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IRREDUCIBILITY OF ACCESSORY PARAMETER FREE SYSTEMS

Dedicated to Professor Kenjiro Okubo on his sixtieth birthday

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Introduction. Systems of differential equations free from accessory parameters are expected to define a new class of special functions. Along the line of Okubo theory, Yokoyama classified such systems: His theorem says that, under a generic condition, if a system of differential equations free from accessory parameters is irreducible, then it falls into one of the eight classes of systems - system (I), (I*), (II), (II*), (III), (III*), (IV) and (IV*). Systems (I) and (I*) are transformed into the generalized hypergeometric equation and the Jordan-Pochhammer equation, respectively, both of which are known to be generically irreducible ([BH], [M1]). System (II) is studied in [ST2], and is shown to be generically irreducible.

In [H2] we have obtained monodromy representations of the systems (J) and (J*) (J=II, III, IV). Using the result, in this paper we show the generic irreducibility of the remaining systems (II*), (III), (III*), (IV), (IV*). Notice that Yokoyama's theorem does not assert the irreducibility of the systems. Our theorems are partial applications of Misaki's pioneering work [M1].

Notation. $e(\alpha) := \exp(2\pi\sqrt{-1}\alpha)$ for $\alpha \in \mathbf{C}$.

§1. System (II*). Let $t_1, t_2, t_3 \in \mathbf{C}$ be mutually distinct points which do not lie on a line. Let $n = 2m$ be an even integer equal to or greater than 4. Take $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbf{C}^m$, $\mu = (\mu_1, \ldots, \mu_{m-1}) \in \mathbf{C}^{m-1}$, $\nu \in \mathbf{C}$, and $\rho = (\rho_1, \rho_2) \in \mathbf{C}^2$ satisfying

$$\lambda_i \neq \lambda_j, \ \mu_i \neq \mu_j, \ \rho_i \neq \rho_j \quad (i \neq j), \tag{1.1}$$

and

$$\sum_{i=1}^{m} \lambda_i + \sum_{i=1}^{m-1} \mu_i + \nu = m\rho_1 + m\rho_2. \tag{1.2}$$

The system $(\mathrm{II}^*)_{\lambda,\mu,\nu,\rho}$ of rank n is the system of differential equations

$$(1.3)\qquad (xI_n - T)\frac{dy}{dx} = Ay$$

with

$$T = \begin{pmatrix} t_1 I_m & & \\ & t_2 I_{m-1} & \\ & & t_3 \end{pmatrix}, \quad A = \left(\begin{array}{ccc|cccc} \lambda_1 & & & & & & \\ & \ddots & & & (\alpha_{ij}) & & \\ & & \lambda_m & & & & \\ \hline & & & \mu_1 & & & \gamma_1 \\ & (\beta_{ij}) & & & \ddots & & \vdots \\ & & & & & \mu_{m-1} & \gamma_{m-1} \\ & & & \delta_1 & \cdots & \delta_{m-1} & \nu \end{array}\right),$$

where

$$\alpha_{ij} = (\lambda_i - \rho_1)(\lambda_i - \rho_2) \prod_{\substack{1\le k\le m \\ k\ne i}} \left(\frac{\lambda_k + \mu_j - \rho_1 - \rho_2}{\lambda_i - \lambda_k}\right) \quad (1 \le i \le m,\ 1 \le j \le m-1),$$

$$\alpha_{im} = (\lambda_i - \rho_1)(\lambda_i - \rho_2) \prod_{\substack{1\le k\le m \\ k\ne i}} \frac{1}{\lambda_i - \lambda_k} \quad (1 \le i \le m),$$

$$\beta_{ij} = \prod_{\substack{1\le \ell\le m-1 \\ \ell\ne i}} \left(\frac{\lambda_j + \mu_\ell - \rho_1 - \rho_2}{\mu_i - \mu_\ell}\right) \quad (1 \le i \le m-1,\ 1 \le j \le m),$$

$$\beta_{mj} = -\prod_{\ell=1}^{m-1} (\lambda_j + \mu_\ell - \rho_1 - \rho_2) \quad (1 \le j \le m),$$

$$\gamma_i = \prod_{\substack{1\le \ell\le m-1 \\ \ell\ne i}} \frac{1}{\mu_i - \mu_\ell} \quad (1 \le i \le m-1),$$

$$\delta_j = -\prod_{k=1}^{m} (\lambda_k + \mu_j - \rho_1 - \rho_2) \quad (1 \le j \le m-1).$$

The monodromy group of system $(\mathrm{II}^*)_{\lambda,\mu,\nu,\rho}$ has been obtained in [H2].

Theorem 1. ([H2, Theorem 7]) *Assume*

$$\begin{cases} \rho_1, \rho_2 \notin \mathbf{Z}_{<0}, \quad \rho_1 - \rho_2 \notin \mathbf{Z}, \\ \lambda_i \notin \mathbf{Z}, \quad \lambda_i - \lambda_j \notin \mathbf{Z} \quad (1 \leq i, j \leq m,\ i \neq j), \\ \mu_i \notin \mathbf{Z}, \quad \mu_i - \mu_j \notin \mathbf{Z} \quad (1 \leq i, j \leq m-1,\ i \neq j), \\ \nu \notin \mathbf{Z}, \end{cases} \tag{1.4}$$

and

$$\begin{cases} \lambda_i - \rho_k \notin \mathbf{Z} \quad (1 \leq i \leq m,\ k = 1, 2), \\ \lambda_i + \mu_j - (\rho_1 + \rho_2) \notin \mathbf{Z} \quad (1 \leq i \leq m,\ 1 \leq j \leq m-1). \end{cases} \tag{1.5}$$

