_2}-1,\\
\xi_3 &= \frac{\psi_3}{\varphi_3}-1, & \xi_4 &= \frac{\psi_4}{\varphi_4}-1,\\
\Phi_1 &= \left(\frac{\psi_1}{\varphi_1}\right)\frac{\psi_4}{\varphi_4}\frac{f_1}{f_2}-1, & \Phi_2 &= \left(\frac{\psi_2}{\varphi_2}\right)\frac{\psi_4}{\varphi_4}\frac{f_1}{f_2}-1,\\
\Psi_1 &= \left(\frac{\psi_1}{\varphi_1}\right)\frac{\psi_3}{\varphi_3}\frac{f_1}{f_2}-1, & \Psi_2 &= \left(\frac{\psi_2}{\varphi_2}\right)\frac{\psi_3}{\varphi_3}\frac{f_1}{f_2}-1.
\end{aligned}$$

We then substitute c_6 into

$$\left\{\begin{array}{l}
\beta_1 b_3+\beta_2 b_6=\varphi_5,\\
\alpha_1 b_3+\alpha_2 b_6+\alpha_3 c_6 f_2=\psi_5
\end{array}\right.$$

and solve them to obtain b_3, b_6 as follows :

$$\begin{aligned}
\omega &= \left[\frac{\xi_4}{\eta_1\eta_2}(\Phi_1-\Phi_2)+\frac{f_2}{f_1}\left(\frac{\Phi_1}{\eta_1}-\frac{\Phi_2}{\eta_2}\right)\right]\alpha_3\gamma_1\gamma_2\\
&+ \left[\frac{\xi_4}{\eta_1\eta_2}\frac{f_1}{\varphi_2}(\Phi_1-\xi_4)-\frac{\xi_4}{\eta_2}\frac{f_2}{\varphi_2}\right]\alpha_3\gamma_1^2 p,\\
b_3 &= \left[\left(\left(\frac{\xi_4}{\eta_2}+\frac{f_2}{f_1}\right)\varphi_5-\frac{\Phi_2}{\eta_2}\psi_5\right)\gamma_2+\frac{\xi_4}{\eta_2}\frac{f_1}{\varphi_2}(\varphi_5-\psi_5)\gamma_1 p\right]/\omega,
\end{aligned}$$

$$b_6 = \left[\frac{\Phi_1}{\eta_1}\psi_5 - \left(\frac{\xi_4}{\eta_1} + \frac{f_2}{f_1}\right)\varphi_5\right]\gamma_1/\omega.$$

From results derived so far, we can easily observe that if one gives any values to 5 parameters, i.e., γ_i, δ_i $(i = 1, 2)$ and α_3, then all constants included in the generators are definitely determined. However, we can only assign any values to 4 parameters. So, one parameter among 5 parameters stated above must be determined by other parameters.

We now consider relations for B_3 in (12) and (13) :

$$(14) \qquad \begin{cases} \alpha_1'\left(\frac{\psi_1}{f_2}\right) + \alpha_3' c_4 = \delta_1\left(\frac{\varphi_1}{f_1}\right), \\ \alpha_1' b_1 + \alpha_2' b_4 = \psi_3, \\ \alpha_1' b_2 + \alpha_2' b_5 + \alpha_3' \psi_4 = q \end{cases}$$

and

$$(15) \qquad \alpha_1' p + \alpha_2'\left(\frac{\psi_2}{f_2}\right) + \alpha_3' c_5 = \delta_1 p + \delta_2\left(\frac{\varphi_2}{f_1}\right),$$

$$(16) \qquad \alpha_1' b_3 + \alpha_2' b_6 + \alpha_3' c_6 f_2 = c_3 f_2.$$

First, we solve (14) :

$$W = \begin{vmatrix} \frac{\psi_1}{f_2} & 0 & c_4 \\ b_1 & b_4 & 0 \\ b_2 & b_5 & \psi_4 \end{vmatrix} = [-\Phi_1]\left(\frac{\varphi_1\varphi_3\varphi_4}{f_1\Delta}\right)\gamma_1,$$

$$\alpha_1' = \begin{vmatrix} c_1 & 0 & c_4 \\ \psi_3 & b_4 & 0 \\ q & b_5 & \psi_4 \end{vmatrix} /W,$$

$$\alpha_2' = \begin{vmatrix} \frac{\psi_1}{f_2} & c_1 & c_4 \\ b_1 & \psi_3 & 0 \\ b_2 & q & \psi_4 \end{vmatrix} /W,$$

$$\alpha_3' = \begin{vmatrix} \frac{\psi_1}{f_2} & 0 & c_1 \\ b_1 & b_4 & \psi_3 \\ b_2 & b_5 & q \end{vmatrix} /W,$$

whence

$$\alpha_1' = \frac{(\xi_4 - \xi_3)}{\Phi_1}\delta_1 + \frac{\xi_3}{\varphi_4}\frac{1}{\Phi_1}q\gamma_1,$$

$$\alpha_2' = \frac{\psi_3}{\varphi_3}\delta_2 - \frac{\psi_4}{\varphi_4}\frac{\Psi_1}{\Phi_1}\left(\frac{\gamma_2\delta_1}{\gamma_1}\right) + \frac{\xi_3}{\varphi_4}\frac{1}{\Phi_1}q\gamma_2,$$

$$\alpha_3' = \frac{\Psi_1}{\Phi_1}\left(\frac{\delta_1}{\gamma_1}\right) - \frac{\psi_1}{\varphi_1}\frac{f_1}{f_2}\frac{\xi_3}{\varphi_4}\frac{1}{\Phi_1}q.$$

Substituting these into (15), we consequently obtain

$$\begin{aligned}\delta_2 &= \frac{\Phi_2}{\Psi_2}\frac{\Psi_1}{\Phi_1}\left(\frac{\gamma_2\delta_1}{\gamma_1}\right) + \frac{f_1}{\varphi_2}\left(\frac{\xi_3 - \xi_4 + \Phi_1 - \Psi_1}{\Phi_1\Psi_2}\right)(\delta_1 p)\\ &\quad + \left(\frac{\psi_1}{\varphi_1} - \frac{\psi_2}{\varphi_2}\right)\frac{f_1}{f_2}\frac{\xi_3}{\varphi_4}\frac{1}{\Phi_1\Psi_2}(\gamma_2 q)\\ &\quad + \frac{f_1}{\varphi_2}\left(\frac{\psi_1}{\varphi_1}\frac{f_1}{f_2} - 1\right)\frac{\xi_3}{\varphi_4}\frac{1}{\Phi_1\Psi_2}(\gamma_1 q p).\end{aligned}$$

This shows that δ_2 can be determined uniquely by γ_1, γ_2 and δ_1. Hence, if 4 parameters $\underline{\gamma_1, \gamma_2, \delta_1}$ and $\underline{\alpha_3}$ are given, then all constants b_i, c_i $(i = 1, 2, 3, 4, 5, 6)$ and d_i $(i = 1, 2, 3, 4)$ are determined explicitly. Then the formula (16) remained to the last is an identity, the fact of which can be proved by the Fuchs relation $e_1e_2e_3e_4e_5 = f_1^3 f_2^2$.

Following the above algorithm, one can make a computer calculate the constants of generators. In the program, c1, c2, d1 and d2 denote γ_1, γ_2, δ_1 and δ_2, respectively.

$$\text{ph}k, \quad \text{ps}k, \quad \text{phi}k, \quad \text{psi}k \quad \Longleftrightarrow \quad \varphi_k, \quad \psi_k, \quad \Phi_k, \quad \Psi_k,$$

$$\text{a}k, \quad \text{b}k, \quad \text{aa}k \quad \Longleftrightarrow \quad \frac{\alpha_k}{\alpha_3}, \quad \frac{\beta_k}{\alpha_3}, \quad \alpha_k'.$$

nb3, nb6 and coeff*k denote numerators of b_3, b_6 and the expressions of constants b_k, c_k, d_k to be required, respectively.

