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NORMALIZER OF MAXIMAL ABELIAN SUBGROUPS OF $GL(n)$ AND GENERAL HYPERGEOMETRIC FUNCTIONS.

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ABSTRACT. Symmetry of the general hypergeomeric functions of confluent type on the Grassmanian $G_{r,n}$ is discussed. This symmetry comes from the action of the normalizer of maximal abelian subgroups of $GL(n)$ on the functions. In particular, in the simplest case, the symmetry provides the well known transformation formulas for the hypergeometric function of Gauss and for the confluent hypergeometric function of Kummer.

1. INTRODUCTION

Inspired by the work of K. Aomoto ([A]) and I.M. Gel'fand et al. ([G], [GRS]) on general hypergeometric function, we introduced in [KHT1] the general hypergeometric function of confluent type (GHF of confluent type, for short). This was defined by extending the well known integral representations for Kummer's confluent hypergeometric function, Bessel function, Hermite-Weber function and Airy function; as for the definition GHF of confluent type, see Section 2.

The purpose of this paper is to give the group of symmetry for GHF of confluent type and to illustrate why hold some transformation formulas for the classical hypergeometric functions of confluent type.

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To present our problem concretely and to make our motivation explicit, we first consider the example of Gauss hypergeometric function. It is given by

$$\begin{aligned} F(a,b,c;x) &= \sum_{k\geq 0} \frac{(a)_k (b)_k}{(c)_k} \frac{x^k}{k!} \\ &= \frac{\Gamma(c)}{\Gamma(c-a)\Gamma(a)} \int_0^1 u^{a-1}(1-u)^{c-a-1}(1-xu)^{-b} du \end{aligned}$$

where $(a)_k = \Gamma(a+k)/\Gamma(a)$. The series converges in the unit disk $|x| < 1$ and defines a holomorphic function there. It satisfies the differential equation on $\mathbb{P}^1$ with singular points $x = 0, 1, \infty$:

$$x(1-x)\frac{d^2y}{dx^2} + \{c-(a+b+1)x\}\frac{dy}{dx} - aby = 0.$$

The equation is called *Gauss hypergeometric differential equation* (HGE, for short). By virtue of the integral representation or of the differential equation, $F(a,b,c;x)$ is extented to the multivalued analytic function in $\mathbb{P}^1 \setminus \{0,1,\infty\}$ by analytic continuation.

At each singular point, HGE has two linearly independent local solutions expressed in terms of $F(a,b,c;x)$. For example, at $x = 0$, we can take

$$F(a,b,c;x), \qquad x^{1-c}F(a-c+1, b-c+1, 2-c; x).$$

Each independent solution has 4 apparently different expressions. For example we have

$$\begin{aligned} F(a,b,c;x) &= (1-x)^{c-a-b}F(c-a, c-b, c; x) \\ &= (1-x)^{-a}F\left(c-b, a, c; \frac{x}{x-1}\right) \\ &= (1-x)^{-b}F\left(c-a, b, c; \frac{x}{x-1}\right) \end{aligned}$$

and similar expressions for $x^{1-c}F(a-c+1, b-c+1, 2-c;\ x)$.

Since the situation is the same at the singular points $x = 1, \infty$, there are 24 solutions in total, which are known as "Kummer's 24 solutions". For the other explicit expressions, refer to [IKSY]. The reason for the existence of such expressions can be explained from the group theoretic point of view when the integral representation for $F(a,b,c;x)$ is rewritten in a more symmetric manner as follows (cf. [G]).

Let $Z_{2,4}$ be the space of 2×4 complex matrices any of whose 2-minor does not vanish and an element $z \in Z_{2,4}$ is denoted as $z = (z_0, \dots, z_3) \in Z_{2,4}$ with column vectors z_i. Put

$$E = \{(t,z) \in \mathbb{P}^1 \times Z_{2,4} \mid tz_i \neq 0 \ \ (0 \leq i \leq 3)\},$$