Then the monodromy group of the system (1.3) *with respect to a fundamental matrix solution is generated by*

$$\begin{aligned} M_1 &= \begin{pmatrix} E_m(\lambda) & (\xi_{ij})_{\substack{1 \leq i \leq m \\ 1 \leq j \leq m}} \\ O & I_m \end{pmatrix}, \\ M_2 &= \begin{pmatrix} I_m & O & O \\ (\eta_{ij})_{\substack{1 \leq i \leq m-1 \\ 1 \leq j \leq m}} & E_{m-1}(\mu) & (\eta_{in})_{1 \leq i \leq m-1} \\ O & O & 1 \end{pmatrix}, \\ M_3 &= \begin{pmatrix} I_{n-1} & O \\ (\zeta_j)_{1 \leq j \leq n-1} & e(\nu) \end{pmatrix}, \end{aligned} \tag{1.6}$$

where

$$E_m(\lambda) = \begin{pmatrix} e(\lambda_1) & & \\ & \ddots & \\ & & e(\lambda_m) \end{pmatrix}, \quad E_{m-1}(\mu) = \begin{pmatrix} e(\mu_1) & & \\ & \ddots & \\ & & e(\mu_{m-1}) \end{pmatrix}, \tag{1.7}$$

$$\begin{aligned} \xi_{ij} &= (e(\lambda_i) - e(\rho_1))(e(\rho_2 - \lambda_i) - 1) \prod_{\substack{1 \leq k \leq m \\ k \neq i}} \frac{e(\mu_j) - e(\rho_1 + \rho_2 - \lambda_k)}{e(\rho_1 + \rho_2 - \lambda_i) - e(\rho_1 + \rho_2 - \lambda_k)} \\ &\qquad (1 \leq i \leq m,\ 1 \leq j \leq m-1), \\ \xi_{im} &= (e(\lambda_i) - e(\rho_1))(e(\rho_2 - \lambda_i) - 1) \prod_{\substack{1 \leq k \leq m \\ k \neq i}} \frac{1}{e(\rho_1 + \rho_2 - \lambda_i) - e(\rho_1 + \rho_2 - \lambda_k)} \\ &\qquad (1 \leq i \leq m), \end{aligned} \tag{1.8}$$

$$
(1.9) \quad
\begin{aligned}
\eta_{ij} &= \prod_{\substack{1\le \ell\le m-1 \\ \ell\ne i}} \frac{e(\rho_1+\rho_2-\lambda_j)-e(\mu_\ell)}{e(\mu_i)-e(\mu_\ell)} \quad (1\le i\le m-1,\ 1\le j\le m), \\
\eta_{in} &= \prod_{\substack{1\le \ell\le m-1 \\ \ell\ne i}} \frac{1}{e(\mu_i)-e(\mu_\ell)} \quad (1\le i\le m-1),
\end{aligned}
$$

and

$$
(1.10) \quad
\begin{aligned}
\zeta_j &= e(\lambda_j+\nu-\rho_1-\rho_2)\prod_{\ell=1}^{m-1}(e(\rho_1+\rho_2-\lambda_j)-e(\mu_\ell)) \quad (1\le j\le m), \\
\zeta_{m+j} &= -\frac{\prod_{k=1}^{m}(e(\mu_j)-e(\rho_1+\rho_2-\lambda_k))}{e(\mu_j)} \quad (1\le j\le m-1).
\end{aligned}
$$

We denote the monodromy group by $G_{\mathrm{II}^*}(\lambda,\mu,\nu,\rho)$:

$$
(1.11) \quad G_{\mathrm{II}^*}(\lambda,\mu,\nu,\rho)=\langle M_1,M_2,M_3\rangle,
$$

where M_1, M_2, M_3 are given by (1.6).

The main result of this section is the following.

Theorem 2. *We assume* (1.4) *and* (1.5). *If moreover*

$$
(1.12) \quad \rho_1\notin \mathbf{Z},\ \rho_2\notin\mathbf{Z},
$$

then the system $(\mathrm{II}^*)_{\lambda,\mu,\nu,\rho}$ *is irreducible.*

Since the system (1.3) is Fuchsian, to show the theorem it is enough to show the irreducibility of $G_{\mathrm{II}^*}(\lambda,\mu,\nu,\rho)$ under the conditions (1.4), (1.5) and (1.12).

Now we assume only (1.4), and study the irreducibility of $G_{\mathrm{II}^*}(\lambda,\mu,\nu,\rho)$, which is well defined by (1.6) - (1.11), while it is not necessarily the monodromy group without the condition (1.5); in this sense we may call it an *apparent monodromy group.*

Proposition 3. *Assume* (1.4). *The group* $G_{\mathrm{II}^*}(\lambda,\mu,\nu,\rho)$ *is irreducible if and only if* (1.5) *and* (1.12) *hold.*

The following is a key lemma for our proof of the proposition.

Lemma 4. *Let* G *be a subgroup of* $\mathrm{GL}(n,\mathbf{C})$*, and let* $M\in G$ *be diagonalizable. Decompose* $V=\mathbf{C}^n$ *into a direct sum of the eigen spaces of* M*:*

$$
(1.13) \quad V=V_1\oplus\cdots\oplus V_\ell,
$$

where V_i and V_j are eigen spaces of M with respect to mutually distinct eigen values if $i \neq j$. Let

$$\pi_i : V \to V_i$$

be the projection onto V_i for $i = 1, \ldots, \ell$. Let W be an invariant subspace of V for G. Then we have

$$\pi_i(W) \subset W$$

for $i = 1, \ldots, \ell$.