Symbolic manipulation

```
on factor;

ph1:=e1-f1$ ph2:=e2-f1$ ph3:=e3-f1$ ph4:=e4-f1$ ph5:=e5-f1$

ps1:=e1-f2$ ps2:=e2-f2$ ps3:=e3-f2$ ps4:=e4-f2$ ps5:=e5-f2$

eta1:=(ps1/ph1)*(f1/f2)-1; eta2:=(ps2/ph2)*(f1/f2)-1;
```

```
xi3:=(ps3/ph3)-1; xi4:=(ps4/ph4)-1;

phi1:=(ps1/ph1)*(ps4/ph4)*(f1/f2)-1;
phi2:=(ps2/ph2)*(ps4/ph4)*(f1/f2)-1;
psi1:=(ps1/ph1)*(ps3/ph3)*(f1/f2)-1;
psi2:=(ps2/ph2)*(ps3/ph3)*(f1/f2)-1;

a1:=(xi4/eta1)*c1;
b1:=(phi1/eta1)*c1;
a2:=(xi4/eta2)*c2+(f1/ph2)*(xi4/eta2)*c1*p;
b2:=(phi2/eta2)*c2+(f1/ph2)*(xi4/eta2)*c1*p;

aa1:=(xi3/ph4)*(1/phi1)*q*c1+((xi4-xi3)/phi1)*d1;
aa2:=(xi3/ph4)*(1/phi1)*q*c2-(ps4/ph4)*(psi1/phi1)*(c2*d1/c1)
    +(ps3/ph3)*d2;
aa3:=-(ps1/ph1)*(f1/f2)*(xi3/ph4)*(1/phi1)*q+(psi1/phi1)*(d1/c1);

d2:=(ps1/ph1-ps2/ph2)*(f1/f2)*(xi3/ph4)*(1/phi1)*(1/psi2)*q*c2
   +(phi2/psi2)*(psi1/phi1)*(c2*d1/c1)
   +(f1/ph2)*((ps1/ph1)*(f1/f2)-1)*(xi3/ph4)*(1/phi1)*(1/psi2)*q*c1*p
   +(f1/ph2)*((xi3-xi4+phi1-psi1)/(phi1*psi2))*d1*p;

nb3:=((xi4/eta2+f2/f1)*ph5-(phi2/eta2)*ps5)*c2
    +(xi4/eta2)*(f1/ph2)*(ph5-ps5)*c1*p;
nb6:=((phi1/eta1)*ps5-(xi4/eta1+f2/f1)*ph5)*c1;

e5:=(f1**3)*(f2**2)/(e1*e2*e3*e4)$
Fuchs:=(aa1+aa3*(f2/f1)*c1-(f2/f1)*d1)*nb3
      +(aa2+aa3*(f2/f1)*c2-(f2/f1)*d2)*nb6;

0 <---- This result proves the validity of our calculation !!!

%% From now on, we seek explicit formulas of constants of generators. %%

omega:=((b1*a2-b2*a1)+(b1*c2-b2*c1)*(f2/f1))*a3;
delta:=d1*c2-d2*c1;
```

```
coeffb1:=(c2*ph3)/delta;
coeffb2:=(q*c2-ph4*d2)/delta;
coeffb3:=((a2+c2*(f2/f1))*ph5-b2*ps5)/omega;
coeffb4:=-(c1*ph3)/delta;
coeffb5:=(d1*ph4-q*c1)/delta;
coeffb6:=(b1*ps5-(a1+c1*(f2/f1))*ph5)/omega;

coeffc1:=d1*(ph1/f1);
coeffc2:=d1*p+d2*(ph2/f1);
coeffc3:=(d1*coeffb3+d2*coeffb6)/f1;
coeffc6:=(c1*coeffb3+c2*coeffb6)/f1;

coeffd1:=b1*a3*(ph1/f1);
coeffd2:=b1*a3*p+b2*a3*(ph2/f1);
coeffd3:=b1*a3*coeffb1+b2*a3*coeffb4;
coeffd4:=b1*a3*coeffb2+b2*a3*coeffb5;

end;
```

We shall here write down explicit formulas of generators of the required monodromy group represented in terms of 4 parameters γ_k, $(k = 1, 2)$, δ_1 and α_3 in the following

General expression

$$
\begin{aligned}
b_1 &= \frac{-(e_1e_4 - f_1f_2)(e_2e_3 - f_1f_2)(e_3 - f_1)}{f_1f_2((e_2 - e_1)\gamma_2 + e_1\gamma_1 p)((e_3 - e_4)\delta_1 + \gamma_1 q)}\gamma_2, \\
b_2 &= \frac{(e_1e_3 - f_1f_2)(e_2e_4 - f_1f_2)(e_4 - f_1)}{f_1f_2((e_2 - e_1)\gamma_2 + e_1\gamma_1 p)((e_3 - e_4)\delta_1 + \gamma_1 q)}\frac{\gamma_2\delta_1}{\gamma_1} \\
&\quad + \frac{(e_1e_4 - f_1f_2)(e_2e_3 - f_1f_2) - f_1f_2(e_4 - f_1)(e_1 - e_2)}{f_1f_2((e_2 - e_1)\gamma_2 + e_1\gamma_1 p)((e_3 - e_4)\delta_1 + \gamma_1 q)}\gamma_2 q \\
&\quad + \frac{e_1(e_4 - f_1)((e_3 - e_4)\delta_1 + \gamma_1 q)}{((e_2 - e_1)\gamma_2 + e_1\gamma_1 p)((e_3 - e_4)\delta_1 + \gamma_1 q)}p, \\
b_3 &= \frac{f_1(e_4 - f_1)((e_1e_3 - f_1f_2)\gamma_2 - e_1e_3\gamma_1 p)}{e_3e_4((e_2 - e_1)\gamma_2 + e_1\gamma_1 p)}\frac{1}{\alpha_3\gamma_1},
\end{aligned}
$$

$$b_4 = \frac{(e_1e_4 - f_1f_2)(e_2e_3 - f_1f_2)(e_3 - f_1)}{f_1f_2((e_2 - e_1)\gamma_2 + e_1\gamma_1 p)((e_3 - e_4)\delta_1 + \gamma_1 q)}\gamma_1,$$

$$b_5 = \frac{-(e_1e_4 - f_1f_2)(e_2e_3 - f_1f_2)((e_4 - f_1)\delta_1 - \gamma_1 q)}{f_1f_2((e_2 - e_1)\gamma_2 + e_1\gamma_1 p)((e_3 - e_4)\delta_1 + \gamma_1 q)},$$

$$b_6 = \frac{-f_1(e_2e_3 - f_1f_2)(e_4 - f_1)}{e_3e_4((e_2 - e_1)\gamma_2 + e_1\gamma_1 p)}\frac{1}{\alpha_3},$$

$$c_1 = \frac{(e_1 - f_1)}{f_1}\delta_1,$$

$$c_2 = \frac{(e_1e_3 - f_1f_2)(e_2e_4 - f_1f_2)(e_2 - f_1)}{f_1(e_1e_4 - f_1f_2)(e_2e_3 - f_1f_2)}\frac{\gamma_2\delta_1}{\gamma_1} + \frac{f_2(e_2 - f_1)((e_2 - e_1)\gamma_2 + e_1\gamma_1 p)}{(e_1e_4 - f_1f_2)(e_2e_3 - f_1f_2)}q$$

$$+ \left(\frac{e_1f_2(e_3 - e_4)(e_2 - f_1)}{(e_1e_4 - f_1f_2)(e_2e_3 - f_1f_2)} + 1\right)\delta_1 p,$$

$$c_3 = \frac{-(e_4 - f_1)(e_4(e_1e_3 - f_1f_2)\delta_1 + f_1f_2\gamma_1 q)}{e_3e_4(e_1e_4 - f_1f_2)}\frac{1}{\alpha_3\gamma_1},$$

$$c_4 = \frac{(e_1 - f_1)}{f_1}\gamma_1,$$

$$c_5 = \frac{(e_2 - f_1)\gamma_2 + f_1\gamma_1 p}{f_1},$$

$$c_6 = \frac{-(e_4 - f_1)}{e_4}\frac{1}{\alpha_3},$$

$$d_1 = \frac{(e_1e_4 - f_1f_2)(e_1 - f_1)}{e_1f_1(e_4 - f_1)}\alpha_3\gamma_1,$$

$$d_2 = \frac{(e_2e_4 - f_1f_2)(e_2 - f_1)}{e_2f_1(e_4 - f_1)}\alpha_3\gamma_2 + \frac{e_1f_2(e_2 - f_1) + e_2(e_1e_4 - f_1f_2)}{e_1e_2(e_4 - f_1)}\alpha_3\gamma_1 p,$$

$$d_3 = \frac{(e_1e_4 - f_1f_2)(e_2e_3 - f_1f_2)(e_3 - f_1)}{e_1e_2(e_4 - f_1)((e_3 - e_4)\delta_1 + \gamma_1 q)}\alpha_3\gamma_1,$$

$$d_4 = \frac{-(e_1e_4 - f_1f_2)(e_2e_4 - f_1f_2)}{e_1e_2((e_3 - e_4)\delta_1 + \gamma_1 q)}\alpha_3\delta_1 + \frac{(e_1e_4 - f_1f_2)(e_2e_3 + e_2e_4 - e_2f_1 - f_1f_2)}{e_1e_2(e_4 - f_1)((e_3 - e_4)\delta_1 + \gamma_1 q)}\gamma_1 q.$$

