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ADMISSIBILITY OF PREDICTION REGIONS IN TWO-DIMENSIONAL NORMAL DISTRIBUTION

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1. Introduction

Suppose that X and Y are independently and identically distributed p-dimensional normal random vectors with mean θ and covariance matrix equal to the identity matrix I_p $(N_p(\theta, I_p))$. This paper deals with the problem of predicting $\mathbf{Y}$ by using a region based on the observed value of $\mathbf{X}$ which is called a prediction region.

A prediction region $S(X)$ is evaluated by its coverage probability $P_\theta\{Y \in S(X)\}$ and its volume with respect to Lebesgue measure μ. The larger its coverage probability and the smaller its volume are, the better the prediction region is. Given a prediction region $S(X)$, consider a function ϕ defined by

$$\phi(x, y) = \begin{cases} 1, & \text{if } y \in S(x), \\ 0, & \text{otherwise.} \end{cases}$$

Then it holds that

$$P_\theta\{Y \in S(X)\} = E_\theta\{\phi(X, Y)\}. \tag{1.1}$$

and

$$E_\theta\{\mu(S(X))\} = E_\theta\{\int \phi(X, y)dy\}, \tag{1.2}$$

where $\mu(S(X))$ denotes the volume of $S(X)$. Conversely, every function ϕ with $0 \le \phi(x, y) \le 1$ define a prediction procedure by which a randomized prediction region is constructed such that (1.1) and (1.2) are satisfied.In the sequel, prediction regions are randomized and identified with such a function ϕ.

Definition 1 *A prediction region ϕ is admissible if there exists no other prediction region ϕ' such that for all θ*

$$E_\theta\{\phi'(X, Y)\} \ge E_\theta\{\phi(X, Y)\} \tag{1.3}$$

and

$$E_\theta\{\int \phi'(X,y)dy\} \le E_\theta\{\int \phi(X,y)dy\} \tag{1.4}$$

and the strict inequlity holds for at least one θ in (1.3) *or in* (1.4).

Definition 2 *A prediction region ϕ is minimax if*

$$\sup_\theta E_\theta\{\int \phi(X,y)dy\} \le \sup_\theta E_\theta\{\int \phi'(X,y)dy\}$$

for any prediction region ϕ' such that

$$\inf_\theta E_\theta\{\phi'(X,Y)\} \ge \inf_\theta E_\theta\{\phi(X,Y)\}.$$

The usual prediction region ϕ_0 is

$$S_0(x) = \{y; \quad |x-y| < h\}, \tag{1.5}$$

where $|x-y|$ denotes the Euclidian distance between x and y. It is easy to see that

$$E_\theta\{\int \phi_0(X,y)dy\} = \frac{\pi^{p/2}h^p}{\Gamma(p/2+1)} = v \quad \text{(say)} \tag{1.6}$$

and

$$E_\theta\{\phi_0(X,Y)\} = \int_0^{h^2/2} \frac{t^{p/2-1}e^{-t/2}}{\Gamma(p/2)2^{p/2}}dt = 1-\alpha \quad \text{(say)}. \tag{1.7}$$

From Theorem 2 of Takada [4] it turns out that ϕ_0 is the best invariant prediction region, that is, ϕ_0 uniformly minimizes (1.2) among the class of prediction regions such that

$$\phi(x+a, y+a) = \phi(x,y) \qquad \text{for any } x,\ y \text{ and } a$$

and the coverage probabilities are not less than a specified value. In Section 2 we shall prove that ϕ_0 is minimax. We [5] proved the admissibility of ϕ_0 for $p=1$. In Section 3 we shall prove the result for $p=2$ by using the method of Joshi [3] to prove the admissibility of confidence regions. For $p \ge 3$ we conjecture that ϕ_0 is not admissible (cf. Joshi [2], Hwang and Casella [1]), but the result has not been proved yet.

2. Minimax Prediction Region

For any prediction region ϕ let

$$L_\phi(x,y) = b\phi(x,\cdot) - \phi(x,y),$$

where $\phi(x,\cdot) = \int \phi(x,y)dy$ and $b = (4\pi)^{-p/2}exp(-h^2/4)$. Then

$$E_\theta\{L_\phi(X,Y)\} = bE_\theta\{\int \phi(X,y)dy\} - E_\theta\{\phi(X,Y)\}. \tag{2.1}$$