Proof. Let λ_i be the eigen value of M corresponding to V_i. Thus we have $\lambda_i \neq \lambda_j$ for $i \neq j$. Take any $x \in W$, and decompose it according to (1.13):

$$x = x_1 + \cdots + x_\ell, \quad x_i = \pi_i(x) \in V_i.$$

We set $M^k x = y_k$ for $k = 0, 1, 2, \ldots$. Then we see $y_k \in W$. On the other hand we have

$$y_k = M^k x = {\lambda_1}^k x_1 + \cdots + {\lambda_\ell}^k x_\ell.$$

Thus we obtain

$$(x_1 \cdots x_\ell) \begin{pmatrix} 1 & \lambda_1 & {\lambda_1}^2 & \cdots & {\lambda_1}^{\ell-1} \\ 1 & \lambda_2 & {\lambda_2}^2 & \cdots & {\lambda_2}^{\ell-1} \\ & & \cdots & & \\ 1 & \lambda_\ell & {\lambda_\ell}^2 & \cdots & {\lambda_\ell}^{\ell-1} \end{pmatrix} = (y_0 \cdots y_{\ell-1}).$$

The determinant of the matrix in the above is the Vandermonde determinant and differs from 0. Then we have

$$(x_1 \cdots x_\ell) = (y_0 \cdots y_{\ell-1}) \begin{pmatrix} 1 & \lambda_1 & {\lambda_1}^2 & \cdots & {\lambda_1}^{\ell-1} \\ 1 & \lambda_2 & {\lambda_2}^2 & \cdots & {\lambda_2}^{\ell-1} \\ & & \cdots & & \\ 1 & \lambda_\ell & {\lambda_\ell}^2 & \cdots & {\lambda_\ell}^{\ell-1} \end{pmatrix}^{-1},$$

which shows $x_i \in W$. Lemma q.e.d.

Proof of Proposition 3. The eigen values of M_1 are $1, e(\lambda_1), \ldots, e(\lambda_m)$. Decompose $V = \mathbf{C}^n$ into a direct sum of the corresponding eigen spaces:

$$V = X_0 \oplus X_1 \oplus \cdots \oplus X_m, \tag{1.14}$$

where X_0 denotes the 1-eigen space of M_1, and X_i denotes the $e(\lambda_i)$-eigen space of M_1 for $i = 1, \ldots, m$. Similarly we decompose V into direct sums of the eigen spaces of M_2 and M_3:

$$V = Y_0 \oplus Y_1 \oplus \cdots \oplus Y_{m-1}, \tag{1.15}$$

where Y_0 denotes the 1-eigen space of M_2, and Y_i denotes the $e(\mu_i)$-eigen space of M_2 for $i = 1, \ldots, m-1$;

$$V = Z_0 \oplus Z_1, \tag{1.16}$$

where Z_0 denotes the 1-eigen space of M_3, and Z_1 denotes the $e(\nu)$-eigen space of M_3. By (1.6) we have

$$\begin{aligned} X_i &= \langle e_i \rangle \quad (1 \leq i \leq m), \\ Y_i &= \langle e_{m+i} \rangle \quad (1 \leq i \leq m-1), \\ Z_1 &= \langle e_n \rangle, \end{aligned} \tag{1.17}$$

where $\{e_1, \ldots, e_n\}$ denotes the standard basis of V. Thus we have another decomposition of V:

$$V = \bigoplus_{i=1}^{m} X_i \oplus \bigoplus_{i=1}^{m-1} Y_i \oplus Z_1. \tag{1.18}$$

Let

$$\begin{aligned} p_i &: V \to X_i \quad (0 \leq i \leq m), \\ q_i &: V \to Y_i \quad (0 \leq i \leq m-1), \\ r_i &: V \to Z_i \quad (i = 0, 1) \end{aligned} \tag{1.19}$$

be projections onto respective eigen spaces. Here we give their explicit forms. Take $v = {}^t(v_1, \ldots, v_n) \in V$. Then we have

$$p_0(v) = \begin{pmatrix} x_1(v) \\ \vdots \\ x_m(v) \\ v_{m+1} \\ \vdots \\ v_n \end{pmatrix}, \quad p_i(v) = (v_i - x_i(v))e_i \quad (1 \leq i \leq m), \tag{1.20}$$

where we have set

$$x_i(v) := \frac{\sum_{k=1}^{m} \xi_{ik} v_{m+k}}{1 - e(\lambda_i)} \quad (1 \le i \le m); \tag{1.21}$$

$$q_0(v) = \begin{pmatrix} v_1 \\ \vdots \\ v_m \\ y_{m+1}(v) \\ \vdots \\ y_{n-1}(v) \\ v_n \end{pmatrix}, \quad q_i(v) = (v_{m+i} - y_{m+i}(v))e_{m+i} \quad (1 \le i \le m-1), \tag{1.22}$$

where

$$y_{m+i}(v) := \frac{\sum_{k=1}^{m} \eta_{ik} v_k + \eta_{in} v_n}{1 - e(\mu_i)} \quad (1 \le i \le m-1); \tag{1.23}$$

$$r_0(v) = \begin{pmatrix} v_1 \\ \vdots \\ v_{n-1} \\ z_n(v) \end{pmatrix}, \quad r_1(v) = (v_n - z_n(v))e_n, \tag{1.24}$$

where

$$z_n(v) := \frac{\sum_{k=1}^{n-1} \zeta_k v_k}{1 - e(\nu)}. \tag{1.25}$$

Now suppose that $G_{\mathrm{II}^*}(\lambda, \mu, \nu, \rho)$ is reducible, and let W be an invariant subspace such that $W \ne \{0\}$, $W \ne V$. Since the dimension of each component of the decomposition (1.18) is 1, there is at least one component A such that $W \cap A = \{0\}$.