Putting $p = q = 0$ in the above formulas, we have

Non-logarithmic case

$$
\begin{aligned}
b_1 &= \frac{(e_1e_4 - f_1f_2)(e_2e_3 - f_1f_2)(e_3 - f_1)}{f_1f_2(e_1 - e_2)(e_3 - e_4)}\frac{1}{\delta_1}, \\
b_2 &= \frac{-(e_1e_3 - f_1f_2)(e_2e_4 - f_1f_2)(e_4 - f_1)}{f_1f_2(e_1 - e_2)(e_3 - e_4)}\frac{1}{\gamma_1}, \\
b_3 &= \frac{-f_1(e_4 - f_1)(e_1e_3 - f_1f_2)}{e_3e_4(e_1 - e_2)}\frac{1}{\alpha_3\gamma_1}, \\
b_4 &= \frac{-(e_1e_4 - f_1f_2)(e_2e_3 - f_1f_2)(e_3 - f_1)}{f_1f_2(e_1 - e_2)(e_3 - e_4)}\frac{\gamma_1}{\gamma_2\delta_1}, \\
b_5 &= \frac{(e_1e_4 - f_1f_2)(e_2e_3 - f_1f_2)(e_4 - f_1)}{f_1f_2(e_1 - e_2)(e_3 - e_4)}\frac{1}{\gamma_2}, \\
b_6 &= \frac{f_1(e_2e_3 - f_1f_2)(e_4 - f_1)}{e_3e_4((e_1 - e_2)}\frac{1}{\alpha_3\gamma_1}, \\
c_1 &= \frac{(e_1 - f_1)}{f_1}\delta_1, \\
c_2 &= \frac{(e_1e_3 - f_1f_2)(e_2e_4 - f_1f_2)(e_2 - f_1)}{f_1(e_1e_4 - f_1f_2)(e_2e_3 - f_1f_2)}\frac{\gamma_2\delta_1}{\gamma_1}, \\
c_3 &= \frac{-(e_1e_3 - f_1f_2)(e_4 - f_1)}{e_3(e_1e_4 - f_1f_2)}\frac{\delta_1}{\alpha_3\gamma_1}, \\
c_4 &= \frac{(e_1 - f_1)}{f_1}\gamma_1, \\
c_5 &= \frac{(e_2 - f_1)}{f_1}\gamma_2, \\
c_6 &= \frac{-(e_4 - f_1)}{e_4}\frac{1}{\alpha_3}, \\
d_1 &= \frac{(e_1e_4 - f_1f_2)(e_1 - f_1)}{e_1f_1(e_4 - f_1)}\alpha_3\gamma_1, \\
d_2 &= \frac{(e_2e_4 - f_1f_2)(e_2 - f_1)}{e_2f_1(e_4 - f_1)}\alpha_3\gamma_2,
\end{aligned}
$$

$$d_3 = \frac{(e_1e_4 - f_1f_2)(e_2e_3 - f_1f_2)(e_3 - f_1)}{e_1e_2(e_3 - e_4)(e_4 - f_1)} \frac{\alpha_3\gamma_1}{\delta_1},$$

$$d_4 = \frac{-(e_1e_4 - f_1f_2)(e_2e_4 - f_1f_2)}{e_1e_2(e_3 - e_4)} \alpha_3.$$

3. 6-th order hypergeometric system

According to K.Okubo's result [3], there are 4 types of 6-th order hypergeometric systems, which are of accessary parameter free, except for Jordan-Pochhammer equation and Generalized hypergeometric equation, as follows : (i) $\{3\cdot 2\cdot 1,\ 4\cdot 1\cdot 1\}$, (ii) $\{3\cdot 2\cdot 1,\ 3\cdot 3\}$, (iii) $\{3\cdot 1\cdot 1\cdot 1,\ 4\cdot 2\}$, (iv) $\{2\cdot 2\cdot 2,\ 4\cdot 2\}$, where each pair denotes the pair $\{L(B), L(A)\}$ or $\{L(A), L(B)\}$. In a non-logarithmic case, among those types, (ii) and (iv) are irreducible. (See [5].)

We shall now explain our method of calculating monodromy groups more explicitly, dealing with 6-th order hypergeometric system of the type (iv).

3.1 $L(B) = 4\cdot 2,\ L(A) = 2\cdot 2\cdot 2$

In this case, we can write generators of the monodromy group in the following form

$$M_1 = \left(\begin{array}{cccc|cc} e_1 & p_1 & p_2 & p_3 & \gamma_1 & \delta_1 \\ 0 & e_2 & p_1' & p_2' & \gamma_2 & \delta_2 \\ 0 & 0 & e_3 & p_1'' & \gamma_3 & \delta_3 \\ 0 & 0 & 0 & e_4 & \gamma_4 & \delta_4 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right),$$

$$M_2 = \left(\begin{array}{cccc|cc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \hline d_1 & d_2 & d_3 & d_4 & e_5 & q' \\ d_5 & d_6 & d_7 & d_8 & 0 & e_6 \end{array}\right),$$

where $p_1,\ p_1',\ p_1'' = 2\pi i e_1$ or zero, $p_2,\ p_2' = (2\pi i)^2 e_1/2!$ or zero and $p_3 = (2\pi i)^3 e_1/3!$ or zero in case $e_i = e_1$ $(i = 2, 3, 4)$, for example,

$$\begin{pmatrix} e_1 & \frac{2\pi i}{1!}e_1 & \frac{(2\pi i)^2}{2!}e_1 & \frac{(2\pi i)^3}{3!}e_1 \\ 0 & e_1 & \frac{2\pi i}{1!}e_1 & \frac{(2\pi i)^2}{2!}e_1 \\ 0 & 0 & e_1 & \frac{2\pi i}{1!}e_1 \\ 0 & 0 & 0 & e_1 \end{pmatrix}$$

in a full-logarithmic case and $q' = 2\pi i e_5$ or zero in case $e_5 = e_6$.

We shall again consider the matrix, which is equivalent to $M_1 - f\,M_2^{-1}$,

$$M_1 - f\,M_2^{-1} \quad \sim \quad \begin{pmatrix} e_1 - f & p_1 & p_2 & p_3 & \gamma_1 & \delta_1 \\ 0 & e_2 - f & p_1' & p_2' & \gamma_2 & \delta_2 \\ 0 & 0 & e_3 - f & p_1'' & \gamma_3 & \delta_3 \\ 0 & 0 & 0 & e_4 - f & \gamma_4 & \delta_4 \\ d_1 & d_2 & d_3 & d_4 & (e_5 - f)/f & q \\ d_5 & d_6 & d_7 & d_8 & 0 & (e_6 - f)/f \end{pmatrix}$$

$$\equiv \quad (A_1,\ A_2,\ A_3,\ A_4,\ A_5,\ A_6)\,,$$

where the A_i denote the corresponding column vectors. Here it is again remarked that $q = 2\pi i$ or zero and the original d_i in M_2 are derived by

$$\begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix}\begin{pmatrix} d_1 & d_2 & d_3 & d_4 \\ d_5 & d_6 & d_7 & d_8 \end{pmatrix}.$$

$L(A) = 2 \cdot 2 \cdot 2$ implies that the rank of $M_1 - f\,M_2^{-1}$ is 4 for each f_i $(i = 1, 2, 3)$. Since it is easy to see that A_k $(k = 1, 2, 3, 4)$ are linearly independent, we can put

$$\text{(17)} \qquad \begin{cases} A_5 \;=\; \alpha_{i\,1}\,A_1 + \alpha_{i\,2}\,A_2 + \alpha_{i\,3}\,A_3 + \alpha_{i\,4}\,A_4, \\ \\ A_6 \;=\; \beta_{i\,1}\,A_1 + \beta_{i\,2}\,A_2 + \beta_{i\,3}\,A_3 + \beta_{i\,4}\,A_4 \end{cases}$$

for each $f = f_i$ $(i = 1, 2, 3)$.

Now we introduce the notations

$$\varphi_i = e_i - f_1, \qquad \psi_i = e_i - f_2, \qquad \xi_i = e_i - f_3 \qquad (i = 1, 2, 3, 4, 5, 6)$$

and

$$\mathcal{D} = \begin{pmatrix} d_1 & d_2 & d_3 & d_4 \\ d_5 & d_6 & d_7 & d_8 \end{pmatrix}, \qquad \Gamma = \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \end{pmatrix}, \qquad \Delta = \begin{pmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{pmatrix},$$

$$\Phi_i = \begin{pmatrix} e_1 - f_i & p_1 & p_2 & p_3 \\ 0 & e_2 - f_i & p_1' & p_2' \\ 0 & 0 & e_3 - f_i & p_1'' \\ 0 & 0 & 0 & e_4 - f_i \end{pmatrix}, \qquad \mathcal{A}_i = \begin{pmatrix} \alpha_{i\,1} \\ \alpha_{i\,2} \\ \alpha_{i\,3} \\ \alpha_{i\,4} \end{pmatrix}, \qquad \mathcal{B}_i = \begin{pmatrix} \beta_{i\,1} \\ \beta_{i\,2} \\ \beta_{i\,3} \\ \beta_{i\,4} \end{pmatrix}$$

for $i = 1, 2, 3$.

