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On the ergodicity of compact abelian group extensions of states on C*-dynamics

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1. Introduction.

In [3], we obtained a characterization for the ergodicity of invariant states on C*-dynamics. In this note, as an application of the characterization, we discuss the ergodicity of compact abelian group extensions of invariant states on C*-dynamics. We give a necessary and sufficient condition for the compact abelian group extension of an invariant state on a C*-dynamics to be ergodic and its consequences.

Let A be a unital C*-algebra. If α is an automorphism of A, then the pair (A, α) is said to be a *C*-dynamics.* A C*-dynamics (B, β) is *conjugate* to (A, α) if there is an isomorphism Φ of A onto B such that $\alpha = \Phi^{-1} \circ \beta \circ \Phi$ on A. A state φ on A is said to be *invariant* if $\varphi \circ \alpha = \varphi$ on A. Let $(\pi_\varphi, \mathcal{H}_\varphi, \xi_\varphi)$ be the cyclic representation of A induced by φ. If φ is an invariant state on (A, α), then it induces a unitary representation u_φ (or simply u) on the Hilbert space $\mathcal{H}_\varphi$ such that $u_\varphi \pi_\varphi(x) u_\varphi^* = \pi_\varphi \circ \alpha(x)$ for every x in A and $u_\varphi \xi_\varphi = \xi_\varphi$. In fact, it is defined by $u_\varphi \pi_\varphi(x) \xi_\varphi = \pi_\varphi(\alpha(x)) \xi_\varphi, x \epsilon A$. A non-commutative version of ergodicity of an invariant state φ on the C*-dynamics (A, α) is that φ is an extreme point in the invariant states. We say φ to be *ergodic* in this case.

An automorphism α of A may be considered as an action α of the group $\mathbf{Z}$ of integers on A. If B is a C*-subalgebra of A which contains the identity of A and B is invariant under the action α, then the restriction β of α to B is an automorphism of B and hence (B, β) is a C*-dynamics. Moreover, if an invariant state φ on (A, α) is ergodic, then an invariant state ψ on (B, β) which is the restriction of φ to B is ergodic. In general, of course, the converse is false. Suppose that ψ is ergodic. What is the condition for φ to be ergodic? We consider this problem in the case that B is the fixed point C*-subalgebra of A under an action of a compact abelian group.

Let (A, α) be a C*-dynamics and let σ be a continuous action of a compact abelian group G on A such that $\sigma_g \circ \alpha = \alpha \circ \sigma_{\kappa(g)}$ for all g in G and some automorphism κ of G. Then α induces an automorphism $\alpha|_{A^\sigma}$ of the fixed point C*-subalgebra A^σ of A under the action

σ. If a C*-dynamics (B,β) is conjugate to the C*-dynamics $(A^\sigma, \alpha|_{A^\sigma})$, we say that (A,α) is a (G,σ)-extension of (B,β) under κ, or simply a G-extension of (B,β). If (A,α) is a G-extension of (B,β), then (B,β) is frequently identified with $(A^\sigma, \alpha|_{A^\sigma})$. Let Γ be the dual group of a compact abelian group G. An element γ of Γ is called *n-periodic* with respect to an automorphism κ of G if $\gamma\kappa \neq \gamma, \dots, \gamma\kappa^{n-1} \neq \gamma$ and $\gamma\kappa^n = \gamma (n \geq 1)$.

Now we give a necessary and sufficient condition for an invariant state φ on the G-extension (A,α) of a C*-dynamics to be ergodic, as a spectral condition.

2. The ergodicity of a state on the G-extension of a C*-dynamics.

As a special case of Theorem 1 ([3]), we have the following characterization for the ergodicity of invariant states on C*-dynamics (A,α).

THEOREM 1. *Let φ be an invariant state on C^*- dynamics (A,α). Then φ is ergodic if and only if*

$$\dim\{\eta \in \mathcal{H}'_\varphi : u_\varphi \eta = \eta\} = 1,$$

where $\mathcal{H}'_\varphi$ is the closed linear span $[\pi_\varphi(A)'\xi_\varphi]$ of $\pi_\varphi(A)'\xi_\varphi$.

