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FIXED POINTS OF HOLOMORPHIC MAPPINGS OF A COMPLEX MANIFOLD

M. A. BASHIR and Aboubakr BAYOUMI

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Abstract: In this paper we consider the set of fixed points of an isometry mapping of a bounded Riemannian complex manifold M into itself. We prove that the set of all fixed points of f is a complex submanifold of M.

1. Introduction:

The importance of the study of fixed points can hardly be exaggerated. This is due to the wide applications of fixed point theorems to many branches of mathematics. The main use is the establishment of existence and uniqueness theorems by a certain fundamental fixed point principle as in nonlinear integral and differential equations. Recently the structure of fixed points on a complex manifold has been studied by several authors [2, 3]. For instance W. Kaup and J. P. Vigue considered among other things the set of all G-symmetric points of a hermitian manifold M [4]. They proved that such a set is a closed real-analytic submanifold of M.

In this paper we consider the structure of fixed points of isometry mappings of a bounded Riemannian complex manifold. Let M be a bounded Riemannian complex manifold and $f : M \to M$ be an isomentry. We define the set Fix f by

$$Fix\ f = \{p \in M : f(p) = p\}$$

We shall prove the following main theorem in section **1**.

Theorem (1.1): Let $f : M \to M$ be an isomentry. Suppose that the set Fix f is non-empty. Then the set Fix f is a submanifold of M.

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Let $B_{r(0)}$ be a ball of radius r in $\mathbb{C}^n$, and let $\bar{f} : B_{r(0)} \to B_{r(0)}$ be a holomorphic mapping, consider the mapping ϕ_n defined by

$$\Phi_n = \frac{1}{n}\sum_{p=0}^{n-1} \bar{f}^p$$

Since $B_{r(0)}$ is convex, ϕ_n is a holomorphic mapping from $B_{r(0)}$ to $B_{r(0)}$. Using Montel's theorem, we can obtain a subsequence ϕ_{n_k} converging to a holomorphic mapping ϕ uniformly on compact subsets of $B_{r(0)}$. As $B_{r(0)}$ is taught, ϕ is a holomorphic mapping from $B_{r(0)}$ to $B_{r(0)}$. We then have the following lemma due to Vigue[5].

Lemma (1.1): Let $a \in Fix\ \bar{f}$. The linear map $\phi'(a)$ is a projection onto $F = \{v \in \mathbb{C}^n : f'(a)v = v\}$. Let G be the kernel of $\phi'(a)$. Then $\bar{f}'(a)G \subset G$, and 1 is not an eigen-value of $\bar{f}'(a)|_G$.

Since the above lemma is fundamental to the proof of our theorem, we shall give a proof here for completion.

2. Proof of Lemma(1.1):

Suppose α is an eigenvalue of $\bar{f}'(a)$ of modulus 1. Put $\bar{f}'(a)$ in Jordan canonical form. Then we claim that the relative submatrix is diagonal. Because if not, then there is a Jordan block of the form

$$\begin{bmatrix} \alpha & 1 & \cdots & \cdots & 0 \\ 0 & \ddots & & & \vdots \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & \alpha \end{bmatrix}$$

Then $(\bar{f}'(a)^n$ has a corresponding block of the form

$$\begin{bmatrix} \alpha & n & & & 0 \\ 0 & \ddots & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & n \\ 0 & & & & \alpha \end{bmatrix}$$

and letting $n \to \infty$ we get a contradiction since $(\bar{f}'(a)^n$ should always have a converging subsequence: [cf. Wu]. This proves our assertion that the relative submatrix corresponding

to the eigenvalue α is in fact diagonal. Therefore the spectral space associated to α is equal to the corresponding eigenspace. This implies that if $\bar{f}'(a)$ is written in a Jordan form, then $\phi'(a)$ is a projection of $\mathbb{C}^n$ onto $F = \{v \in \mathbb{C}^n \mid \bar{f}'(a).\ v = v\}$.
Now let $z \in \mathbb{C}^n$ be in F then $\bar{f}'\phi'(z) = \phi'(z) = \phi'\bar{f}'(z)$.
i. e. $\bar{f}'(a)$ commutes with $\phi'(a)$. If w belongs to the kernel G of $\phi'(a)$, then

$$\phi'(a)\bar{f}'(a)(w) = \bar{f}'(a)\phi'(a)(w) = 0.$$

This implies that $\bar{f}'(a)(w)$ also belongs to G whenever w does. i. e. the kernel G of $\phi'(a)$ is stable under $\bar{f}'(a)$ $(\bar{f}'(a)G \subset G)$.