Then, the relations (17) can be expressed as

$$\Phi_i \mathcal{A}_i = \Gamma \qquad (i = 1, 2, 3), \tag{18}$$

$$\mathcal{D}\mathcal{A}_1 = \begin{pmatrix} \frac{\varphi_5}{f_1} \\ 0 \end{pmatrix}, \qquad \mathcal{D}\mathcal{A}_2 = \begin{pmatrix} \frac{\psi_5}{f_2} \\ 0 \end{pmatrix}, \qquad \mathcal{D}\mathcal{A}_3 = \begin{pmatrix} \frac{\xi_5}{f_3} \\ 0 \end{pmatrix} \tag{19}$$

and

$$\Phi_i \mathcal{B}_i = \Delta \qquad (i = 1, 2, 3), \tag{20}$$

$$\mathcal{D}\mathcal{B}_1 = \begin{pmatrix} q \\ \frac{\varphi_6}{f_1} \end{pmatrix}, \qquad \mathcal{D}\mathcal{B}_2 = \begin{pmatrix} q \\ \frac{\psi_6}{f_2} \end{pmatrix}, \qquad \mathcal{D}\mathcal{B}_3 = \begin{pmatrix} q \\ \frac{\xi_6}{f_3} \end{pmatrix}. \tag{21}$$

The combination of (19) and (21) leads to

$$\begin{pmatrix} \alpha_{1\,1} & \alpha_{1\,2} & \alpha_{1\,3} & \alpha_{1\,4} \\ \alpha_{2\,1} & \alpha_{2\,2} & \alpha_{2\,3} & \alpha_{2\,4} \\ \alpha_{3\,1} & \alpha_{3\,2} & \alpha_{3\,3} & \alpha_{3\,4} \\ \beta_{1\,1} & \beta_{1\,2} & \beta_{1\,3} & \beta_{1\,4} \\ \hline \beta_{2\,1} & \beta_{2\,2} & \beta_{2\,3} & \beta_{2\,4} \\ \beta_{3\,1} & \beta_{3\,2} & \beta_{3\,3} & \beta_{3\,4} \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{pmatrix} = \begin{pmatrix} \frac{\varphi_5}{f_1} \\ \frac{\psi_5}{f_2} \\ \frac{\xi_5}{f_3} \\ q \\ \hline q \\ q \end{pmatrix} \tag{22}$$

and

$$(23)\qquad \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \alpha_{24} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & \alpha_{34} \\ \beta_{11} & \beta_{12} & \beta_{13} & \beta_{14} \\ \hline \beta_{21} & \beta_{22} & \beta_{23} & \beta_{24} \\ \beta_{31} & \beta_{32} & \beta_{33} & \beta_{34} \end{pmatrix} \begin{pmatrix} d_5 \\ d_6 \\ d_7 \\ d_8 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{\varphi_6}{f_1} \\ \hline \frac{\psi_6}{f_2} \\ \frac{\xi_6}{f_3} \end{pmatrix}.$$

Now, assigning any values to 5 parameters among 16 parameters in generators, we have to determine the remaining 11 by the above formulas. To see this, let γ_i $(i = 1, 2, 3, 4)$ and δ_4 be given. Then, from (18) the α_{ij} are completely determined and from (20) the β_{ij} can be expressed in terms of three parameters δ_k $(k = 1, 2, 3)$. From relations of the first four rows of (22) and (23), we can then derive the d_i, which depend on δ_k $(k = 1, 2, 3)$, that is, if three δ_k $(k = 1, 2, 3)$ are known, then the required constants d_i are completely determined. Now, four relations are remaining in (22) and (23). In order to determine δ_k $(k = 1, 2, 3)$, we use three relations, and one relation remained to the last is just the Fuchs relation.

Non-logarithmic case

We shall here apply the above principle to a non-logarithmic case. In this case, we have

$$\Phi_1 = \begin{pmatrix} \varphi_1 & 0 & 0 & 0 \\ 0 & \varphi_2 & 0 & 0 \\ 0 & 0 & \varphi_3 & 0 \\ 0 & 0 & 0 & \varphi_4 \end{pmatrix}, \ \Phi_2 = \begin{pmatrix} \psi_1 & 0 & 0 & 0 \\ 0 & \psi_2 & 0 & 0 \\ 0 & 0 & \psi_3 & 0 \\ 0 & 0 & 0 & \psi_4 \end{pmatrix}, \ \Phi_3 = \begin{pmatrix} \xi_1 & 0 & 0 & 0 \\ 0 & \xi_2 & 0 & 0 \\ 0 & 0 & \xi_3 & 0 \\ 0 & 0 & 0 & \xi_4 \end{pmatrix}.$$

Then, (22) and (23) can be rewritten in the following form :

$$(24)\qquad \begin{pmatrix} \frac{1}{\varphi_1} & \frac{1}{\varphi_2} & \frac{1}{\varphi_3} \\ \frac{1}{\psi_1} & \frac{1}{\psi_2} & \frac{1}{\psi_3} \\ \frac{1}{\xi_1} & \frac{1}{\xi_2} & \frac{1}{\xi_3} \end{pmatrix} \begin{pmatrix} \gamma_1 d_1 \\ \gamma_2 d_2 \\ \gamma_3 d_3 \end{pmatrix} = \begin{pmatrix} \frac{\varphi_5}{f_1} - \frac{1}{\varphi_4}\theta_1 \\ \frac{\psi_5}{f_2} - \frac{1}{\psi_4}\theta_1 \\ \frac{\xi_5}{f_3} - \frac{1}{\xi_4}\theta_1 \end{pmatrix},$$

$$(25)\qquad \begin{pmatrix} \frac{1}{\varphi_1} & \frac{1}{\varphi_2} & \frac{1}{\varphi_3} \\ \frac{1}{\psi_1} & \frac{1}{\psi_2} & \frac{1}{\psi_3} \\ \frac{1}{\xi_1} & \frac{1}{\xi_2} & \frac{1}{\xi_3} \end{pmatrix} \begin{pmatrix} \gamma_1 d_5 \\ \gamma_2 d_6 \\ \gamma_3 d_7 \end{pmatrix} = \begin{pmatrix} -\frac{1}{\varphi_4}\theta_2 \\ -\frac{1}{\psi_4}\theta_2 \\ -\frac{1}{\xi_4}\theta_2 \end{pmatrix},$$

$$(26)\qquad \begin{pmatrix} \frac{1}{\varphi_1} & \frac{1}{\varphi_2} & \frac{1}{\varphi_3} \\ \frac{1}{\psi_1} & \frac{1}{\psi_2} & \frac{1}{\psi_3} \\ \frac{1}{\xi_1} & \frac{1}{\xi_2} & \frac{1}{\xi_3} \end{pmatrix} \begin{pmatrix} \delta_1 d_1 \\ \delta_2 d_2 \\ \delta_3 d_3 \end{pmatrix} = \begin{pmatrix} -\frac{1}{\varphi_4}\theta_3 \\ -\frac{1}{\psi_4}\theta_3 \\ -\frac{1}{\xi_4}\theta_3 \end{pmatrix},$$

(27)
$$\begin{pmatrix} \frac{1}{\varphi_1} & \frac{1}{\varphi_2} & \frac{1}{\varphi_3} \\ \frac{1}{\psi_1} & \frac{1}{\psi_2} & \frac{1}{\psi_3} \\ \frac{1}{\xi_1} & \frac{1}{\xi_2} & \frac{1}{\xi_3} \end{pmatrix} \begin{pmatrix} \delta_1 \, d_5 \\ \delta_2 \, d_6 \\ \delta_3 \, d_7 \end{pmatrix} = \begin{pmatrix} \frac{\varphi_6}{f_1} - \frac{1}{\varphi_4} \theta_4 \\ \frac{\psi_6}{f_2} - \frac{1}{\psi_4} \theta_4 \\ \frac{\xi_6}{f_3} - \frac{1}{\xi_4} \theta_4 \end{pmatrix},$$

where we have put

$$\theta_1 = \gamma_4 \, d_4, \qquad \theta_2 = \gamma_4 \, d_8, \qquad \theta_3 = \delta_4 \, d_4, \qquad \theta_4 = \delta_4 \, d_8.$$

Now, we first solve (24) and (25) to obtain d_i $(i = 1, 2, 3, 5, 6, 7)$. Then, we substitute those values into the formula, which is derived directly from (26) and (27), as follows :

$$\begin{pmatrix} -\frac{1}{\varphi_4} \theta_3 \\ -\frac{1}{\psi_4} \theta_3 \\ -\frac{1}{\xi_4} \theta_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{\varphi_1} & \frac{1}{\varphi_2} & \frac{1}{\varphi_3} \\ \frac{1}{\psi_1} & \frac{1}{\psi_2} & \frac{1}{\psi_3} \\ \frac{1}{\xi_1} & \frac{1}{\xi_2} & \frac{1}{\xi_3} \end{pmatrix} \begin{pmatrix} d_1 \, d_5^{-1} & 0 & 0 \\ 0 & d_2 \, d_6^{-1} & 0 \\ 0 & 0 & d_3 \, d_7^{-1} \end{pmatrix}$$
$$\times \begin{pmatrix} \frac{1}{\varphi_1} & \frac{1}{\varphi_2} & \frac{1}{\varphi_3} \\ \frac{1}{\psi_1} & \frac{1}{\psi_2} & \frac{1}{\psi_3} \\ \frac{1}{\xi_1} & \frac{1}{\xi_2} & \frac{1}{\xi_3} \end{pmatrix}^{-1} \begin{pmatrix} \frac{\varphi_6}{f_1} - \frac{1}{\varphi_4} \theta_4 \\ \frac{\psi_6}{f_2} - \frac{1}{\psi_4} \theta_4 \\ \frac{\xi_6}{f_3} - \frac{1}{\xi_4} \theta_4 \end{pmatrix}.$$

Taking account of the relation $\theta_2\theta_3 = \theta_1\theta_4$, we can obtain θ_1 and θ_4 by means of the solution of the first two rows in the above formula. Under the Fuchs relation $e_1e_2e_3e_4e_5e_6 = f_1^2 f_2^2 f_3^2$, the last row then becomes an identity, which plays an important role in checking whether all calculations done are valid or not.