where $t = (t_0, t_1)$ is the homogeneous coordinate of $\mathbb{P}^1$ and $tz_i := (t_0, t_1)z_i$ denotes a homogeneous linear function of t_0 and t_1 defined by the vector z_i. It is known ([H]) that the projection to the second factor $\pi : E \to Z_{2,4}$ defines a C^∞ fiber bundle and, in particular, a topologically locally trivial fibration. The fiber $E(z) := \pi^{-1}(z)$ is

$$E(z) = \mathbb{P}^1 \setminus \{z_0^*, \dots, z_3^*\},$$

where z_i^* denotes the point in $\mathbb{P}^1$ defined by the condition $tz_i = 0$. Take the parameter $\alpha = (\alpha_0, \dots, \alpha_3) \in \mathbb{C}^4$ satisfying the condition

$$\alpha_0 + \cdots + \alpha_3 = -2$$

and consider the multivalued 1-form on E:

$$\omega(t, z, \alpha) := \prod_{0 \le i \le 3} (tz_i)^{\alpha_i} (t_0 dt_1 - t_1 dt_0)$$

Note that the 1-form $\omega(t, z, \alpha)$ is invariant by the homothety $t \to ct, c \in \mathbb{C}^\times$, and therefore it really defines a multivalued 1-form on E. Let $\mathcal{L}$ be the 1-dimensional local system on E such that each branch of $\prod_{0 \le i \le 3}(tz_i)^{\alpha_i}$ determines a horizontal section of $\mathcal{L}$. Let $\mathcal{L}^\vee$ be the dual local system of $\mathcal{L}$ and consider the homology group $H_1(E(z), \mathcal{L}^\vee|_{E(z)})$ with coefficient in the local system $\mathcal{L}^\vee|_{E(z)}$. By virtue of the local triviality of the fibration $\pi : E \to Z_{2,4}$, we see that $\mathcal{H} := \cup_{z \in Z_{2,4}} H_1(E(z), \mathcal{L}^\vee|_{E(z)})$ is again a local system on $Z_{2,4}$ whose fibers are 2-dimensional vector spaces.

Take a horizontal section Δ of $\mathcal{H}$ with $\Delta(z) \in H_1(E(z), \mathcal{L}^\vee|_{E(z)})$, which is called a twisted cycle (see [Kit] for the details about twisted cycles). Then the general hypergeometric function due to Gel'fand is defined by the integral

$$\Phi(z, \alpha, \Delta) = \int_{\Delta(z)} \omega(t, z, \alpha).$$

In the following, we consider $\Phi(z, \alpha, \Delta)$ for some fixed twisted cycle Δ. Therefore, in order to avoid the cumbersome notation, we omit Δ in Φ and simply write $\Phi(z; \alpha)$.

The function $\Phi(z; \alpha)$ satisfies very simple but substantial properties. Let H be the Cartan subgroup of the complex general linear group $GL(4)$ consisting of diagonal matrices. It is easy to check that if $z \in Z_{2,4}$ we have $gzh \in Z_{2,4}$ for any $g \in GL(2)$ and $h \in H$. Hence we can consider the action of $GL(2) \times H$ on $Z_{2,4}$ by the left and right matrix multiplication:

$$\begin{array}{ccc} GL(2) \times Z_{2,4} \times H & \to & Z_{2,4} \\ (g, z, h) & \mapsto & gzh. \end{array}$$

This action can be lifted to the action of $\tilde{GL}(2) \times \tilde{H}$ on the universal covering space $\tilde{Z}_{2,4}$ of $Z_{2,4}$, where $\tilde{GL}(2)$ and $\tilde{H}$ are the universal covering group of $GL(2)$ and H, respectively. For this action $\Phi(z;\alpha)$ behaves as

$$\Phi(gz;\alpha) = (\det g)^{-1}\Phi(z;\alpha), \quad g \in \tilde{GL}(2)$$
$$\Phi(zh;\alpha) = \chi(h)\Phi(z;\alpha), \quad h \in \tilde{H} \tag{1.1}$$

where $\chi(h) := \prod_i h_i^{\alpha_i}$. By virtue of this property, we can relate the function $\Phi(z;\alpha)$ to Gauss hypergeometric function in the following manner. Put $M := GL(2)\backslash Z_{2,4}/H$. It is seen that M is an complex affine algebraic manifold biholomorphic to $\mathbb{P}^1 \setminus \{0,1,\infty\}$. This is established by finding a normal form of $z \in Z_{2,4}$ by the above group action as