As an application of Theorem 1, we prove the following theorem, which gives a characterization for an invariant state on the G-extension of a C*-dynamics to be ergodic.

THEOREM 2. *Let (A,α) be a (G,σ)-extension of a C^*-dynamics (B,β) under κ and let φ be an invariant state on A under α and σ. Suppose that the restriction ψ of φ to B is ergodic. Then φ is not ergodic if and only if there exist a positive integer n and a γ in Γ, n-periodic with respect to κ and not equal to 1, and a ξ_γ in $\mathcal{H}'_\varphi$, $\xi_\gamma \neq 0$ such that $u_\varphi^n \xi_\gamma = \xi_\gamma$ and $v_g \xi_\gamma = \langle g, \gamma \rangle \xi_\gamma$ for all g in G, where $\mathcal{H}'_\varphi$ is the closed linear span $[\pi_\varphi(A)'\xi_\varphi]$ of $\pi_\varphi(A)'\xi_\varphi$ and v is a unitary representation of G on $\mathcal{H}_\varphi$ defined by $v_g \pi_\varphi(x)\xi_\varphi = \pi_\varphi(\sigma_g(x))\xi_\varphi$ for x in A.*

To prove Theorem 2, we need the following lemma, as in [2].

LEMMA 3. *Let v be a unitary representation of a compact abelian group G on a Hilbert space $\mathcal{H}$ and u a unitary operator on $\mathcal{H}$ such that $v_g u = u v_{\kappa(g)}$ for all g in G and some automorphism κ of G. For γ in Γ, let $\mathcal{U}_\gamma$ be the set of all ξ in $\mathcal{H}$ such that $v_g \xi = \langle g, \gamma \rangle \xi$ for all g in G. Then we have*

(1)
$$\mathcal{H} = \sum_{\gamma \in \Gamma}^{\oplus} \mathcal{U}_\gamma$$

and

(2) *if* $\xi \in \mathcal{U}_\gamma$, *then* $u\xi \in \mathcal{U}_{\gamma\kappa}$.

PROOF. (1) For ξ in $\mathcal{U}_\gamma$ and ξ' in $\mathcal{U}_{\gamma'}$, we have

$$\begin{aligned}(\xi|\xi') &= (v_g\xi|v_g\xi') \\ &= \langle g,\gamma\rangle\overline{\langle g,\gamma'\rangle}(\xi|\xi'),\end{aligned}$$

for all g in G. If $\gamma \neq \gamma'$, $\langle g,\gamma\rangle\overline{\langle g,\gamma'\rangle} \neq 1$ for some g in G, and so ξ is orthogonal to ξ'. Suppose that a ξ in $\mathcal{H}$ is orthogonal to any vector in $\bigcup_{\gamma\in\Gamma}\mathcal{U}_\gamma$. Put $\xi_\gamma = \int_G \overline{\langle g,\gamma\rangle}v_g\xi \mathrm{d}g$ for each γ in Γ. Then ξ_γ belongs to $\mathcal{U}_\gamma$, and we have

$$\begin{aligned}(\xi_\gamma|\xi_\gamma) &= \int_G\int_G \overline{\langle gh^{-1},\gamma\rangle}(v_{gh^{-1}}\xi|\xi)\mathrm{d}g\mathrm{d}h \\ &= \int_G \overline{\langle g,\gamma\rangle}(v_g\xi|\xi)\mathrm{d}g \\ &= (\xi_\gamma|\xi) = 0.\end{aligned}$$

Hence $\xi_\gamma = 0$ for all γ in Γ. If we put $f(g) = (v_g\xi|\xi)$ for each g in G, the function f on G is integrable and positive definite. Hence we have

$$\begin{aligned}(v_g\xi|\xi) &= \int_\Gamma \langle g,\gamma\rangle\hat{f}(\gamma)\mathrm{d}\gamma \\ &= \int_\Gamma \langle g,\gamma\rangle\int_G \overline{\langle h,\gamma\rangle}(v_h\xi|\xi)\mathrm{d}h\mathrm{d}\gamma \\ &= \int_\Gamma \langle g,\gamma\rangle(\xi_\gamma|\xi)\mathrm{d}\gamma \\ &= 0\end{aligned}$$

for all g in G. Thus $(\xi|\xi) = 0$, and so $\xi = 0$. This implies the assertion (1).