Denoting

$$F = (f_1 - f_2)(f_1 - f_3)(f_2 - f_3),$$

we now put

$$\gamma_1 = \frac{\varphi_1\psi_1\xi_1}{(e_1 - e_2)(e_1 - e_3)} \frac{1}{F},$$

$$\gamma_2 = \frac{\varphi_2\psi_2\xi_2}{(e_1 - e_2)(e_2 - e_3)} \frac{1}{F},$$

$$\gamma_3 = \frac{\varphi_3\psi_3\xi_3}{(e_1 - e_3)(e_2 - e_3)} \frac{1}{F},$$

$$\gamma_4 = \frac{\varphi_4\psi_4\xi_4}{(e_1 - e_4)(e_2 - e_4)(e_3 - e_4)} \frac{1}{F}$$

and obtain

$$d_1 = \frac{(e_2e_3e_5 - f_1f_2f_3)}{f_1f_2f_3} F - \frac{(e_2 - e_4)(e_3 - e_4)}{\varphi_4\psi_4\xi_4} F \theta_1,$$

$$d_2 = -\frac{(e_1e_3e_5 - f_1f_2f_3)}{f_1f_2f_3}F + \frac{(e_1-e_4)(e_3-e_4)}{\varphi_4\psi_4\xi_4}F\,\theta_1,$$

$$d_3 = \frac{(e_1e_2e_5 - f_1f_2f_3)}{f_1f_2f_3}F - \frac{(e_1-e_4)(e_2-e_4)}{\varphi_4\psi_4\xi_4}F\,\theta_1,$$

$$d_5 = -\frac{(e_2-e_4)(e_3-e_4)}{\varphi_4\psi_4\xi_4}F\,\theta_2 = -\left(\frac{1}{e_1-e_4}\right)d_8,$$

$$d_6 = \frac{(e_1-e_4)(e_3-e_4)}{\varphi_4\psi_4\xi_4}F\,\theta_2 = \left(\frac{1}{e_2-e_4}\right)d_8,$$

$$d_7 = -\frac{(e_1-e_4)(e_2-e_4)}{\varphi_4\psi_4\xi_4}F\,\theta_2 = -\left(\frac{1}{e_3-e_4}\right)d_8.$$

Solving θ_1 and θ_4, we have

$$\theta_1 = \gamma_4\,d_4 = \frac{\left\{h_4\left(\frac{\varphi_5}{f_1}\right) + h(e_5-e_6)\right\}f_1\varphi_4\psi_4\xi_4}{(e_1-e_4)(e_2-e_4)(e_3-e_4)(e_5-e_6)\varphi_6},$$

$$\theta_4 = \delta_4\,d_8 = -\frac{h_4\,\varphi_4\psi_4\xi_4}{(e_1-e_4)(e_2-e_4)(e_3-e_4)(e_5-e_6)},$$

where

$$h_4 = \left(\frac{1}{e_1}+\frac{1}{e_2}+\frac{1}{e_3}+\frac{e_6}{e_4e_5}\right)f_1f_2f_3 - (e_1e_6+e_2e_6+e_3e_6+e_4e_5),$$

$$h = \left(e_1-\frac{f_2f_3}{e_1}\right)+\left(e_2-\frac{f_2f_3}{e_2}\right)+\left(e_3-\frac{f_2f_3}{e_3}\right)-\left(e_4-\frac{f_2f_3}{e_4}\right)$$
$$+\left(\frac{e_4e_5}{f_1}-\frac{f_1f_2f_3}{e_4e_5}\right)+\left(\frac{e_4e_6}{f_1}-\frac{f_1f_2f_3}{e_4e_6}\right).$$

Here, as a value of the fifth parameter to which any value can be assigned, we put

$$\delta_4 = -\frac{h_4\,\varphi_4\psi_4\xi_4}{(e_1-e_4)(e_2-e_4)(e_3-e_4)},$$

which immediately yields

$$d_8 = \frac{1}{(e_5-e_6)}.$$

We have thus determined all the constants d_i in the generator M_2 :

$$d_1 = -\frac{(e_2e_3e_5-f_1f_2f_3)(e_2e_4e_5-f_1f_2f_3)(e_3e_4e_5-f_1f_2f_3)e_1}{(e_1e_2e_3e_4e_5^2-f_1^2f_2^2f_3^2)(e_1-e_4)f_1f_2f_3}F,$$

$$d_2 = \frac{(e_1e_3e_5-f_1f_2f_3)(e_1e_4e_5-f_1f_2f_3)(e_3e_4e_5-f_1f_2f_3)e_2}{(e_1e_2e_3e_4e_5^2-f_1^2f_2^2f_3^2)(e_2-e_4)f_1f_2f_3}F,$$

$$d_3 = -\frac{(e_1e_2e_5 - f_1f_2f_3)(e_1e_4e_5 - f_1f_2f_3)(e_2e_4e_5 - f_1f_2f_3)e_3}{(e_1e_2e_3e_4e_5^2 - f_1^2f_2^2f_3^2)(e_3 - e_4)f_1f_2f_3}F,$$

$$d_4 = -\frac{(e_1e_2e_5 - f_1f_2f_3)(e_1e_3e_5 - f_1f_2f_3)(e_2e_3e_5 - f_1f_2f_3)e_4}{(e_1e_2e_3e_4e_5^2 - f_1^2f_2^2f_3^2)(e_1 - e_4)f_1f_2f_3}F,$$

$$d_5 = -\frac{1}{(e_1 - e_4)(e_5 - e_6)},$$

$$d_6 = \frac{1}{(e_2 - e_4)(e_5 - e_6)},$$

$$d_7 = -\frac{1}{(e_3 - e_4)(e_5 - e_6)}$$

From (26) or (27), we consequently obtain

$$\delta_1 = -\frac{h_1\varphi_1\psi_1\xi_1}{(e_1 - e_2)(e_1 - e_3)},$$

$$\delta_2 = -\frac{h_2\varphi_2\psi_2\xi_2}{(e_1 - e_2)(e_2 - e_3)},$$

$$\delta_3 = -\frac{h_3\varphi_3\psi_3\xi_3}{(e_1 - e_3)(e_2 - e_3)},$$

where

$$h_1 = \left(\frac{1}{e_2} + \frac{1}{e_3} + \frac{1}{e_4} + \frac{e_6}{e_1e_5}\right) f_1f_2f_3 - (e_2e_6 + e_3e_6 + e_4e_6 + e_1e_5),$$

$$h_2 = \left(\frac{1}{e_1} + \frac{1}{e_3} + \frac{1}{e_4} + \frac{e_6}{e_2e_5}\right) f_1f_2f_3 - (e_1e_6 + e_3e_6 + e_4e_6 + e_2e_5),$$

$$h_3 = \left(\frac{1}{e_1} + \frac{1}{e_2} + \frac{1}{e_4} + \frac{e_6}{e_3e_5}\right) f_1f_2f_3 - (e_1e_6 + e_2e_6 + e_4e_6 + e_3e_5).$$

3.2 $L(B) = 2\cdot 2\cdot 2,\ L(A) = 4\cdot 2$

Just as in the case $L(B) = 2\cdot 2\cdot 1$, $L(A) = 3\cdot 2$ of the 5-th order hypergeometric system, generators are represented in the form

$$M_1 = \left(\begin{array}{cc|cccc} e_1 & p_1 & b_1 & b_2 & b_3 & b_7 \\ 0 & e_2 & b_4 & b_5 & b_6 & b_8 \\ \hline & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ 0 & & 0 & & & 1 \end{array}\right),$$

$$M_2 = \left(\begin{array}{cc|cc|cc} 1 & & 0 & & 0 & \\ & 1 & & & & \\ \hline c_1 & c_2 & e_3 & q & c_3 & c_7 \\ c_4 & c_5 & 0 & e_4 & c_6 & c_8 \\ \hline & & & & 1 & \\ 0 & & 0 & & & 1 \end{array}\right), \qquad M_3 = \left(\begin{array}{cccc|cc} 1 & & & & 0 & \\ & 1 & & & & \\ & & 1 & & & \\ 0 & & & 1 & & \\ \hline d_1 & d_2 & d_3 & d_4 & e_5 & q_1 \\ d_5 & d_6 & d_7 & d_8 & 0 & e_6 \end{array}\right),$$

where $p_1 = 2\pi i e_1$ or zero in case $e_1 = e_2$, $q = 2\pi i e_3$ or zero in case $e_3 = e_4$ and $q_1 = 2\pi i e_5$ or zero in case $e_5 = e_6$.

So, we have only to consider the matrix

$$\begin{pmatrix} (e_1 - f)/f & p & b_1 & b_2 & b_3 & b_7 \\ 0 & (e_2 - f)/f & b_4 & b_5 & b_6 & b_8 \\ c_1 & c_2 & e_3 - f & q & c_3 f & c_7 f \\ c_4 & c_5 & 0 & e_4 - f & c_6 f & c_8 f \\ d_1 & d_2 & d_3 & d_4 & e_5 - f & q_1 \\ d_5 & d_6 & d_7 & d_8 & 0 & e_6 - f \end{pmatrix} = \begin{pmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \\ B_5 \\ B_6 \end{pmatrix},$$

where $p = 2\pi i$ or zero, and the B_i denote the corresponding row vectors. Again as for the relation between the original b_i in M_1 and the b_i in the above matrix, the same remark as stated in the preceding sections is made.

Now, the respective ranks of the above matrix for $f = f_1$ and $f = f_2$ are 2 and 4. We take B_1, B_2 for $f = f_1$ and B_1, B_2, B_4, B_6 for $f = f_2$ as linearly independent row vectors.