First we assume that

$$W \cap X_i = \{0\} \tag{1.26}$$

for some $i \in \{1, \ldots, m\}$. Take $v \in W \setminus \{0\}$, and decompose v according to (1.15) as

$$v = y_0 + y_1 + \cdots + y_{m-1}, \quad y_j \in Y_j \quad (0 \le j \le m-1). \tag{1.27}$$

By Lemma 4 we see

$$y_j \in W \quad (0 \le j \le m-1). \tag{1.28}$$

If $y_j \neq 0$ for some $j \in \{1, \ldots, m-1\}$, $W \cap Y_j \neq \{0\}$, which implies $e_{m+j} \in W$ by (1.17). Then, again using Lemma 4, we have $p_i(e_{m+j}) \in W$, while $p_i(e_{m+j}) \in X_i$, so that $p_i(e_{m+j}) \in W \cap X_i$. Thus it follows from (1.26) and (1.20) that

$$0 = p_i(e_{m+j}) = \frac{\xi_{ij}}{e(\lambda_i) - 1} e_i,$$

and hence we obtain

$$\xi_{ij} = 0. \tag{1.29}$$

If $y_j = 0$ for all $j \in \{1, \ldots, m-1\}$, $v = y_0 \neq 0$, and $y_0 \in W$. Decompose y_0 according to (1.16) as

$$y_0 = z_0 + z_1, \quad z_0 \in Z_0, \ z_1 \in Z_1. \tag{1.30}$$

If $z_1 \neq 0$, in the same way as above we have

$$\xi_{im} = 0. \tag{1.31}$$

If $z_1 = 0$, we have

$$v = y_0 = z_0 \in Y_0 \cap Z_0. \tag{1.32}$$

In this case we consider $w = M_1 v \in W$, and apply the above argument to w to obtain (1.29) or (1.31) or

$$M_1 v \in Y_0 \cap Z_0. \tag{1.33}$$

Now we assume (1.32) and (1.33). From $v \in Y_0$ and $v \in Z_0$ it follows that

$$\sum_{\ell=1}^{m} \eta_{k\ell} v_\ell + (e(\mu_k) - 1) v_{m+k} + \eta_{kn} v_n = 0 \quad (1 \leq k \leq m-1) \tag{1.34}$$

and

$$\sum_{\ell=1}^{n-1} \zeta_\ell v_\ell + (e(\nu) - 1) v_n = 0, \tag{1.35}$$

respectively. From $M_1 v \in Y_0$ and $M_1 v \in Z_0$ with the help of (1.34) and (1.35) it follows that

$$\sum_{\ell=1}^{m} \eta_{k\ell} \left\{ (e(\lambda_\ell) - 1) v_\ell + \sum_{p=1}^{m} \xi_{\ell p} v_{m+p} \right\} = 0 \quad (1 \leq k \leq m-1) \tag{1.36}$$

and

$$\sum_{\ell=1}^{m}\zeta_\ell\left\{(e(\lambda_\ell)-1)v_\ell+\sum_{p=1}^{m}\xi_{\ell p}v_{m+p}\right\}=0. \tag{1.37}$$

Set

$$P=\begin{pmatrix}(\eta_{k\ell})_{\substack{1\le k\le m-1\\ 1\le\ell\le m}}\\ (\zeta_\ell)_{1\le\ell\le m}\end{pmatrix},\quad R=\begin{pmatrix}E_m(\lambda)-I_m & (\xi_{\ell p}) & \\ (\eta_{k\ell}) & E_{m-1}(\mu)-I_{m-1} & (\eta_{kn})\\ (\zeta_\ell) & (\zeta_{m+\ell}) & e(\nu)-1\end{pmatrix}. \tag{1.38}$$

Then we can sum up (1.34), (1.35), (1.36) and (1.37) into

$$\begin{pmatrix}P & \\ & I_m\end{pmatrix}Rv=0. \tag{1.39}$$

Lemma 5.

$$\det P=e(\nu)\cdot\frac{\prod_{k=1}^{m-1}(-e(\mu_k))\prod_{1\le q<\ell\le m}(e(\rho_1+\rho_2-\lambda_q)-e(\rho_1+\rho_2-\lambda_\ell))}{\prod_{\ell=1}^{m}e(\rho_1+\rho_2-\lambda_\ell)\prod_{1\le k<p\le m-1}(e(\mu_p)-e(\mu_k))}.$$

Proof. Set $e(\rho_1+\rho_2-\lambda_\ell)=a_\ell$, $e(\mu_k)=b_k$ for $1\le\ell\le m$, $1\le k\le m-1$. Suppose for a moment that

$$a_\ell-b_k\neq 0\quad(\forall\ell,\ \forall k). \tag{1.40}$$

Then we have

$$\begin{aligned}P=&\begin{pmatrix}\prod_{q\neq1}\frac{1}{b_1-b_q} & & & \\ & \ddots & & \\ & & \prod_{q\neq m-1}\frac{1}{b_{m-1}-b_q} & \\ & & & e(\nu)\end{pmatrix}\\ &\times\left(\frac{1}{a_\ell-b_k}\right)_{1\le k,\ell\le m}\begin{pmatrix}\prod_{q=1}^{m-1}(a_1-b_q) & & \\ & \ddots & \\ & & \prod_{q=1}^{m-1}(a_m-b_q)\end{pmatrix},\end{aligned} \tag{1.41}$$

where we have set $b_m=0$. Applying Cauchy's lemma ([W, Lemma (7.6.A)]), we calculate the determinant of the middle matrix in the right hand side of (1.41), from which we obtain

$$\det P=e(\nu)\cdot\frac{\prod_{k=1}^{m-1}(-b_k)\prod_{1\le q<\ell\le m}(a_q-a_\ell)}{\prod_{\ell=1}^{m}a_\ell\prod_{1\le k<p\le m-1}(b_p-b_k)}. \tag{1.42}$$

As a rational function in the a_ℓ, b_k, (1.42) holds without the restriction (1.40), and this completes the proof. Lemma 5 q.e.d.