We know that the spectral space associated to α is equal to the corresponding eigenspace. This shows that 1 is not an eigenvalue of $\bar{f}'(a)|_G$ since $z \in G$ means $\bar{f}'(a) = \underline{0}$.

Now consider the sequence of iterates of ϕ:

$$\Psi_n = \phi^n$$

Using Lemma (1.1) and Cauchy's inequalities one can prove that [cf. 5] Ψ_n converges uniformly on compact subset of $B_{r(0)}$ to a holomorphic mapping Ψ which is a holomorphic retraction (i. e. $\Psi^2 = \Psi$) satisfying $\Psi \circ \bar{f} = \Psi$ and Fix $\bar{f} \subset Fix\ \Psi$.

These consequences will be utilized in the proof of the main theorem in section **3**.

3. Proof of the main theorem:

Let M be a bounded Riemannian complex manifold of complex dimension n and let $f : M \to M$ be an isometry mapping. Suppose that a is a fixed point of f. Take a coordinate system u, α around $a = f(a)$ in M such that $\alpha(a) = \alpha(f(a)) = 0$. The neighborhood u maybe taken to be a small ball centered at a. (If u is not originally a ball then a small ball centered at a inside u may be considered.) Now take a ball $B_{r(0)}$ of radius r in $\alpha(u) \equiv u \subset \mathbb{C}^n$. Then $B_{r(0)}$ is a bounded convex domain in $\mathbb{C}^n$. Since f is an isometry one can define the map $\bar{f} \equiv \alpha \circ f \circ \alpha^{-1}$ such that

$$\bar{f} : B_{r(0)} \to B_{r(0)}$$

with the property that $\bar{f}(0) = 0$. We shall first prove that the set Fix $\bar{f}$ is a complex submanifold of $B_{r(0)}$. The proof is as follows:

Consider the mapping Ψ which is constructed from $\bar{f}$ at the end of section **2**. Since this Ψ is a holomorphic retraction, by a theorem of H. Cartan [1] we can find a neighborhood K of the fixed point 0 and a local chart $k :\ K \to \bar{K}$ such that $k(0) = 0$ and Ψ is a linear projection P in this local chart k. Assume now we are in the local chart k. Let $F = Im\,P$, $G = ker\,P$.

We can use coordinates $z = (x, y)$ with $x \in F$ and $y \in G$. Now

$$\Psi \circ \bar{f} = \Psi, \qquad \Psi(x, y) = x$$

This implies that we can write

$$\bar{f}(x, y) = (x, \bar{f}_2(x, y)), \quad x \in F$$

By Lemma(1.1), 1 is not an eigenvalue of $\bar{f}'(0)|_G$ this means that 1 is not an eigenvalue of $\frac{\partial \bar{f}_2}{\partial y}(0, 0)$. This is true for every x belonging to a small neighborhood W of 0 in F. So $\det(\mathrm{id}|_{\bar{G}} - \frac{\partial \bar{f}_2}{\partial y}(x, 0)) \neq 0$

Now let $g = id|_{\bar{G}} - \bar{f}_2(x, .)$. Using the inverse function theorem, the image of g contains an open set in G.

For $x = 0$, $g(0) = -\bar{f}_2(0, 0) = 0$, i.e. 0 belongs to this image. The same is true for every x in a small neighborhood of 0 in G. Now

$$g^{-1}(0) = \{(x, 0) : I_G(0) - \bar{f}_2(x, 0) = 0\} = \{(x, 0) : \bar{f}_2(x, 0) = 0\}$$

is a submanifold of $B_{r(0)}$. For any $(x, 0)$ in this submanifold we have $\bar{f}(x, 0) = (x, 0)$, $x \in F$. i.e. it is a submanifold of fixed points of f. Then we use analytic continuation to extend the above situation to the whole of $B_{r(0)}$. Notice that for any $(x, y) \in B_{r(0)}$ we have $\Psi(x, y) = (x, 0)$ i.e. $\Psi(B_{r(0)}) = Fix\ \bar{f}$ which means that Fix $\bar{f}$ is a complex submanifold of $B_{r(0)}$.