$$z = gz'h, \quad z' = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & -x \end{pmatrix},$$

and the entry x of z' gives an affine coordinate of M. Then, by taking the twisted cycle $\Delta(z)$ given by the path in $E(z)$ connecting the point z_1^* to z_2^*, we have

$$\Phi(z;\alpha) = (\det g)^{-1}\chi(h)\Phi(z';\alpha),$$
$$\Phi(z';\alpha) = \frac{\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)}{\Gamma(\alpha_1+\alpha_2+2)}F(\alpha_1+1,-\alpha_3,\alpha_1+\alpha_2+2;x). \tag{1.2}$$

Now we are ready to explain the group theoretic aspect of Kummer's 24 solutions. Let $N_G(H)$ be the normalizer of the Cartan subgroup H in $GL(4)$. It is the semi-direct product

$$N_G(H) = H \rtimes W.$$

Here W is the subgroup of $N_G(H)$ consisting of all permutation matrices which is isomorphic to the symmetric group $\mathfrak{S}_4$ of 4 letters by the correspondence

$$P_\sigma = (\delta_{i\sigma(j)})_{0\le i,j<4} \leftrightarrow \quad \sigma \in \mathfrak{S}_4$$

and is identified with the Weyl group of $GL(4)$. In the followings of this section, we regard W as $\mathfrak{S}_4$ by the above identification. Define the action $\rho : W \to Aut(Z_{2,4})$ by

$$\rho(\sigma)z = zP_\sigma := (z_{\sigma(0)},\dots,z_{\sigma(3)}) \quad \text{for} \quad z = (z_0,\dots,z_3).$$

We see easily that the function $\Phi(z;\alpha)$ satisfies

$$\Phi(zP_\sigma;\alpha P_\sigma) = \Phi(z;\alpha) \quad \text{for} \quad \sigma \in W. \tag{1.3}$$

On the other hand, since $P_\sigma \in N_G(H)$, the above action induces a representation of W in the group of automorphisms of $M = GL(2)\backslash Z_{2,4}/H$:

$$\bar{\rho} : W \to Aut(M), \quad \bar{\rho}(\sigma)[z] = [zP_\sigma].$$

Interpretting the relations (1.3) as those for Gauss hypergeometric function using (1.2), we obtain $24(=\sharp W)$ solutions of Kummer.

The purpose of the present paper is to extend the above result about the Gauss hypergeometric function for the GHF of confluent type introduced in [KHT1].

This paper is organized as follows. In Section 2 we recall briefly the definition of GHF of confluent type. The GHF is defined as the "Radon transform" of a character for the universal covering group of the maximal abelian subgroup H_λ of $GL(n)$, which is parameterized by a partition λ of n. In Sections 3 and 4, we determine the structure of normalizer $N_{GL(n)}(H_\lambda)$ and $W := N_{GL(n)}(H_\lambda)/H_\lambda$. The latter group can be regarded as the analogue of Weyl group for $GL(n)$. In Section 5 we consider the action of the normalizer $N_G(H_\lambda)$ (and of the group W) on the affine manifolds $Z_{r,n}$ and $M = GL(r)\backslash Z_{r,n}/H_\lambda$ on which GHF of type λ is defined. Then we study the effect of the action on the GHF of type λ. In Section 6, we treat the hypergeometric system of Airy type, namely the system of differential equations for GHF of type $\lambda = (n)$. We give the transformation formula for the system under the action of the group W. The last section is devoted to establishing the connection between GHF on $Z_{2,4}, Z_{2,5}$ and the classical hypergeometric functions of one and two variables, respectively.

2. General hypergeometric function

Let n be a positive integer. There is given a partition $\lambda = (\lambda_0, \dots, \lambda_{l-1})$ of n, namely, a sequence of positive integers $\lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_{l-1}$ satisfying $|\lambda| = \lambda_0 + \cdots + \lambda_{l-1} = n$. We visualize λ by the figure called Young diagram as illustrated in the Figure 1. We sometimes say " Young diagram λ " instead of saying "partition λ".