Lemma 6.

$$\det R = (e(\rho_1) - 1)^m (e(\rho_2) - 1)^m.$$

Proof. In the proof of Theorem 1 we have used the fact that the eigen values of the matrix $M_\infty := M_3 M_2 M_1$ are $e(\rho_1)$ (m-ple) and $e(\rho_2)$ (m-ple). Thus we have

$$\det(tI_n - M_\infty) = (t - e(\rho_1))^m (t - e(\rho_2))^m. \tag{1.43}$$

Set

$$Q = \begin{pmatrix} I_m & & \\ -(\eta_{k\ell}) & I_{m-1} & \\ -(\zeta_\ell) & -(\zeta_{m+\ell}) & 1 \end{pmatrix},$$

then we see $\det Q = 1$, so that

$$\det[Q(tI_n - M_\infty)] = (t - e(\rho_1))^m (t - e(\rho_2))^m \tag{1.44}$$

by (1.43). If we put $t = 1$, we see that the left hand side of (1.44) coincides with $\det[-R]$, from which the lemma follows. Lemma 6 q.e.d.

By the above two lemmas, on the assumption (1.4)

$$e(\rho_1) = 1, \text{ or } e(\rho_2) = 1 \tag{1.45}$$

follows from (1.39). To sum up, if we assume (1.26), we have (1.29) or (1.31) or (1.45). Taking (1.8) into consideration, we see that these conditions are equivalent to

$$\lambda_i - \rho_1 \in \mathbf{Z}, \text{ or } \lambda_i - \rho_2 \in \mathbf{Z}, \text{ or } \lambda_i + \mu_k - \rho_1 - \rho_2 \in \mathbf{Z}, \text{ or } \rho_1 \in \mathbf{Z}, \text{ or } \rho_2 \in \mathbf{Z}. \tag{1.46}$$

Assuming $W \cap Y_i = \{0\}$ $(1 \le \exists i \le m-1)$ or $W \cap Z_1 = \{0\}$ in place of (1.26), we also obtain (1.46).

Conversely we assume (1.46). We set $V_i := \langle e_i \rangle$ $(1 \le i \le n)$. If $\lambda_i - \rho_1 \in \mathbf{Z}$ or $\lambda_i - \rho_2 \in \mathbf{Z}$ for some $i \in \{1, \ldots, m\}$, we see easily that $\bigoplus_{j \neq i} V_j$ is an invariant subspace for $G_{\mathrm{II}^*}(\lambda, \mu, \nu, \rho)$. If $\lambda_i + \mu_j - \rho_1 - \rho_2 \in \mathbf{Z}$ for some $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, m-1\}$, $V_i \oplus V_{m+j}$ becomes an invariant subspace for $G_{\mathrm{II}^*}(\lambda, \mu, \nu, \rho)$. If $\rho_1 \in \mathbf{Z}$ or $\rho_2 \in \mathbf{Z}$, we have $\det R = 0$ by Lemma 6, and in this case the 0-eigen space of R is an invariant subspace for $G_{\mathrm{II}^*}(\lambda, \mu, \nu, \rho)$. Thus in any case $G_{\mathrm{II}^*}(\lambda, \mu, \nu, \rho)$ is reducible.

Hence we have shown that $G_{\mathrm{II}^*}(\lambda, \mu, \nu, \rho)$ is reducible if and only if (1.46) holds, which completes the proof. Proposition 3 q.e.d.

Proof of Theorem 2. Combine Theorem 1 and Proposition 3 with the remark after Theorem 2 to show the theorem. Theorem 2 q.e.d.

§2. System (III). Let $t_1, t_2 \in \mathbf{C}$ be mutually distinct points. Let $n = 2m+1$ be an odd integer equal to or greater than 5. Take $\lambda = (\lambda_1, \dots, \lambda_{m+1}) \in \mathbf{C}^{m+1}$, $\mu = (\mu_1, \dots, \mu_m) \in \mathbf{C}^m$ and $\rho = (\rho_1, \rho_2, \rho_3) \in \mathbf{C}^3$ satisfying

$$\lambda_i \neq \lambda_j,\ \mu_i \neq \mu_j,\ \rho_i \neq \rho_j \quad (i \neq j), \tag{2.1}$$

and

$$\sum_{i=1}^{m+1} \lambda_i + \sum_{i=1}^{m} \mu_i = m\rho_1 + m\rho_2 + \rho_3. \tag{2.2}$$

The system $(\mathrm{III})_{\lambda,\mu,\rho}$ of rank n is the system of differential equations

$$(xI_n - T)\frac{dy}{dx} = Ay \tag{2.3}$$

with

$$T = \begin{pmatrix} t_1 I_{m+1} & \\ & t_2 I_m \end{pmatrix}, \quad A = \left(\begin{array}{ccc|ccc} \lambda_1 & & & & & \\ & \ddots & & & (\alpha_{ij}) & \\ & & \lambda_{m+1} & & & \\ \hline & & & \mu_1 & & \\ & (\beta_{ij}) & & & \ddots & \\ & & & & & \mu_{m-1} \end{array}\right),$$

where

$$\alpha_{ij} = (\lambda_i - \rho_1)(\lambda_i - \rho_2) \prod_{\substack{1 \leq k \leq m+1 \\ k \neq i}} \left(\frac{\lambda_k + \mu_j - \rho_1 - \rho_2}{\lambda_i - \lambda_k} \right)$$

$$(1 \leq i \leq m+1,\ 1 \leq j \leq m),$$

$$\beta_{ij} = \prod_{\substack{1 \leq \ell \leq m \\ \ell \neq i}} \left(\frac{\lambda_j + \mu_\ell - \rho_1 - \rho_2}{\mu_i - \mu_\ell} \right) \quad (1 \leq i \leq m,\ 1 \leq j \leq m+1).$$

Our result is the following.