We put

$$(28) \qquad \begin{cases} B_3 = \delta_1 B_1 + \delta_2 B_2, \\ B_4 = \gamma_1 B_1 + \gamma_2 B_2, \\ B_5 = \beta_1 B_1 + \beta_2 B_2, \\ B_6 = \beta_1' B_1 + \beta_2' B_2 \end{cases}$$

for $f = f_1$ and

$$(29)\qquad \begin{cases} B_3 = \alpha'_1 B_1 + \alpha'_2 B_2 + \alpha'_3 B_4 + \alpha'_4 B_6, \\ B_5 = \alpha_1 B_1 + \alpha_2 B_2 + \alpha_3 B_4 + \alpha_4 B_6 \end{cases}$$

for $f = f_2$.

We shall now determine all the coefficients b_i, c_i, d_i or all parameters in the linear combinations (28) and (29) by means of assigning any values to 5 parameters among them. Although there are many methods or orders of calculations, we here merely follow the calculation done in the case $L(B) = 2\cdot 2\cdot 1$, $L(A) = 3\cdot 2$. We also use the same notations. And hereafter, a symbol attached by an overline will denote a value derived in § 2.2, i.e., for example, $\overline{b_1}$ and $\overline{\beta_1}$ are equal to $(\varphi_3\gamma_2)/\Delta$ and $(\Phi_1/\eta_1)\alpha_3\gamma_1$, respectively.

Solving relations in B_3, B_4, we have the same formulas for b_1, b_4 and b_2, b_5 :

$$b_1 = \overline{b_1} = \frac{\varphi_3\gamma_2}{\Delta}, \qquad b_4 = \overline{b_4} = -\frac{\varphi_3\gamma_1}{\Delta},$$

$$b_2 = \overline{b_2} = \frac{q\gamma_2 - \varphi_4\delta_2}{\Delta}, \qquad b_5 = \overline{b_5} = -\frac{q\gamma_1 - \varphi_4\delta_1}{\Delta}.$$

Again, solving the relations for d_i $(i = 1, 2, 3, 4)$ in B_5 of (28) and (29), we see that α_1, α_2 take the same values in § 2.2, i.e., $\alpha_1 = \overline{\alpha_1}$ and $\alpha_2 = \overline{\alpha_2}$, and β_1, β_2 are expressed in terms of $\overline{\beta_1}$, $\overline{\beta_2}$ in § 2.2 and β'_1, β'_2 as follows :

$$\beta_1 = \overline{\beta_1} + \alpha_4\beta'_1, \qquad \beta_2 = \overline{\beta_2} + \alpha_4\beta'_2.$$

Substituting the relations for c_6 and c_8 in B_4, we then solve

$$\begin{cases} \beta_1 b_3 + \beta_2 b_6 = \varphi_5, \\ \left(\alpha_1 + \alpha_3\gamma_1\frac{f_2}{f_1}\right) b_3 + \left(\alpha_2 + \alpha_3\gamma_2\frac{f_2}{f_1}\right) b_6 = \psi_5. \end{cases}$$

For simplicity, we hereafter put

$$z = \alpha_3\gamma_1, \qquad x = \frac{\gamma_2}{\gamma_1}$$

and then obtain :

$$\begin{aligned} \Omega &= \beta_1\left(\alpha_2 + \alpha_3\gamma_2\frac{f_2}{f_1}\right) - \beta_2\left(\alpha_1 + \alpha_3\gamma_1\frac{f_2}{f_1}\right) \\ &= \omega\alpha_3 + \left[(\alpha_2\beta'_1 - \alpha_1\beta'_2)\frac{1}{\alpha_3} + (\beta'_1\gamma_2 - \beta'_2\gamma_1)\frac{f_2}{f_1}\right]\alpha_3\alpha_4 \end{aligned}$$

$$= \widehat{\omega} z + \alpha_4 \beta_1' \left\{ \left(\frac{\xi_4}{\eta_2} + \frac{f_2}{f_1} \right) x + \frac{f_1}{\varphi_2} \frac{\xi_4}{\eta_2} p \right\} z - \alpha_4 \beta_2' \left(\frac{\xi_4}{\eta_1} + \frac{f_2}{f_1} \right) z$$

$$b_3 = \frac{\widehat{b_3} z - \psi_5 \alpha_4 \beta_2'}{\Omega},$$

$$b_6 = \frac{\widehat{b_6} z + \psi_5 \alpha_4 \beta_1'}{\Omega},$$

where $\widehat{\omega}$ and $\widehat{b_3}$, $\widehat{b_6}$ denote the following values

$$\begin{aligned} \widehat{\omega} &= \left[\frac{\xi_4}{\eta_1 \eta_2} (\Phi_1 - \Phi_2) + \frac{f_2}{f_1} \left(\frac{\Phi_1}{\eta_1} - \frac{\Phi_2}{\eta_2} \right) \right] xz \\ &+ \left[\frac{\xi_4}{\eta_1 \eta_2} \frac{f_1}{\varphi_2} (\Phi_1 - \xi_4) - \frac{\xi_4}{\eta_2} \frac{f_2}{\varphi_2} \right] pz, \\ \widehat{b_3} &= \left(\left(\frac{\xi_4}{\eta_2} + \frac{f_2}{f_1} \right) \varphi_5 - \frac{\Phi_2}{\eta_2} \psi_5 \right) x + \frac{\xi_4}{\eta_2} \frac{f_1}{\varphi_2} (\varphi_5 - \psi_5) p, \\ \widehat{b_6} &= \left(\frac{\Phi_1}{\eta_1} \psi_5 - \left(\frac{\xi_4}{\eta_1} + \frac{f_2}{f_1} \right) \varphi_5 \right). \end{aligned}$$

Similarly, we also solve

$$\begin{cases} \beta_1 b_7 + \beta_2 b_8 = q_1, \\ \left(\alpha_1 + \alpha_3 \gamma_1 \frac{f_2}{f_1} \right) b_7 + \left(\alpha_2 + \alpha_3 \gamma_2 \frac{f_2}{f_1} \right) b_8 = q_1 - \psi_6 \alpha_4, \end{cases}$$

obtaining

$$b_7 = \frac{\widehat{b_7} z + (\psi_6 \alpha_4 - q_1) \alpha_4 \beta_2'}{\Omega},$$

$$b_8 = \frac{\widehat{b_8} z - (\psi_6 \alpha_4 - q_1) \alpha_4 \beta_1'}{\Omega},$$

where we have put

$$\widehat{b_7} = \left[\frac{\Phi_2}{\eta_2} x + \frac{\xi_4}{\eta_2} \frac{f_1}{\varphi_2} p \right] \psi_6 \alpha_4 + \left(\frac{f_2}{f_1} - \frac{\psi_4}{\varphi_4} \right) q_1 x,$$

$$\widehat{b_8} = -\frac{\Phi_1}{\eta_1} \psi_6 \alpha_4 - \left(\frac{f_2}{f_1} - \frac{\psi_4}{\varphi_4} \right) q_1.$$

The substitution of these values into relations in B_6

$$\begin{cases} \beta_1' b_3 + \beta_2' b_6 = 0, \\ \beta_1' b_7 + \beta_2' b_8 = \varphi_6 \end{cases}$$

immediately leads to

$$\begin{cases} \beta_1' \widehat{b}_3 + \beta_2' \widehat{b}_6 = 0, \\ \beta_1' \widetilde{b}_7 + \beta_2' \widetilde{b}_8 = \varphi_6 \widehat{\omega}, \end{cases}$$

where

$$\begin{aligned} \widetilde{b}_7 &= \left[\left(\frac{\Phi_2}{\eta_2}\psi_6 - \left(\frac{\xi_4}{\eta_2} + \frac{f_2}{f_1} \right) \varphi_6 \right) x + (\psi_6 - \varphi_6) \left(\frac{\xi_4}{\eta_2} \frac{f_1}{\varphi_2} \right) p \right] \alpha_4 \\ &\quad + \left(\frac{f_2}{f_1} - \frac{\psi_4}{\varphi_4} \right) q_1 x, \\ \widetilde{b}_8 &= \left(-\frac{\Phi_1}{\eta_1}\psi_6 + \left(\frac{\xi_4}{\eta_1} + \frac{f_2}{f_1} \right) \varphi_6 \right) \alpha_4 - \left(\frac{f_2}{f_1} - \frac{\psi_4}{\varphi_4} \right) q_1. \end{aligned}$$

Hence we have

$$\begin{cases} \beta_1' = \dfrac{-\widehat{b}_6 \varphi_6 \widehat{\omega}}{\widehat{b}_3 \widetilde{b}_8 - \widehat{b}_6 \widetilde{b}_7}, \\ \beta_2' = \dfrac{\widehat{b}_3 \varphi_6 \widehat{\omega}}{\widehat{b}_3 \widetilde{b}_8 - \widehat{b}_6 \widetilde{b}_7}, \end{cases} \tag{30}$$

It is remaked that $\widehat{b}_3$, $\widehat{b}_6$ and $\widetilde{b}_7$, $\widetilde{b}_8$ are depending only on x and α_4, and hence β_i' $(i = 1, 2)$ are expressed in terms of x, z and α_4, i.e., γ_i $(i = 1, 2)$, α_3 and α_4. This fact implies that if one gives any values to 6 parameters γ_i, δ_i $(i = 1, 2)$ and α_3, α_4, then one can determine all constants included in the generators. We shall now prove that one of them depends on the other.