From (1.6) and (1.7) it follows that

$$E_\theta\{L_{\phi_0}(X,Y)\} = bv - (1-\alpha). \tag{2.2}$$

Suppose that a prior distribution ξ of θ is $N_p(0,\tau I_p)$ and let

$$R(\tau,\phi) = \int \{E_\theta L_\phi(X,Y)\}\xi(d\theta).$$

Then it follows that

$$R(\tau,\phi) = \int f_\tau(x)\{\int (b - f_\tau(y|x))\phi(x,y)dy\}dx, \tag{2.3}$$

where $f_\tau(x)$ is the marginal density of X and $f_\tau(y|x)$ is the conditional density of Y given X=x. It is easy to see that the conditional distribution of Y given X=x is $N_p(\mu(x),\rho I_p)$, where $\mu(x) = \tau x/(1+\tau)$ and $\rho = (2\tau+1)/(\tau+1)$. Hence $f_\tau(y|x) > b$ if and only if $|y-\mu(x)| < c$, where $c^2 = p(2/k)\log k + h^2/k$ and $k = 2/\rho$. So the prediction region ϕ_τ given by

$$S_\tau(x) = \{y; |y-\mu(x)| < c\}$$

minimizes $R(\tau,\phi)$ among all prediction regions and

$$R(\tau,\phi_\tau) = \frac{bc^p\pi^{p/2}}{\Gamma(p/2+1)} - \int_0^{c^2/\rho} \frac{t^{p/2-1}e^{-t/2}}{\Gamma(p/2)2^{p/2}}dt. \tag{2.4}$$

Theorem 1 *The usual prediction region ϕ_0 is minimax.*

Proof. From (2.2) and (2.4) it follows that

$$\lim_{\tau\to\infty} R(\tau,\phi_\tau) = bv - (1-\alpha). \tag{2.5}$$

Since

$$\sup_\theta E_\theta\{L_\phi(X,Y)\} \geq R(\tau,\phi),$$

from (2.5) we get

$$\sup_\theta E_\theta\{L_\phi(X,Y)\} \geq bv - (1-\alpha). \tag{2.6}$$

From the inequality that

$$\sup_\theta E_\theta\{L_\phi(X,Y)\} \leq b \sup_\theta E_\theta\{\int \phi(X,y)dy\} - \inf_\theta E_\theta\{\phi(X,Y)\}$$

and (2.6), it follows that if

$$\inf_\theta\{\phi(X,Y)\} \geq 1-\alpha,$$

then

$$\sup_\theta E_\theta\{\int \phi(X,y)dy\} \geq v,$$

which completes the proof.

3. Admissibility

In this section we shall prove the admissibility of the usual prediction region ϕ_0 of (1.5) for $p=2$. The method of the proof is almost the same as that of the proof of the admissibility of the usual confidence region given by Joshi [3].

From (2.4) we get

$$\begin{aligned} R(\tau,\phi_\tau) &= bc^2\pi - 1 + exp(-kc^2/4) \\ &= bk^{-1}v + 4\pi bk^{-1}\log k - 1 + k^{-1}\alpha. \end{aligned}$$

Since $k > 1$ and $1 - k^{-1} < (2\tau)^{-1}$, from (2.2)

$$R(\tau,\phi_0) - R(\tau,\phi_\tau) < (bv+\alpha)/(2\tau). \tag{3.1}$$

Theorem 2 *The usual prediction region ϕ_0 is admissible for $p=2$.*

Proof. Suppose that there exists a prediction region ϕ_1 such that for all θ

$$E_\theta\{\phi_1(X,Y)\} \geq 1-\alpha$$

and

$$E_\theta\{\int \phi_1(X,y)dy\} \le v.$$

Then from (2.1) we get

$$E_\theta\{L_{\phi_1}(x,Y)\} \le E_\theta\{L_{\phi_0}(X,Y)\}$$

for all θ, so that for any τ

$$R(\tau,\phi_1) \le R(\tau,\phi_0). \tag{3.2}$$

Let $f(x,y) = (4\pi)^{-1}exp\{-|x-y|^2/4\}$. Then it follows from (1.5) that

$$\phi_0(x,y) = \begin{cases} 1, & \text{if } f(x,y) > b, \\ 0, & \text{otherwise.} \end{cases} \tag{3.3}$$

Define two functions by

$$U_i(x) = b\phi_i(x,\cdot) - \int \phi_i(x,y)f(x,y)dy, \quad i = 0,1.$$

Then we get

$$U_1(x) - U_0(x) = \int (b - f(x,y))(\phi_1(x,y) - \phi_0(x,y))dy, \tag{3.4}$$

and hence from (3.3)

$$U_1(x) \ge U_0(x) \quad \text{for any } x \tag{3.5}$$

Let

$$M = \int (U_1(x) - U_0(x))dx.$$

Then $0 \le M \le \infty$. We shall prove that $M < \infty$.