Next we continue the proof of our theorem at the level of $f : M \to M$. So let $\bar{\Omega} = Fix\ \bar{f}$ and let $\Omega = \alpha^{-1}(\bar{\Omega}) \subset u$. Then we shall prove the following statements.

(i) Ω belongs to Fix f and $\Omega = \underset{a}{u}$ Fix f.

(ii) Ω, $\pi \circ \alpha|_\Omega$ is a coordinate system around $a \in Fix\ f$, where π is the projection

$$\pi(z^1, \ldots, z^m, z^{m+1}, \ldots, z^r) = (z^1, z^2, \ldots, z^m)$$

The statement (i) follows from the following argument:

Write $\alpha \circ f \circ \alpha^{-1}(\bar{\Omega}) = \bar{f}(\bar{\Omega}) = \bar{\Omega}$

This implies that

$$f \circ \bar{\alpha}^1(\bar{\Omega}) = \bar{\alpha}^1(\bar{\Omega}) \text{ i.e. } f(\Omega) = \Omega$$

so $\Omega \subset Fix\ f$. Since $\Omega \subset u$ we get $\Omega = \underset{a}{u}$ Fix f.

To prove statement (ii) let $p \in \Omega$. Then $p = \alpha^{-1}(\bar{z})$ for some $\bar{z}$ in $\bar{\Omega}$. Notice that since $\bar{\Omega} = Fix\ \bar{f}$ is a complex submanifold of $\mathbb{C}^n$, then one can write $\bar{z} = (z^1, z^2, \ldots, z^m, 0, \ldots, 0)$ for some integer $m < n$.

Thus $\pi \circ \alpha(p) = \pi(\bar{z}) = (z^1, z^2, \ldots, z^m)$ which proves (ii).

Now let a_i be an arbitrary point belonging to Fix f in M. Repeating exactly the above argument we have Ω_i, $\pi \circ \alpha_i|_\Omega$ a coordinate system around $a_i \in Fix\ f$. To show that Fix f is a submanifold of M of dim $m < n$ we need to show that $(\Omega_i,\ \pi \circ \alpha_i|_{\Omega_i})$ is an atlas for Fix f. With the help of (i) and (ii) we need only show that if a belongs to two overlapping coordinate neighborhoods Ω_i, $\pi \circ \alpha_i$ and Ω_j, $\pi \circ \alpha_j$, then the mapping

$(\pi \circ \alpha_i) \circ (\pi \circ \alpha_j)^{-1} : \pi \circ \alpha_j(\Omega_i \wedge \Omega_j) \to \pi \circ \alpha_i(\Omega_i \wedge \Omega_j)$ is analytic. But this is obvious since $(\pi \circ \alpha_i) \circ (\pi \circ \alpha_j)^{-1} = \pi \circ \alpha_i \circ \alpha_j^{-1} \circ \pi^{-1}$ is analytic by virtue of $\alpha_i \circ \alpha_j^{-1}$ being analytic as M is a complex manifold, and π, π^{-1} also analytic.

References:

1. H. Cartan, Sur les retraction d'une variete, C. R. Acad. Sci, Paris Ser, 1, Math. **303** (1986) 715–716.
2. C. Earle and R. Hamilton, A fixed point theorem for holomorphic mappings. Proc. Sympos. Pure Math. no **16**, Amer. Math. Soc., Providence, R. I, (1970) 61–65.
3. T. Franzoni and E. Vesentini, Holomorphic maps and invariant distances, Mathematical Studies, no. **40**, North-Holland, Amsterdam, 1980.
4. W. Kaup and J. P. Vigue, Symmetry and local conjugacy on complex manifolds, Math. Ann. **286**, (1990) 329–340.
5. J. P. Vigue, Fixed points of holomorphic mappings in a bounded convex domain in $\mathbb{C}^n$, Proc. Sympos in Pure Math, Vol. **52** (1991). Part 2, 579–582.

Department of Mathemactics, College of Science,
King Saud University, P. O. Box 2455, Riyadh
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