Theorem 7. *We assume*

$$\begin{cases} \rho_k \notin \mathbf{Z}_{<0}, & \rho_k - \rho_\ell \notin \mathbf{Z} \quad (1 \le k, \ell \le 3,\ k \ne \ell), \\ \lambda_i \notin \mathbf{Z}, & \lambda_i - \lambda_j \notin \mathbf{Z} \quad (1 \le i, j \le m+1,\ i \ne j), \\ \mu_i \notin \mathbf{Z}, & \mu_i - \mu_j \notin \mathbf{Z} \quad (1 \le i, j \le m,\ i \ne j), \end{cases} \tag{2.4}$$

and

$$\begin{cases} \lambda_i - \rho_k \notin \mathbf{Z} \quad (1 \le i \le m+1,\ k = 1, 2), \\ \lambda_i + \mu_j - (\rho_1 + \rho_2) \notin \mathbf{Z} \quad (1 \le i \le m+1,\ 1 \le j \le m). \end{cases} \tag{2.5}$$

If moreover

$$\rho_1 \notin \mathbf{Z},\ \rho_2 \notin \mathbf{Z},\ \rho_3 \notin \mathbf{Z}, \tag{2.6}$$

then the system $(\mathrm{III})_{\lambda,\mu,\rho}$ *is irreducible.*

The proof is similar to that of Theorem 2, and is omitted.

Remark. In Theorem 3 in our previous paper [H2], which is used for the proof of the above theorem, we have made too much assumptions (the assumption (2.3) in [H2]). Please replace it by the assumption (2.5) in the above theorem.

§3. System (III*). Let $t_1, t_2, t_3 \in \mathbf{C}$ be mutually distinct points which do not lie on a line. Let $n = 2m + 1$ be an odd integer equal to or greater than 5. Take $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbf{C}^m$, $\mu = (\mu_1, \ldots, \mu_m) \in \mathbf{C}^m$, $\nu \in \mathbf{C}$ and $\rho = (\rho_1, \rho_2) \in \mathbf{C}^2$ satisfying

$$\lambda_i \ne \lambda_j,\ \mu_i \ne \mu_j,\ \rho_i \ne \rho_j \quad (i \ne j), \tag{3.1}$$

and

$$\sum_{i=1}^{m} \lambda_i + \sum_{i=1}^{m} \mu_i + \nu = (m+1)\rho_1 + m\rho_2. \tag{3.2}$$

The system $(\mathrm{III}^*)_{\lambda,\mu,\nu,\rho}$ of rank n is the system of differential equations

$$(xI_n - T)\frac{dy}{dx} = Ay \tag{3.3}$$

with

$$T=\begin{pmatrix} t_1 I_m & & \\ & t_2 & \\ & & t_3 I_m \end{pmatrix},\quad A=\begin{pmatrix} \lambda_1 & & & \gamma_1 & & & \\ & \ddots & & \vdots & & (\alpha_{ij}) & \\ & & \lambda_m & \gamma_m & & & \\ \sigma_1 & \cdots & \sigma_m & \nu & \tau_1 & \cdots & \tau_m \\ & & & \theta_1 & \mu_1 & & \\ & (\beta_{ij}) & & \vdots & & \ddots & \\ & & & \theta_m & & & \mu_{m-1} \end{pmatrix},$$

where

$$\alpha_{ij}=(\lambda_i-\rho_1)\prod_{\substack{1\le k\le m\\ k\ne i}}\left(\frac{\rho_1+\rho_2-\lambda_k-\mu_j}{\lambda_i-\lambda_k}\right)\quad(1\le i,j\le m),$$

$$\beta_{ij}=(\mu_i-\rho_1)\prod_{\substack{1\le \ell\le m\\ \ell\ne i}}\left(\frac{\rho_1+\rho_2-\lambda_j-\mu_\ell}{\mu_i-\mu_\ell}\right)\quad(1\le i,j\le m),$$

$$\gamma_i=\frac{\lambda_i-\rho_1}{\prod_{\substack{1\le k\le m\\ k\ne i}}(\lambda_k-\lambda_i)}\quad(1\le i\le m),$$

$$\theta_i=\frac{\mu_i-\rho_1}{\prod_{\substack{1\le \ell\le m\\ \ell\ne i}}(\mu_i-\mu_\ell)}\quad(1\le i\le m),$$

$$\sigma_j=\prod_{\ell=1}^{m}(\rho_1+\rho_2-\lambda_j-\mu_\ell)\quad(1\le j\le m),$$

$$\tau_j=-\prod_{k=1}^{m}(\lambda_k+\mu_j-\rho_1-\rho_2)\quad(1\le j\le m).$$

Our result is the following.

Theorem 8. *We assume*

$$\begin{cases} \rho_1,\rho_2\notin\mathbf{Z}_{<0}, \quad \rho_1-\rho_2\notin\mathbf{Z}, \\ \lambda_i\notin\mathbf{Z}, \quad \lambda_i-\lambda_j\notin\mathbf{Z}\quad(1\le i,j\le m,\ i\ne j), \\ \mu_i\notin\mathbf{Z}, \quad \mu_i-\mu_j\notin\mathbf{Z}\quad(1\le i,j\le m,\ i\ne j), \\ \nu\notin\mathbf{Z}, \end{cases}\tag{3.4}$$

and

$$\begin{cases} \lambda_i-\rho_1\notin\mathbf{Z}\quad(1\le i\le m), \\ \mu_i-\rho_1\notin\mathbf{Z}\quad(1\le i\le m), \\ \lambda_i+\mu_j-(\rho_1+\rho_2)\notin\mathbf{Z}\quad(1\le i,j\le m). \end{cases}\tag{3.5}$$

If moreover

$$\rho_1 \notin \mathbf{Z},\ \rho_2 \notin \mathbf{Z}, \tag{3.6}$$

then the system $(\mathrm{III}^*)_{\lambda,\mu,\nu,\rho}$ *is irreducible.*

The proof is similar to that of Theorem 2, and is omitted.