Since the marginal distribution of X is $N_2(0,(1+\tau)I_2)$, from (2.3) we get

$$R(\tau,\phi_1) - R(\tau,\phi_0) = (2\pi(1+\tau))^{-1}\int G_\tau(x)dx, \tag{3.6}$$

where

$$\begin{aligned} G_\tau(x) &= exp\{-|x|^2/(2(1+\tau))\}\{[b\phi_1(x,\cdot) - \int \phi_1(x,y)f_\tau(y|x)dy] \\ &- [b\phi_0(x,\cdot) - \int \phi_0(x,y)f_\tau(y|x)dy]\} \end{aligned}$$

It is easy to see that

$$\lim_{\tau\to\infty} G_\tau(x) = U_1(x) - U_0(x)$$

and

$$\begin{aligned} |G_\tau(x)| &\le b\phi_1(x,\cdot) + 2 + bv \\ &= G(x) \quad \text{(say)}. \end{aligned}$$

Let $T_a = \{x; |x| \le a\}$. Then

$$\int_{T_a} G(x)dx \le (bv+2)\pi a^2 + b\int_{T_a} \phi_1(x,\cdot)dx$$

and

$$\begin{aligned} \int_{T_a} \phi_1(x,\cdot)dx &\le 2\pi e^{a^2/2} b \int_{T_a} \phi_1(x,\cdot)p_0(x)dx \\ &\le 2\pi e^{a^2/2} bv, \end{aligned}$$

where $p_0(x) = (2\pi)^{-1}exp(-|x|^2/2)$. So we get

$$\int_{T_a} G(x)dx < \infty.$$

By the dominated convergence theorem

$$\begin{aligned} \lim_{\tau\to\infty}\int_{T_a} G_\tau(x)dx &= \int_{T_a} (U_1(x) - U_0(x))dx \\ &= k_a \quad \text{(say)}, \end{aligned}$$

which implies that for any $\epsilon > 0$ there exsits τ_0 such that

$$\int_{T_a} G_\tau(x)dx \ge k_a - \epsilon \quad \text{for } \tau \ge \tau_0 \tag{3.7}$$

Since for any prediction region ϕ

$$b\phi(x,\cdot) - \int \phi(x,y)f_\tau(y|x)dy \ge b\phi_\tau(x,\cdot) - \int \phi_\tau(x,y)f_\tau(y|x)dy,$$

we get

$$(2\pi(1+\tau))^{-1}\int_{T_a^c} G_\tau(x)dx$$

$$\geq (2\pi(1+\tau))^{-1} \int_{T_a^c} exp(-|x|^2/2(1+\tau))\{[b\phi_\tau(x,\cdot)$$
$$- \int \phi_\tau(x,y) f_\tau(y|x)dy] - [b\phi_0(x,\cdot) - \int \phi_0(x,y) f_\tau(y|x)dy]\}dx$$
$$\geq R(\tau,\phi_\tau) - R(\tau,\phi_0)$$
$$\geq -(bv+\alpha)/(2\tau),$$

where the last inequality follows from (3.1). Hence from (3.6) and (3.7) for $\tau \geq \tau_0$

$$R(\tau,\phi_1) - R(\tau,\phi_0) \geq \frac{k_a - \epsilon}{2\pi(1+\tau)} - \frac{bv+\alpha}{2\tau},$$

so that from (3.2)

$$\frac{\pi(1+\tau)(bv+\alpha)}{\tau} + \epsilon \geq k_a,$$

and hence for any $a > 0$

$$k_a \leq 2\pi(bv+\alpha).$$

Therefore

$$\lim_{a\to\infty} \int_{T_a} (U_1(x) - U_0(x))dx = M < \infty.$$

It can be shown that $M = 0$, but the proof is tedious and so is omitted. See the section 6 of Joshi [3]. Hence from (3.5)

$$U_1(x) = U_0(x) \quad a.e..$$

It follows from (3.4) that

$$U_1(x) - U_0(x) = \int_{S_0(x)} (f(x,y) - b)(1 - \phi_1(x,y))dy + \int_{S_0(x)^c} (b - f(x,y))\phi_1(x,y)dy,$$

and hence from (3.3) for any x

$$\phi_1(x,y) = \phi_0(x,y) \quad a.e. \quad y.$$

Therefore by Fubini's theorem

$$\phi_1(x,y) = \phi_0(x,y) \quad a.e.,$$

which completes the proof.

References

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