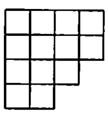

FIGURE 1. Young diagram for $\lambda = (4,4,3,2)$.

To such λ, we associate a maximal abelian subgroup of $GL(n) := GL(n, \mathbb{C})$:

$$H_\lambda = J(\lambda_0) \times \cdots \times J(\lambda_{l-1}) \subset GL(n),$$

where $J(\lambda_k)$ is the Jordan group of size λ_k defined by

$$J(\lambda_k) := \left\{ \sum_{0\le i<\lambda_k} h_i\Lambda^i \mid h_i \in \mathbb{C}, h_0 \neq 0 \right\},$$

where Λ is the shift matrix of size λ_k:

$$\Lambda = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}. \tag{2.1}$$

We use also the notation $[h_0, \dots, h_{\lambda_k-1}]$ to denote the element $\sum_{0\le i<\lambda_k} h_i\Lambda^i \in J(\lambda_k)$. Define the biholomorphic map

$$\iota : H_\lambda \to \prod_k (\mathbb{C}^\times \times \mathbb{C}^{\lambda_k-1}) \tag{2.2}$$

by associating $h = (h^{(0)}, \dots, h^{(l-1)})$, $h^{(k)} = [h_0^{(k)}, \dots, h_{\lambda_k-1}^{(k)}]$ with

$$\iota(h) = (h_0^{(0)}, \dots, h_{\lambda_0-1}^{(0)}, \dots, h_0^{(l-1)}, \dots, h_{\lambda_{l-1}-1}^{(l-1)}).$$

Let $\tilde{H}_\lambda$ be the universal convering group of H_λ:

$$\tilde{H}_\lambda = \tilde{J}(\lambda_0) \times \cdots \times \tilde{J}(\lambda_{l-1}).$$

Then the map ι can be lifted to the map $\tilde{H}_{\lambda_k} \to \prod_k (\tilde{\mathbb{C}}^\times \times \mathbb{C}^{\lambda_k-1})$, which will be also denoted by ι. We give the explicit form of the characters of $\tilde{H}_\lambda$, namely, the complex analytic homomorphisms from $\tilde{H}_\lambda$ to the complex torus $\mathbb{C}^\times$. To this end, we define the functions $\theta_i(v)$ $(i \ge 0)$ of $v = (v_0, v_1, v_2, \dots)$ by the generating function

$$\begin{aligned} \sum_{i=0}^{\infty} \theta_i(v) T^i &= \log(v_0 + v_1 T + v_2 T^2 + \cdots) \\ &= \log v_0 + \log\left(1 + \frac{v_1}{v_0}T + \frac{v_2}{v_0}T^2 + \cdots\right). \end{aligned}$$

Notice that $\theta_0(v) = \log v_0$ and $\theta_i(v)$ $(i \ge 1)$ is a weighted homogeneous polynomial of $v_1/v_0, \dots, v_i/v_0$ of total weight i when the weight of v_j is defined to be j.

Proposition 2.1 ([GRS]).

(1) Let $\chi_m : \tilde{J}(m) \to \mathbb{C}^\times$ be a character. Then, for some complex constants $\alpha = (\alpha_0, \dots, \alpha_{m-1}) \in \mathbb{C}^m$, we have

$$\chi_m(h) = \exp\left(\sum_{0\le i<m} \alpha_i\theta_i(\iota(h))\right) = h_0^{\alpha_0}\exp\left(\sum_{1\le i<m} \alpha_i\theta_i(\iota(h))\right),$$

where $h = [h_0, \dots, h_{m-1}]$. This character will be denoted by $\chi_m(\cdot\,;\alpha)$ to indicate the dependence on α.

(2) Let χ be a character of $\tilde{H}_\lambda$. Then there are complex constants $\alpha = (\alpha^{(0)}, \dots, \alpha^{(l-1)}) \in \mathbb{C}^n$, $\alpha^{(k)} \in \mathbb{C}^{\lambda_k}$ such that

$$\chi(h) = \prod_{0\le k<l} \chi_{\lambda_k}(h^{(k)};\alpha^{(k)}),$$

where $h = (h^{(0)}, \dots, h^{(l-1)}) \in \tilde{H}_\lambda$ and $h^{(k)} \in \tilde{J}(\lambda_k)$. The character χ with parameter α will be denoted by $\chi(\cdot\,;\alpha)$.