As done in § 2.2, we first seek α_1', α_2' and α_3' by the relations

$$\begin{cases} \alpha_1' \left(\dfrac{\psi_1}{f_2} \right) + \alpha_3' c_4 = c_1 - \alpha_4' d_5, \\ \alpha_1' b_1 + \alpha_2' b_4 = \psi_3 - \alpha_4' d_7, \\ \alpha_1' b_2 + \alpha_2' b_5 + \alpha_3' \psi_4 = q - \alpha_4' d_8. \end{cases} \tag{31}$$

This system of linear equations is just the same as (14) except for terms in the right hand side, i.e., if $\alpha_4' = 0$, then $\alpha_i' = \overline{\alpha_i'}$ $(i = 1, 2, 3)$.

Solving (31) :

$$\alpha_1' = \begin{vmatrix} c_1 - \alpha_4' d_5 & 0 & c_4 \\ \psi_3 - \alpha_4' d_7 & b_4 & 0 \\ q - \alpha_4' d_8 & b_5 & \psi_4 \end{vmatrix} /W,$$

$$\alpha_2' = \begin{vmatrix} \frac{\psi_1}{f_2} & c_1 - \alpha_4' d_5 & c_4 \\ b_1 & \psi_3 - \alpha_4' d_7 & 0 \\ b_2 & q - \alpha_4' d_8 & \psi_4 \end{vmatrix} /W,$$

$$\alpha_3' = \begin{vmatrix} \frac{\psi_1}{f_2} & 0 & c_1 - \alpha_4' d_5 \\ b_1 & b_4 & \psi_3 - \alpha_4' d_7 \\ b_2 & b_5 & q - \alpha_4' d_8 \end{vmatrix} /W,$$

where W is a value derived in § 2.2, we consequently obtain

$$\alpha_1' = \overline{\alpha_1'} - \left(\frac{\xi_4}{\Phi_1}\beta_1'\right)\alpha_4',$$

$$\alpha_2' = \overline{\alpha_2'} - \left(\beta_2' - \frac{\eta_1}{\Phi_1}\frac{\psi_4}{\varphi_4}\frac{\gamma_2}{\gamma_1}\beta_1'\right)\alpha_4',$$

$$\alpha_3' = \overline{\alpha_3'} - \left(\frac{\eta_1}{\Phi_1}\frac{1}{\gamma_1}\beta_1'\right)\alpha_4'.$$

Now we substitute those values into

$$\alpha_1' b_3 + \alpha_2' b_6 + \alpha_3' c_6 f_2 = c_3 f_2,$$

and obtain

$$\alpha_4'\left[\left(\frac{\xi_4}{\Phi_1}\beta_1'\right) b_3 + \left(\beta_2' - \frac{\eta_1}{\Phi_1}\frac{\psi_4}{\varphi_4}\frac{\gamma_2}{\gamma_1}\beta_1'\right) b_6 + \left(\frac{\eta_1}{\Phi_1}\frac{1}{\gamma_1}\beta_1'\right) c_6 f_2\right] =$$

$$\alpha_4'\left[\left(\frac{\xi_4}{\Phi_1} + \frac{\eta_1}{\Phi_1}\frac{f_2}{f_1}\right)\beta_1' b_3 + \left\{\beta_2' - \frac{\eta_1}{\Phi_1}\left(\frac{\psi_4}{\varphi_4} - \frac{f_2}{f_1}\right) x\beta_1'\right\} b_6\right]$$

$$= \overline{\alpha_1'} b_3 + \overline{\alpha_2'} b_6 + \overline{\alpha_3'} c_6 f_2 - c_3 f_2$$

$$= \left(\overline{\alpha_1'} + \overline{\alpha_3'}\frac{f_2}{f_1}\frac{z}{\alpha_3} - \frac{f_2}{f_1}\delta_1\right) b_3 + \left(\overline{\alpha_2'} + \overline{\alpha_3'}\frac{f_2}{f_1}\frac{z}{\alpha_3}x - \frac{f_2}{f_1}\delta_2\right) b_6.$$

Hence, we have

$$\alpha_4' = \frac{(e_4 - f_1)(\alpha_4(e_5 - e_6) + q_1)\cdot \mathrm{Num}}{e_4 f_1 f_2 (e_3 - f_1)(e_6 - f_1)(e_1 e_4 e_5 - f_1^2 f_2)((e_1 - e_2)x - e_1 p)\alpha_3 z},$$

where

$$\begin{aligned}\mathrm{Num} &= e_2(e_1e_4e_5 - f_1^2f_2)\{-\alpha_3\delta_2e_3(e_1e_4 - f_1f_2) + \alpha_3\delta_1e_4(e_1e_3 - f_1f_2)x + f_1f_2qxz\} \\ &\quad -e_1f_1f_2(\alpha_3\delta_1(e_3 - e_4) + qz)\{(e_2e_4e_5 - f_1^2f_2)x + f_1^2f_2p\}.\end{aligned}$$

Combining this with

$$\begin{aligned}\beta_1' &= \frac{(e_6 - f_1)(e_1e_4e_5 - f_1^2f_2)z}{e_1f_1(e_4 - f_1)(\alpha_4(e_5 - e_6) + q_1)}, \\ \beta_2' &= \frac{(e_6 - f_1)(e_2e_4e_5 + f_1^2f_2p - f_1^2f_2x)z}{e_2f_1(e_4 - f_1)(\alpha_4(e_5 - e_6) + q_1)},\end{aligned}$$

we have thus determined all α_i' $(i = 1, 2, 3, 4)$.

Now, substituting above values of the α_i' into

$$\alpha_1'p + \alpha_2'\left(\frac{\psi_2}{f_2}\right) + \alpha_3'c_5 + \alpha_4'd_6 = c_2,$$

we consequently obtain the required relation

$$\begin{aligned}\delta_2 &= \frac{(e_1e_3e_5 - f_1^2f_2)(e_2e_4e_5 - f_1^2f_2)}{(e_1e_4e_5 - f_1^2f_2)(e_2e_3e_5 - f_1^2f_2)}\delta_1x \\ &\quad + \frac{e_1e_5(e_3 - e_4)f_1^2f_2}{(e_1e_4e_5 - f_1^2f_2)(e_2e_3e_5 - f_1^2f_2)}\delta_1p \\ &\quad + \frac{e_5f_1^2f_2(e_1pq - (e_1 - e_2)qx)}{(e_1e_4e_5 - f_1^2f_2)(e_2e_3e_5 - f_1^2f_2)}\left(\frac{z}{\alpha_3}\right).\end{aligned}$$

This shows that δ_2 can be determined uniquely by γ_1, γ_2, δ_1 and α_3. Hence, if 5 parameters γ_1, γ_2, δ_1 and α_3, α_4 are given, then all constants b_i, c_i, d_i $(i = 1, 2, 3, 4, 5, 6, 7, 8)$ are determined explicitly.

As a matter of course, the formula remained to the last :

$$\alpha_1'b_7 + \alpha_2'b_8 + \alpha_3'c_8f_2 + \alpha_4'\psi_6 = c_7f_2$$

becomes an identity, which is verified by the Fuchs relation $e_1e_2e_3e_4e_5e_6 = f_1^4f_2^2$.

We shall summarize all results derived in the following

General expression

$$b_1 = -\frac{(e_3 - f_1)(e_1e_4e_5 - f_1^2f_2)(e_2e_3e_5 - f_1^2f_2)\alpha_3x}{e_5f_1^2f_2((e_3 - e_4)\alpha_3\delta_1 + qz)(e_1p + (e_2 - e_1)x)},$$

$$b_2 = \frac{(e_4-f_1)[(e_1e_3e_5-f_1^2f_2)(e_2e_4e_5-f_1^2f_2)x+e_1e_5f_1^2f_2(e_3-e_4)p]\alpha_3^2\delta_1}{e_5f_1^2f_2((e_3-e_4)\alpha_3\delta_1+qz)(e_1p+(e_2-e_1)x)z}$$

$$+\frac{(e_4-f_1)\alpha_3 q}{((e_3-e_4)\alpha_3\delta_1+qz)}-\frac{(e_1e_4e_5-f_1^2f_2)(e_2e_3e_5-f_1^2f_2)\alpha_3 qx}{e_5f_1^2f_2((e_3-e_4)\alpha_3\delta_1+qz)(e_1p+(e_2-e_1)x)},$$

$$b_3 = -\frac{e_1(e_4-f_1)[(e_2e_4e_5-f_1^2f_2)x+f_1^2f_2p]}{e_4f_1f_2(e_1p+(e_2-e_1)x)z},$$

$$b_4 = \frac{(e_3-f_1)(e_1e_4e_5-f_1^2f_2)(e_2e_3e_5-f_1^2f_2)\alpha_3}{e_5f_1^2f_2((e_3-e_4)\alpha_3\delta_1+qz)(e_1p+(e_2-e_1)x)},$$

$$b_5 = -\frac{((e_4-f_1)\alpha_3\delta_1-qz)(e_1e_4e_5-f_1^2f_2)(e_2e_3e_5-f_1^2f_2)\alpha_3}{e_5f_1^2f_2((e_3-e_4)\alpha_3\delta_1+qz)(e_1p+(e_2-e_1)x)z},$$