§4. System (IV). Let $t_1, t_2 \in \mathbf{C}$ be mutually distinct points. Take $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4) \in \mathbf{C}^4$, $\mu = (\mu_1, \mu_2) \in \mathbf{C}^2$ and $\rho = (\rho_1, \rho_2, \rho_3) \in \mathbf{C}^3$ satisfying

$$\lambda_i \neq \lambda_j,\ \mu_i \neq \mu_j,\ \rho_i \neq \rho_j \quad (i \neq j), \tag{4.1}$$

and

$$\sum_{i=1}^{4} \lambda_i + \sum_{i=1}^{2} \mu_i = 2\rho_1 + 2\rho_2 + 2\rho_3. \tag{4.2}$$

The system $(\mathrm{IV})_{\lambda,\mu,\rho}$ is the system of differential equations

$$(xI_6 - T)\frac{dy}{dx} = Ay \tag{4.3}$$

of rank 6 with

$$T = \begin{pmatrix} t_1 I_4 & \\ & t_2 I_2 \end{pmatrix},\quad A = \begin{pmatrix} \lambda_1 & & & & \alpha_{11} & \alpha_{12} \\ & \lambda_2 & & & \alpha_{21} & \alpha_{22} \\ & & \lambda_3 & & \alpha_{31} & \alpha_{32} \\ & & & \lambda_4 & \alpha_{41} & \alpha_{42} \\ \beta_{11} & \beta_{12} & \beta_{13} & \beta_{14} & \mu_1 & \\ \beta_{21} & \beta_{22} & \beta_{23} & \beta_{24} & & \mu_2 \end{pmatrix},$$

where

$$\alpha_{ij} = \frac{\prod_{\ell=1,2,3}(\lambda_i - \rho_\ell)}{\prod_{\substack{1\le k\le 4 \\ k\neq i}}(\lambda_i - \lambda_k)} \cdot a_{ij} \quad (1 \le i \le 4,\ j = 1, 2),$$

$$\beta_{ij} = \frac{1}{\mu_i - \mu_{i'}} \cdot b_{ij} \quad (i = 1, 2,\ 1 \le j \le 4,\ \{i, i'\} = \{1, 2\}),$$

$$a_{11} = \prod_{k=2}^{4}(\lambda_1 + \lambda_k + \mu_2 - \rho_1 - \rho_2 - \rho_3),$$

$$a_{12} = \prod_{k=2}^{4}(\lambda_1 + \lambda_k + \mu_1 - \rho_1 - \rho_2 - \rho_3),$$

$$a_{ij} = \lambda_1 + \lambda_i + \mu_{j'} - \rho_1 - \rho_2 - \rho_3 \quad (i = 2, 3, 4,\ j = 1, 2,\ \{j, j'\} = \{1, 2\}),$$

$$b_{11} = b_{21} = 1,$$

$$b_{ij} = \prod_{\substack{k=2,3,4 \\ k \neq j}} (\lambda_1 + \lambda_k + \mu_i - \rho_1 - \rho_2 - \rho_3) \quad (i = 1, 2,\ j = 2, 3, 4).$$

Our result is the following.

Theorem 9. *We assume*

$$\begin{cases} \rho_k \notin \mathbf{Z}_{<0}, \quad \rho_k - \rho_\ell \notin \mathbf{Z} \quad (k, \ell = 1, 2, 3,\ k \neq \ell), \\ \lambda_i \notin \mathbf{Z}, \quad \lambda_i - \lambda_j \notin \mathbf{Z} \quad (1 \leq i, j \leq 4,\ i \neq j), \\ \mu_1, \mu_2 \notin \mathbf{Z},\ \mu_1 - \mu_2 \notin \mathbf{Z}, \end{cases} \tag{4.4}$$

and

$$\begin{cases} \lambda_i - \rho_k \notin \mathbf{Z} \quad (1 \leq i \leq 4,\ k = 1, 2, 3), \\ \lambda_i + \lambda_j + \mu_k - (\rho_1 + \rho_2 + \rho_3) \notin \mathbf{Z} \quad (1 \leq i, j \leq 4,\ i \neq j,\ k = 1, 2). \end{cases} \tag{4.5}$$

If moreover

$$\rho_1 \notin \mathbf{Z},\ \rho_2 \notin \mathbf{Z},\ \rho_3 \notin \mathbf{Z}, \tag{4.6}$$

then the system $(\mathrm{IV})_{\lambda,\mu,\rho}$ *is irreducible.*

The proof is similar to that of Theorem 2, and is omitted.

§5. System (IV*). Let $t_1, t_2, t_3 \in \mathbf{C}$ be mutually distinct points which do not lie on a line. Take $\lambda = (\lambda_1, \lambda_2) \in \mathbf{C}^2$, $\mu = (\mu_1, \mu_2) \in \mathbf{C}^2$, $\nu = (\nu_1, \nu_2) \in \mathbf{C}^2$ and $\rho = (\rho_1, \rho_2) \in \mathbf{C}^2$ satisfying

$$\lambda_1 \neq \lambda_2,\ \mu_1 \neq \mu_2,\ \nu_1 \neq \nu_2,\ \rho_1 \neq \rho_2, \tag{5.1}$$

and

$$\lambda_1 + \lambda_2 + \mu_1 + \mu_2 + \nu_1 + \nu_2 = 4\rho_1 + 2\rho_2. \tag{5.2}$$