For the sake of simplicity, we write $\theta_i(h)$ instead of $\theta_i(\iota(h))$ by abuse of notation. To give the space on which the general hypergeometric funciton is defined, we introduce the following terminology.

Definition 2.2. Let $\lambda = (\lambda_0, \dots, \lambda_{l-1})$ be a *Young diagram* of weight n. A subdiagram of λ is a sequence of nonnegative integers $\lambda = (\mu_0, \dots, \mu_{l-1})$ such that

$$0 \le \mu_i \le \lambda_i, \quad 0 \le i < l$$

and is denoted as $\mu \subset \lambda$. The integer $|\mu| := \mu_0 + \cdots + \mu_{l-1}$ is called the *weight* of μ.

Let $M(r,n)$ be the set of $r \times n$ complex matrices. The set $M(r,r)$ will be denoted simply as $M(r)$. For an element $z \in M(r,n)$, we write

$$z = (z^{(0)}, \dots, z^{(l-1)}), \quad z^{(k)} = (z_0^{(k)}, \dots, z_{\lambda_k-1}^{(k)})$$

where $z_i^{(k)}$ are column vectors. For a subdiagram μ of λ of weight r, we put

$$z_\mu = (z_0^{(0)}, \dots, z_{\mu_0-1}^{(0)}, \dots, z_0^{(l-1)}, \dots, z_{\mu_{l-1}-1}^{(l-1)}) \in M(r).$$

Definition 2.3. The set

$$Z_{r,n} = \{z \in M(r,n) \mid \det z_\mu \neq 0 \text{ for any subdiagram } \mu \subset \lambda \text{ of weight } r\}$$

is called the *generic stratum* of $M(r,n)$ with respect to H_λ.

Note that the generic stratum $Z_{r,n}$ is a Zariski open subset of $M(r,n)$. Put

$$E = \{(t,z) \in \mathbb{P}^{r-1} \times Z_{r,n} \mid tz_0^{(k)} \neq 0 \ \ (0 \leq i < l)\},$$

where $t = (t_0, \dots, t_{r-1})$ is the homogeneous coordinate of $\mathbb{P}^{r-1}$ and $tz_0^{(k)}$ denotes the homogeneous linear function of t defined by the column vector $z_0^{(k)}$. Let $\pi : E \to Z_{r,n}$ be the projection map defined by $\pi(t,z) = z$. Then we know that E is a C^∞ fiber bundle over $Z_{r,n}$ with the fiber

$$E(z) := \mathbb{P}^{r-1} \setminus \bigcup_k H^{(k)}(z),$$

where $H^{(k)}(z)$ is the hyperplane in $\mathbb{P}^{r-1}$ defined by the equation $tz_0^{(k)} = 0$. Note that these hyperplanes are in general position by virtue of the definition of $Z_{r,n}$.

Let $\alpha = (\alpha^{(0)}, \dots, \alpha^{(l-1)}) \in \mathbb{C}^n$, $\alpha^{(k)} \in \mathbb{C}^{\lambda_k}$ be complex constants satisfying

$$\alpha_0^{(0)} + \cdots + \alpha_0^{(\ell-1)} = -r. \tag{2.3}$$

And let $\chi(\cdot\,;\alpha)$ be the character of $\tilde{H}_\lambda$. Put

$$\omega(t,z,\alpha) := \chi(\iota^{-1}(tz);\alpha) \cdot \tau,$$

where

$$\tau = \sum_{0 \leq i < n} (-1)^i t_i dt_0 \wedge \cdots \wedge \widehat{dt_i} \wedge \cdots \wedge dt_{r-1},$$

is the $(r-1)$-form obtained by taking the inner product of Euler vector field and the r-form $dt_0 \wedge \cdots \wedge dt_{r-1}$, the symbol $\widehat{dt_i}$ implies that this term is omitted in the expression of τ. And ι is the map defined by (2.2). Note that by virtue of the condition (2.3), we see that the $(r-1)$-form $\omega(t,z,\alpha)$ is homogeneous of degree 0 with respect to t and hence it defines a multivalued $(r-1)$-form on E.