(2) If ξ is in $\mathcal{U}_\gamma$, we have

$$\begin{aligned}v_g(u\xi) &= u(v_{\kappa(g)}\xi) \\ &= \langle\kappa(g),\gamma\rangle u\xi \\ &= \langle g,\gamma\kappa\rangle u\xi.\end{aligned}$$

This implies the assertion (2).

Thus the proof is complete.

PROOF OF THEOREM 2. Since (A,α) is a (G,σ)-extension of (B,β), we may identify (B,β) with $(A^\sigma,\alpha|_{A^\sigma})$. Let $(\pi_\psi,\mathcal{H}_\psi,\xi_\psi)$ be the cyclic representation of B induced by ψ, where $\psi = \varphi|_B$, and u_ψ be a unitary operator on $\mathcal{H}_\psi$ defined by $u_\psi \pi_\psi(y)\xi_\psi = \pi_\psi(\beta(y))\xi_\psi$ for all y in B. By the above identification of (B, β) and $(A^\sigma, \alpha|_{A^\sigma})$, we have $\mathcal{H}_\psi \subset \mathcal{H}_\varphi, \xi_\psi = \xi_\varphi$ and $u_\psi = u_\varphi|_{\mathcal{H}_\psi}$. Suppose that ξ_γ satisfies the conditions of Theorem 2. Put $\xi = \sum_{i=0}^{n-1} u_\varphi^i \xi_\gamma$. Then ξ belongs to $\sum_{i=0}^{n-1\oplus} \mathcal{U}_{\gamma\kappa^i}$ and $u_\varphi \xi = \xi$, where $\mathcal{U}_\gamma = \{\eta \in \mathcal{H}'_\varphi | v_g \eta =< g, \gamma > \eta \ (g \in G)\}$. Since ξ does not belong to $\mathbf{C}\xi_\varphi$, the dimension of $\{\xi \in \mathcal{H}'_\varphi : u_\varphi \xi = \xi\}$ is greater than 1. Hence, by Theorem 1, φ is not ergodic. Conversely, suppose that φ is not ergodic. Then, by Theorem 1, there exists a vector ξ in $\mathcal{H}'_\varphi$ not belonging to $\mathbf{C}\xi_\varphi$ such that $u_\varphi \xi = \xi$. By Lemma 3, we have the direct sum decomposition $\xi = \sum_{\gamma \in \Gamma} \xi_\gamma$ with ξ_γ in $\mathcal{U}_\gamma = \{\eta \in \mathcal{H}'_\varphi | v_g \eta = \langle g, \gamma \rangle \eta (g \in G)\}$, and also $u_\varphi \xi = \sum_{\gamma \in \Gamma} u_\varphi \xi_\gamma$ and $u_\varphi \xi_\gamma$ is in $\mathcal{U}_{\gamma\kappa}$. Since $u_\varphi \xi = \xi$, we have $u_\varphi \xi_\gamma = \xi_{\gamma\kappa}$ and $||\xi_\gamma||_\varphi = ||\xi_{\gamma\kappa}||_\varphi$ for all γ in Γ. From the orthogonality of ξ_γ's, we have $\xi_\gamma = 0$ if γ is not periodic with respect to κ. Now we note that for a vector η in $\mathcal{H}'_\varphi$, if η is u_φ-invariant and $\{v_g : g \in G\}$-invariant, then it belongs to $\mathcal{H}'_\psi$ and is u_ψ-invariant. If $\xi_\gamma = 0$ for all $\gamma \neq 1$, then $\xi = \xi_1$, and so ξ is $\{v_g : g \in G\}$-invariant and u_φ-invariant. Hence ξ belongs to $\mathcal{H}'_\psi$ and u_ψ-invariant, and thus ξ belongs to $\mathbf{C}\xi_\psi$ from the assumption of the ergodicity for ψ. Thus ξ belongs to $\mathbf{C}\xi_\varphi$ which contradicts the assumption of ξ. Therefore there exists a γ in $\Gamma, \gamma \neq 1$ such that $\xi_\gamma \neq 0$. From the above, this γ is n-periodic with respect to κ for some positive integer n, and then we have $u_\varphi^n \xi_\gamma = \xi_{\gamma\kappa^n} = \xi_\gamma$.