$$b_6 = \frac{e_2(e_4-f_1)(e_1e_4e_5-f_1^2f_2)}{e_4f_1f_2(e_1p+(e_2-e_1)x)z},$$

$$b_7 = \frac{e_1(e_4-f_1)[(e_2e_4e_6-f_1^2f_2)\alpha_4x+f_1^2f_2\alpha_4p-e_2e_4q_1x]}{e_4f_1f_2(e_1p+(e_2-e_1)x)z},$$

$$b_8 = -\frac{e_2(e_4-f_1)[(e_1e_4e_6-f_1^2f_2)\alpha_4x-e_1e_4q_1]}{e_4f_1f_2(e_1p+(e_2-e_1)x)z},$$

$$c_1 = \frac{(e_1-f_1)\delta_1}{f_1},$$

$$c_2 = \frac{(e_2-f_1)[(e_1e_3e_5-f_1^2f_2)(e_2e_4e_5-f_1^2f_2)x+e_1e_5f_1^2f_2(e_3-e_4)p]\delta_1}{f_1(e_1e_4e_5-f_1^2f_2)(e_2e_3e_5-f_1^2f_2)}$$

$$+\frac{e_5f_1f_2(e_2-f_1)(e_1p+(e_2-e_1)x)qz}{(e_1e_4e_5-f_1^2f_2)(e_2e_3e_5-f_1^2f_2)\alpha_3}+\delta_1p,$$

$$c_3 = -\frac{(e_4-f_1)[(e_2e_4e_5-f_1^2f_2)\alpha_3\delta_1-e_2e_5qz]}{e_4(e_2e_3e_5-f_1^2f_2)\alpha_3z},$$

$$c_4 = \frac{(e_1-f_1)z}{f_1\alpha_3},$$

$$c_5 = \frac{((e_2-f_1)x+f_1p)z}{f_1\alpha_3},$$

$$c_6 = -\frac{(e_4 - f_1)}{e_4\alpha_3},$$

$$c_7 = \frac{e_5(e_4 - f_1)[\{e_2(e_1e_3e_5 - f_1^2f_2) + e_1(e_2e_4e_6 - f_1^2f_2)\}\alpha_4\delta_1 + e_1e_2(e_3 - e_4)\delta_1q_1]}{(e_1e_4e_5 - f_1^2f_2)(e_2e_3e_5 - f_1^2f_2)z}$$

$$-\frac{e_2e_5(e_4 - f_1)[(e_1e_4e_6 - f_1^2f_2)\alpha_4 - e_1e_4q_1]q}{e_4(e_1e_4e_5 - f_1^2f_2)(e_2e_3e_5 - f_1^2f_2)\alpha_3},$$

$$c_8 = \frac{(e_4 - f_1)\alpha_4}{e_4\alpha_3},$$

$$d_1 = \frac{(e_1 - f_1)[\alpha_4(e_5 - f_1)(e_1e_4e_6 - f_1^2f_2) + f_1(e_1e_4 - f_1f_2)q_1]z}{e_1f_1^2(e_4 - f_1)((e_5 - e_6)\alpha_4 + q_1)},$$

$$d_2 = \frac{[(e_5 - f_1)(e_1e_4e_6 - f_1^2f_2)\alpha_4 + f_1(e_1e_4 - f_1f_2)q_1]pz}{e_1f_1(e_4 - f_1)((e_5 - e_6)\alpha_4 + q_1)}$$

$$+\frac{(e_2 - f_1)(e_5 - f_1)[(e_2e_4e_6 - f_1^2f_2)x + f_1^2f_2p]\alpha_4z}{e_2f_1^2(e_4 - f_1)((e_5 - e_6)\alpha_4 + q_1)}$$

$$+\frac{(e_2 - f_1)[(e_2e_4 - f_1f_2)x + f_1f_2p]q_1z}{e_2f_1(e_4 - f_1)((e_5 - e_6)\alpha_4 + q_1)},$$

$$d_3 = \frac{(e_3 - f_1)(e_1e_4e_5 - f_1^2f_2)(e_2e_3e_5 - f_1^2f_2)(\alpha_4(e_5 - f_1) + q_1)\alpha_3z}{e_1e_2e_5f_1(e_4 - f_1)((e_5 - e_6)\alpha_4 + q_1)((e_3 - e_4)\alpha_3\delta_1 + qz)},$$

$$d_4 = -\frac{e_4(e_5 - f_1)\{e_1(e_2e_3e_5 - f_1^2f_2) + e_2(e_1e_4e_6 - f_1^2f_2)\}\alpha_3^2\alpha_4\delta_1}{e_1e_2f_1((e_5 - e_6)\alpha_4 + q_1)((e_3 - e_4)\alpha_3\delta_1 + qz)}$$

$$+\frac{[(e_2e_4e_5 - f_1^2f_2)f_1^2f_2 - e_1e_4e_5f_1(e_2e_4 - f_1f_2) - e_1e_2e_3e_4e_5(e_5 - f_1)]\alpha_3^2\delta_1q_1}{e_1e_2e_5f_1((e_5 - e_6)\alpha_4 + q_1)((e_3 - e_4)\alpha_3\delta_1 + qz)}$$

$$+\frac{(e_5 - f_1)[e_2e_3(e_1e_4e_5 - f_1^2f_2) + e_1e_4(e_2e_4e_6 - f_1^2f_2)]\alpha_3\alpha_4qz}{e_1e_2f_1(e_4 - f_1)((e_5 - e_6)\alpha_4 + q_1)((e_3 - e_4)\alpha_3\delta_1 + qz)}$$

$$-\frac{[e_2e_5(e_1e_4e_6 - f_1^2f_2) + f_1f_2(e_2e_4e_5 - f_1^2f_2)]\alpha_3\alpha_4qz}{e_1e_2e_5(e_4 - f_1)((e_5 - e_6)\alpha_4 + q_1)((e_3 - e_4)\alpha_3\delta_1 + qz)}$$

$$+\frac{[e_1e_4e_5(e_2e_3e_5 - f_1^2f_2) - f_1^2f_2(e_2e_4e_5 - f_1^2f_2)]\alpha_3qq_1z}{e_1e_2e_5f_1(e_4 - f_1)((e_5 - e_6)\alpha_4 + q_1)((e_3 - e_4)\alpha_3\delta_1 + qz)}$$

$$+\frac{[e_1e_4(e_4 - f_1) - f_1f_2(e_3 - f_1)]\alpha_3qq_1z}{e_1(e_4 - f_1)((e_5 - e_6)\alpha_4 + q_1)((e_3 - e_4)\alpha_3\delta_1 + qz)},$$

$$d_5 = \frac{(e_1 - f_1)(e_6 - f_1)(e_1e_4e_5 - f_1^2 f_2)z}{e_1 f_1^2 (e_4 - f_1)((e_5 - e_6)\alpha_4 + q_1)},$$

$$d_6 = \frac{(e_6 - f_1)[e_1(e_2 - f_1)\{(e_2e_4e_5 - f_1^2 f_2)x + f_1^2 f_2 p\} + e_2 f_1(e_1e_4e_5 - f_1^2 f_2)p]z}{e_1e_2 f_1^2 (e_4 - f_1)((e_5 - e_6)\alpha_4 + q_1)},$$

$$d_7 = \frac{(e_3 - f_1)(e_6 - f_1)(e_1e_4e_5 - f_1^2 f_2)(e_2e_3e_5 - f_1^2 f_2)\alpha_3 z}{e_1e_2e_5 f_1(e_4 - f_1)((e_5 - e_6)\alpha_4 + q_1)((e_3 - e_4)\alpha_3\delta_1 + qz)},$$

$$d_8 = -\frac{(e_4 - f_1)(e_6 - f_1)(e_1e_4e_5 - f_1^2 f_2)(e_2e_4e_5 - f_1^2 f_2)\alpha_3^2\delta_1}{e_1e_2e_5 f_1(e_4 - f_1)((e_5 - e_6)\alpha_4 + q_1)((e_3 - e_4)\alpha_3\delta_1 + qz)}$$

$$+\frac{(e_6 - f_1)(e_1e_4e_5 - f_1^2 f_2)\{(e_2e_3e_5 - f_1^2 f_2) + e_2e_5(e_4 - f_1)\}\alpha_3 q z}{e_1e_2e_5 f_1(e_4 - f_1)((e_5 - e_6)\alpha_4 + q_1)((e_3 - e_4)\alpha_3\delta_1 + qz)}.$$

References

[1] M. Kohno : *A note on the calculation of monodromy groups of hypergeometric systems*, RIMS Kokyuroku **663**, 1988, 83-104.

[2] M. Kohno : *Global analysis in linear differential equations*, Preprint.

[3] K. Okubo : *On the group of Fuchsian equations*, Seminor Report of Tokyo Metropolitan University, 1987.

[4] K. Tsurumaru : *A study on the group of differential equations*, Master Thesis of Kumamoto University, 1990.

[5] T. Yokoyama : *On an irreducibility condition for hypergeometric systems*, to appear in Funkcial. Ekvac.

Department of Mathematics
Faculty of Science
Kumamoto University
Kurokami 2-39-1
Kumamoto 860