Now we turn to the description of chains over which the integral of $\omega(t,z,\alpha)$ is considered. Put

$$\chi(\iota^{-1}(tz);\alpha) = U_1(t,z)U_2(t,z),$$

where

$$U_1(t,z) = \prod_{0 \leq k < l} (tz_0^{(k)})^{\alpha_0^{(k)}},$$

$$U_2(t,z) = \prod_{0 \leq k < l} \exp\left(\sum_{1 \leq i < \lambda_k} \alpha_i^{(k)} \theta_i(tz^{(k)})\right).$$

Note that U_1 is an element of $\mathcal{O}_{E/Z_{r,n}}(-r)$ and U_2 is a single valued holomorphic function on E with the essential singularities along

$$\bigcup_{k,\lambda_k\geq 2} H^{(k)}, \quad H^{(k)} = \bigcup_{z\in Z_{r,n}} H^{(k)}(z).$$

Let $\mathcal{L}$ be the 1-dimensional local system on E such that each branch of U_1 determines a horizontal section of $\mathcal{L}$. We denote by $\mathcal{L}^\vee$ the dual local system of $\mathcal{L}$. Moreover let Ψ be the family of supports in E defined by the function U_2, namely, Ψ is the family of closed sets A of E such that the real part $\Re U_2(t,z)$ tends to $-\infty$ as the point $(t,z)\in A$ approaches to $\cup_{k,\lambda_k\geq 2} H^{(k)}$. The restriction of Ψ to the fiber $E(z)$ of the fiber bundle $\pi: E\to Z_{r,n}$ defines again a family of supports on $E(z)$, which will be also denoted by the symbol Ψ. Consider the homology group $H_{r-1}({}_\Psi E(z);\mathcal{L}^\vee|_{E(z)})$ with family of supports Ψ and with coefficients in the local system $\mathcal{L}^\vee$. Put

$$\mathcal{H} := \bigcup_{z\in Z_{r,n}} H^{\Psi}_{r-1}(E(z);\mathcal{L}^\vee|_{E(z)}).$$

A horizontal section Δ of $\mathcal{H}$ will be called the twisted cycle.

Definition 2.4. The function defined by the integral

$$\Phi(z,\alpha,\Delta) = \int_{\Delta(z)} \omega(t,z,\alpha)$$

will be called the *general hypergeometric function of type* λ (GHF of type λ, for short).

In the following, we consider $\Phi(z;\alpha)$ for some fixed twisted cycle Δ. Therefore, we omit Δ in Φ and simply write $\Phi(z;\alpha)$.

Remark 2.5. When $\lambda = (1,\dots,1)$, GHF of type λ coincides with the general hypergeometric function defined in [G].

3. Normalizer of abelian group H_λ

In the followings of this paper we use the different notation about the partition of n from that in the previous sections.

There is given a partition of n:

$$\lambda = (\underbrace{\lambda_1,\dots,\lambda_1}_{p_1 \text{ times}},\underbrace{\lambda_2,\dots,\lambda_2}_{p_2 \text{ times}},\dots,\underbrace{\lambda_s,\dots,\lambda_s}_{p_s \text{ times}}), \quad \lambda_i\neq\lambda_j,$$

namely, a sequence of positive integers $\lambda_1>\cdots>\lambda_s$ which appear $p_1,\dots,p_s$ times, respectively, and satisfy

$$p_1\lambda_1+\cdots+p_s\lambda_s = n.$$

We associate such λ with the abelian group H_λ defined by

$$H_\lambda = \prod_{1\le i\le s} H_i, \quad H_i = \prod_{p_i \text{ times}} J(\lambda_i),$$

where $J(\lambda_i)$ is the Jordan group of size λ_i. The group $H_i = \prod_{p_i \text{ times}} J(\lambda_i)$ is considered as a subgroup of $GL(p_i\lambda_i)$ by the correspondence

$$H_i \ni (X_1, \dots, X_{p_i}) \mapsto diag(X_1, \dots, X_{p_i}) \in GL(p_i\lambda_i),$$

where $X_j \in J(\lambda_i)$. Similarly, $H_\lambda = H_1 \times \cdots \times H_s$ is regarded as a subgroup of $GL(n)$ by the correspondence

$$H_1 \times \cdots \times H_s \ni (X_1, \dots, X_s) \mapsto diag(X_1, \dots, X_s) \in GL(n).$$