This completes the proof.

Also, in particular, if κ is the identity automorphism of G, then we have the following

COROLLARY 4. *Let $(A, \alpha), (B, \beta), \varphi, \psi, v_g, g \in G$ be as in Theorem 2. If κ is the identity automorphism of G, then φ is not ergodic if and only if there exist a γ in Γ, not equal to 1, and a ξ_γ in $\mathcal{H}'_\varphi$, $\xi_\gamma \neq 0$ such that $u_\varphi \xi_\gamma = \xi_\gamma$ and $v_g \xi_\gamma =< g, \gamma > \xi_\gamma$ for all g in G.*

COROLLARY 5. *Let σ be a continuous action of a compact abelian group G on a unital C^*-algebra A and φ be an extremal G-invariant state on A. Then φ is not an extremal state, i.e. not a pure state on A if and only if there exist a γ in Γ, not equal to 1, and a ξ_γ in $\mathcal{H}'_\varphi$, $\xi_\gamma \neq 0$ such that $v_g \xi_\gamma =< g, \gamma > \xi_\gamma$ for all g in G.*

Since it is clear that if ξ_φ is separating for $\pi_\varphi(A)''$, then ξ_ψ is separating for $\pi_\psi(B)''$, we have the following

COROLLARY 6. *Let $(A, \alpha), (B, \beta), \varphi, \psi, v_g, g \in G$ be as in Theorem 2. If ξ_φ is separating for $\pi_\varphi(A)''$, then φ is not ergodic if and only if there exist a positive integer n and a γ in Γ, n-periodic with respect to κ, and not equal to 1, and ξ_γ in $\mathcal{H}_\varphi$, $\xi_\gamma \neq 0$ such that $u_\varphi^n \xi_\gamma = \xi_\gamma$ and $v_g \xi_\gamma =< g, \gamma > \xi_\gamma$ for all g in G.*

The condition of Corollary 6 is rather fitting in the following property (*) of φ:

$$\dim\{\xi \in \mathcal{H}_\varphi : u_\varphi \xi = \xi\} = 1 \qquad (*)$$

It is known that the property (*) implies the ergodicity of φ, but, in general, the ergodicity of φ does not imply the property (*) ([1],[3], etc.). For the property (*) of the G-extension, we have the following

THEOREM 7. *Let $(A, \alpha), (B, \beta), \varphi, \psi, v_g, g \in G$ be as in Theorem 2. Suppose the restriction ψ of φ to B has the property (*). Then φ does not have the property (*) if and only if there exist a positive integer n and a γ in Γ, n-periodic with respect to κ and not equal to 1, and a ξ_γ in $\mathcal{H}_\varphi$, $\xi_\gamma \neq 0$ such that $u_\varphi^n \xi_\gamma = \xi_\gamma$ and $v_g \xi_\gamma = < g, \gamma > \xi_\gamma$ for all g in G.*

The proof of Theorem 7 is almost parallel to one of Theorem 2, so is omitted. Since if ξ_φ is separating for $\pi_\varphi(A)''$, then the ergodicity implies the property (*) ([1],[3], etc.), Corollary 6 follows again from Theorem 7.

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