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SOME IDENTITIES FOR COMMUTATIVE QUASI-GROUPS

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1. Introduction

Let S be a system of elements a, b, c,......... closed with respect to a multiplicative operation. For any a, b, $c \in S$, consider the following postulates:

(A) $(a\,b)\,a = b$ (A′) $a\,(b\,a) = b$

(B) $a\,(a\,b) = b$

(C) $(b\,a)\,a = b$

(D) $c\,(b\,(a\,(a\,c))) = b$

Following Higman and Neumann [2], Padmanabhan [3] and Sholander [4], Agrawal [1] also formulated an identity (D) (single-equational axiom) to characterize a commutative quasi-group. In this paper, we have established some more identities to characterize commutative quasi-groups individually as follows:

(E) $(((c\,a)\,a)\,b)\,c = b$

(F) $(a\,(a\,c))(b\,c) = b$

(G) $(c\,b)((c\,a)\,a) = b$

(H) $(((a\,b)\,c)\,c)\,a = b$

(I) $a\,(c\,(c\,(b\,a))) = b$

(J) $(b\,c)(a\,(a\,c)) = b$

(K) $((c\,a)\,a)(c\,b) = b$

2. Some propositions

The following prososition were proved [1]:

Proposition (2.1) $(A) \Longleftrightarrow (A')$

Proposition (2.2) $(A, B)^{*} \Longleftrightarrow (A, C)$

Proposition (2.3) $(B, C) \Longleftrightarrow (A, B)$

*Here and onwards, the notation (A, B) will mean the postulates (A) and (B) together.

3. Characterization of commutative quasi-groups

Theorem (3.1) It has been proved that [1], the system S is a commutative quasi-group, if any one of the following is satisfied:

(I) (A, B) (II) (A, C) (III) (B, C)

Theorem (3.2) The system S togeher with the identity relation (E) characterizes a commutative quasi-group.

Proof Putting $c = ba$ in (E), we find

$$((((b\,a)\,a)\,a)\,b)\,(b\,a) = b \tag{3.1}$$

Now, if we replace first b by a and then c by b in (E), we get

$$(((b\,a)\,a)\,a)\,b = a \tag{3.2}$$

Thus, (3.1) and (3.2) yield

$$a\,(b\,a) = b \tag{3.3}$$

which, in view of the Proposition (2.1), imply

$$(a\,b)\,a = b \tag{3.4}$$

Again, replacing a by bc in (E) and applying (3.3), we find

$$(((c\,(b\,c))(b\,c))\,b)\,c = b,$$

or

$$((b\,(b\,c))\,b)\,c = b$$

which, in view of (3.4), yields

$$(b\,c)\,c = b \tag{3.5}$$

Thus, we conclude that (3.4) and (3.5) are the same as the postulates (A) and (C) and therefore, in view of the Theorem (3.1), the system S forms a commutative quasi-group.

Theorem (3.3) The system S together with the identity relation (F) characterizes a commutative quasi-group.

Proof Putting $b = a$ in (F), we find

$$(a\,(a\,c))\,(a\,c) = a \tag{3.6}$$

Replacing ac by d in (3.6), we obtain

$$(a\,d)\,d = a \tag{3.7}$$

Again, replacing a by bc in (F) and applying (3.7), we have

$$((b\,c)\,((b\,c)\,c))\,(b\,c) = b,$$

or

$$((b\,c)\,b)\,(b\,c) = b \tag{3.8}$$

Further, replacing bc by a in (3.8), we get

$$(a\,b)\,a = b \tag{3.9}$$

Thus, we observe that (3.9) and (3.7) are the same as the postulates (A) and (C) and therefore, in view of the Theorem (3.1), the system S forms a commutative quasi-group.

Theorem (3.4) The system S together with the identity relation (G) characterizes a commutative quasi-group.

Proof Putting $b = a$ in (G), we have

$$(c\,a)\,((c\,a)\,a) = a \tag{3.10}$$

Replacing ca by b in (3.10), we find

$$b\,(b\,a) = a \tag{3.11}$$

Again, replacing a by cb in (G) and applying (3.11), we obtain

$$(cb)\,((c\,(c\,b))\,c\,b) = b,$$

or

$$(c\,b)\,(b\,(c\,b)) = b \tag{3.12}$$

Now, replacing cb by a in (3.12), we get

(3.13) $$a\,(b\,a) = b$$

which, in view of the Proposition (2.1), imply

(3.14) $$(a\,b)\,a = b$$

Thus, we notice that (3.14) and (3.11) are the same as the postulates (A) and (B) and therefore, in view of the Theorem (3.1), the system S forms a commutative quasi-group.

Therorem (3.5) The system S together with the identity relation (H) characterizes a commutative quasi-group.

Proof Replacing a by ca in (H), we find

(3.15) $$((((c\,a)\,b)\,c)\,c)\,(c\,a) = b$$

Again, replacing b by c in (3.15) and applying (H), we get

$$((((c\,a)\,c)\,c)\,c)\,(c\,a) = c,$$

or

(3.16) $$a\,(c\,a) = c$$

which, in view of the Proposition (2.1), imply

(3.17) $$(a\,c)\,a = c$$

Now, replacing c by a in (H) and applying (3.17), we obtain

$$(((a\,b)\,a)\,a)\,a = b,$$

or

(3.18) $$(b\,a)\,a = b$$

Thus, we conclude that (3.17) and (3.18) are the same as the postulates (A) and (C) and therefore, in view of the Theorem (3.1), the system S forms a commutative quasi-group.

Theorem (3.6) The system S together with the identity relation (I) characterizes a commutative quasi-group.

Proof Replacing a by ac in (I), we find

$$(ac)(c(c(b(ac)))) = b \tag{3.19}$$

Replacing b by c in (3.19) and applying (I), we get

$$(ac)(c(c(c(ac)))) = b,$$

or

$$(ac)a = c \tag{3.20}$$

Again, replacing c by a in (I) and applying (3.20), we obtain

$$a(a(a(ba))) = b$$

or

$$a(ab) = b \tag{3.21}$$

Thus, we observe that (3.20) and (3.21) are the same as the postulates (A) and (B) and therefore, in view of the Theorem (3.1), the system S forms a commutative quasi-group.

Theorem (3.7) The system S together with the identity relation (J) characterizes a commutative quasi-group.

Proof. Putting $b = a$ in (J), we find

$$(ac)(a(ac)) = a \tag{3.22}$$

Replacing ac by b in (3.22), we get

$$b(ab) = a \tag{3.23}$$

which, in view of the Proposition (2.1), imply

$$(ba)b = a \tag{3.24}$$

Now, replacing c by a in (J) and applying (3.23), we obtain

$$(ba)(a(aa)) = b,$$

or

$$(ba)a = b \tag{3.25}$$

Thus, we notice that (3.24) and (3.25) are the same as the postulates (A) and (C) and therefore, in view of the Theorem (3.1), the system S forms a commutative quasi-group.

Theorem (3.8) The system S together with the identity relation (K) characterizes a commutative quasi-group.

Proof Putting $b = a$ in (K), we find

$$((c\,a)\,a)\,(c\,a) = a \tag{3.26}$$

Replacing ca by b in (3.26), we get

$$(b\,a)\,b = a \tag{3.27}$$

Again, replacing c by a in (K) and applying (3.27), we obtain

$$((a\,a)\,a)\,(a\,b) = b,$$

or

$$a\,(a\,b) = b \tag{3.28}$$

Thus, we conclude that (3.27) and (3.28) are the same as the postulates (A) and (B) and therefore, in view of the Theorem (3.1), the system S forms a commutative quasi-group.

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