By this identification, H_λ is a maximal abelian subgroup of $GL(n)$. We write G and H instead of $GL(n)$ and H_λ, respectively, if there is no fear of confusion.

Theorem 3.1. *Let $N_G(H)$ be the normalizer of H in G.*

(1) The correspondence

$$\prod_{1\le i\le s} N_{GL(p_i\lambda_i)}(H_i) \ni (X_1, \dots, X_s) \mapsto diag(X_1, \dots, X_s) \in G$$

gives an isomorphism

$$N_G(H) \simeq \prod_{1\le i\le s} N_{GL(p_i\lambda_i)}(H_i).$$

(2) For any i, we have

$$N_{GL(p_i\lambda_i)}(H_i) \simeq \left(\prod_{p_i \text{ times}} N_{GL(\lambda_i)}(J(\lambda_i)) \right) \rtimes \mathfrak{S}_{p_i}.$$

Precisely, any $X \in N_{GL(p_i\lambda_i)}(H_i)$ is decomposed uniquely as

$$X = diag(X_1, \dots, X_{p_i}) \cdot P_\sigma,$$

where $X_j \in N_{GL(\lambda_i)}(J(\lambda_i))$ and $P_\sigma \in GL(p_i\lambda_i), \sigma \in \mathfrak{S}_{p_i}$, is the permutation matrix which has, when decomposed into blocks of square matrices of size λ_i, the (j,k)-block equal to $\delta_{j\sigma(k)}1_{\lambda_i}$. Let $S_i := \{P_\sigma \mid \sigma \in \mathfrak{S}_{p_i}\}$ be the subgroup of $GL(p_i\lambda_i)$ isomorphic to $\mathfrak{S}_{p_i}$. Then we have

$$N_{GL(p_i\lambda_i)}(H_i) = \left(\prod_{p_i \text{ times}} N_{GL(\lambda_i)}(J(\lambda_i)) \right) \rtimes S_i.$$

In the particular case, we have the following well known result:

Corollary 3.2. *When $\lambda = (1, \dots, 1)$, the group $H = H_\lambda$ is the Cartan subgroup of $GL(n)$ consisting of the diagonal matrices and its normalizer $N_G(H)$ is*

$$N_G(H) \simeq H \rtimes W,$$

where $W = N_G(H)/H \simeq \mathfrak{S}_n$ is the Weyl group of $GL(n)$.

Proof of Theorem 3.1. Let $X \in N_G(H)$. Decompose X into blocks as

$$X = (X_{ij})_{1\le i,j\le s}, \quad X_{ij} \in M(m_i\lambda_i, m_j\lambda_j)$$

according as the product structure $H = H_1 \times \cdots \times H_s$. Since $X \in N_G(H)$, for arbitrary chosen $A = diag(A_1, \dots, A_s) \in H$, we have $XAX^{-1} =: B \in H$. Put $B = (B_1, \dots, B_s)$, $B_i \in H_i$. The condition $XAX^{-1} = B$ is written as

$$X_{ij}A_j = B_iX_{ij}, \quad 1 \le i, j \le s. \tag{3.1}$$

We assert $X_{ij} = 0$ if $i \ne j$. Note that $A_i \in H_i = \prod_{p_i \text{ times}} J(\lambda_i)$ and its each component belonging to $J(\lambda_i)$ can be chosen arbitrary. Note also that $\lambda_1, \dots, \lambda_s$ are distinct positive integers. Since the structure of Jordan normal form is preserved by the conjugation $A \mapsto XAX^{-1}$, for generic A, the eigenvalues of A_i coincides with those of B_i counting their multiplicities. It follows from (3.1) that $X_{ij} = 0$ if $i \ne j$ and $X_{ii} \in N_{GL(p_i\lambda_i)}(H_i)$. This proves the assertion (1).