The system $(\mathrm{IV}^*)_{\lambda,\mu,\nu,\rho}$ is the system of differential equations

$$(xI_6 - T)\frac{dy}{dx} = Ay \tag{5.3}$$

of rank 6 with

$$T = \begin{pmatrix} t_1 I_2 & & \\ & t_2 I_2 & \\ & & t_3 I_2 \end{pmatrix}, \quad A = \begin{pmatrix} \lambda_1 & & \alpha_{13} & \alpha_{14} & \alpha_{15} & \alpha_{16} \\ & \lambda_2 & \alpha_{23} & \alpha_{24} & \alpha_{25} & \alpha_{26} \\ \beta_{11} & \beta_{12} & \mu_1 & & \beta_{15} & \beta_{16} \\ \beta_{21} & \beta_{22} & & \mu_2 & \beta_{25} & \beta_{26} \\ \gamma_{11} & \gamma_{12} & \gamma_{13} & \gamma_{14} & \nu_1 & \\ \gamma_{21} & \gamma_{22} & \gamma_{23} & \gamma_{24} & & \nu_2 \end{pmatrix},$$

where

$$\alpha_{ij} = \frac{\lambda_i - \rho_1}{\lambda_i - \lambda_{i'}} \cdot a_{ij} \quad (i = 1, 2,\ \{i, i'\} = \{1, 2\},\ j = 3, 4, 5, 6),$$
$$\beta_{ij} = \frac{\mu_i - \rho_1}{\mu_i - \mu_{i'}} \cdot b_{ij} \quad (i = 1, 2,\ \{i, i'\} = \{1, 2\},\ j = 1, 2, 5, 6),$$
$$\gamma_{ij} = \frac{\nu_i - \rho_1}{\nu_i - \nu_{i'}} \cdot c_{ij} \quad (i = 1, 2,\ \{i, i'\} = \{1, 2\},\ j = 1, 2, 3, 4),$$
$$a_{13} = \lambda_1 + \mu_2 + \nu_1 - 2\rho_1 - \rho_2, \quad a_{14} = \lambda_1 + \mu_1 + \nu_2 - 2\rho_1 - \rho_2,$$
$$a_{15} = \lambda_2 + \mu_2 + \nu_1 - 2\rho_1 - \rho_2, \quad a_{16} = \lambda_2 + \mu_1 + \nu_2 - 2\rho_1 - \rho_2,$$
$$a_{23} = \lambda_2 + \mu_2 + \nu_2 - 2\rho_1 - \rho_2, \quad a_{24} = \lambda_2 + \mu_1 + \nu_1 - 2\rho_1 - \rho_2,$$
$$a_{25} = \lambda_2 + \mu_2 + \nu_2 - 2\rho_1 - \rho_2, \quad a_{26} = \lambda_2 + \mu_1 + \nu_1 - 2\rho_1 - \rho_2,$$
$$b_{11} = \lambda_2 + \mu_1 + \nu_1 - 2\rho_1 - \rho_2, \quad b_{12} = \lambda_1 + \mu_1 + \nu_2 - 2\rho_1 - \rho_2,$$
$$b_{15} = \lambda_1 + \mu_1 + \nu_2 - 2\rho_1 - \rho_2, \quad b_{16} = \lambda_1 + \mu_2 + \nu_2 - 2\rho_1 - \rho_2,$$
$$b_{21} = \lambda_2 + \mu_2 + \nu_2 - 2\rho_1 - \rho_2, \quad b_{22} = \lambda_1 + \mu_2 + \nu_1 - 2\rho_1 - \rho_2,$$
$$b_{25} = \lambda_1 + \mu_1 + \nu_1 - 2\rho_1 - \rho_2, \quad b_{26} = \lambda_1 + \mu_2 + \nu_1 - 2\rho_1 - \rho_2,$$
$$c_{11} = \lambda_1 + \mu_2 + \nu_2 - 2\rho_1 - \rho_2, \quad c_{12} = \lambda_1 + \mu_2 + \nu_1 - 2\rho_1 - \rho_2,$$
$$c_{13} = \lambda_1 + \mu_2 + \nu_1 - 2\rho_1 - \rho_2, \quad c_{14} = \lambda_1 + \mu_2 + \nu_2 - 2\rho_1 - \rho_2,$$
$$c_{21} = \lambda_1 + \mu_1 + \nu_1 - 2\rho_1 - \rho_2, \quad c_{22} = \lambda_1 + \mu_1 + \nu_2 - 2\rho_1 - \rho_2,$$
$$c_{23} = \lambda_1 + \mu_1 + \nu_1 - 2\rho_1 - \rho_2, \quad c_{24} = \lambda_1 + \mu_1 + \nu_2 - 2\rho_1 - \rho_2.$$

Our result is the following.

Theorem 10. *We assume*

$$\begin{cases} \rho_1, \rho_2 \notin \mathbf{Z}_{<0}, & \rho_1 - \rho_2 \notin \mathbf{Z}, \\ \lambda_1, \lambda_2 \notin \mathbf{Z}, & \lambda_1 - \lambda_2 \notin \mathbf{Z}, \\ \mu_1, \mu_2 \notin \mathbf{Z}, & \mu_1 - \mu_2 \notin \mathbf{Z}, \\ \nu_1, \nu_2 \notin \mathbf{Z}, & \nu_1 - \nu_2 \notin \mathbf{Z}. \end{cases} \tag{5.4}$$

and

$$\begin{cases} \lambda_i - \rho_1 \notin \mathbf{Z} & (i = 1, 2), \\ \mu_i - \rho_1 \notin \mathbf{Z} & (i = 1, 2), \\ \nu_i - \rho_1 \notin \mathbf{Z} & (i = 1, 2), \\ \lambda_i + \mu_j + \nu_k - (2\rho_1 + \rho_2) \notin \mathbf{Z} & (i, j, k = 1, 2). \end{cases} \tag{5.5}$$

If moreover

$$\rho_1 \notin \mathbf{Z}, \ \rho_2 \notin \mathbf{Z}, \tag{5.6}$$

then the system $(\mathrm{IV}^*)_{\lambda,\mu,\nu,\rho}$ *is irreducible.*

The proof is similar to that of Theorem 2, and is omitted.

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