The assertion (2) is shown in the same way. Let $Y \in N_{GL(p_i\lambda_i)}(H_i)$. Set $p = p_i$ for simplicity. Decompose Y as $Y = (Y_{jk})_{1\le j,k\le p}$, where Y_{jk} is a $\lambda_i \times \lambda_i$-matrix. As above, for any $C = diag(C_1, \dots, C_p) \in H_i$, there exists $D = (D_1, \dots, D_p) \in H_i$ such that $YCY^{-1} = D$. Note that the eigenvalues $c_1, \dots, c_p$ of $C_1, \dots, C_p$ can be chosen so that they are all distinct. Since the eigenvalues are preserved by conjugation $C \mapsto YCY^{-1}$ counting their multiplicities, the eigenvalues $d_1, \dots, d_p$ of $D_1, \dots, D_p$ are obtained by rearranging $c_1, \dots, c_p$, say,

$$(d_1, \dots, d_p) = (c_{\sigma^{-1}(1)}, \dots, c_{\sigma^{-1}(p)})$$

for some $\sigma \in \mathfrak{S}_p$. We assert

$$Y_{jk} = 0 \quad \text{for} \quad j \ne \sigma(k)$$

and

$$Y_{\sigma(k)k} \in N_{GL(\lambda_i)}(J(\lambda_i)). \tag{3.2}$$

In fact, the relation $YCY^{-1} = D$ is written as

$$Y_{jk}C_k = D_jY_{jk}, \quad 1 \le j, k \le p; \tag{3.3}$$

when $j \neq \sigma(k)$, the eigenvalues of C_k and D_j are different and the equation (3.3) implies $Y_{jk} = 0$. Moreover the equation (3.3) for $j = \sigma(k)$ implies (3.2). Put $Y_{\sigma(k)} := Y_{\sigma(k)k}$ $(1 \leq k \leq p)$. Then we see that

$$Y = diag(Y_1, \ldots, Y_p) \cdot P_\sigma,$$

where P_σ denotes the permutation matrix in $GL(p\lambda_i)$ whose (j,k)-block is $\delta_{j\sigma(k)} \cdot 1_{\lambda_i}$. This establishes the assertion (2). □

4. NOMALIZER OF THE JORDAN GROUP

Put $G = GL(m)$ and let $J = J(m)$ be the Jordan group of size m. We use the notation $h = [h_0, h_1, \ldots, h_{m-1}]$ to denote the element $h = \sum_{0 \leq i < m} h_i \Lambda^i \in J$. In this section we investigate the structure of the normalizer $N_G(J)$ of the Jordan group.

Let $x = (x_1, x_2, \ldots)$ be the variables and consider the formal power series in T:

$$f(x,T) = \sum_{i \geq 1} x_i T^i.$$

Define the polynomials $\phi_{ij}(x)$ $(i,j \geq 0)$ in the variables x by

$$f(x,T)^i = \sum_{j \geq 0} \phi_{ij}(x) T^j, \quad i \geq 0. \tag{4.1}$$

We set, by definition, $f(x,T)^0 = 1$ and therefore,

$$\phi_{0,j}(x) = \begin{cases} 1 & \text{if } j = 0, \\ 0 & \text{if } j \neq 0. \end{cases}$$

The polynomials $\phi_{ij}(x)$ are given by

$$\phi_{ij}(x) = \sum_{\nu_1 + \cdots + \nu_i = j} x_{\nu_1} \cdots x_{\nu_i}, \tag{4.2}$$

where in the summation $\nu_1, \ldots, \nu_i$ run over the positive integers satisfying $\nu_1 + \cdots + \nu_i = j$. Define the weight of x_i to be equal to i. We see from the expression (4.2) that if the polynomial $\phi_{ij}(x)$ is nonzero, it is a weighted homogeneous polynomial in x of weight j. It follows that $\phi_{ij}(x)$ is a function of $x_1, \ldots, x_j$.

Remark 4.1. The expression (4.2) is written in terms of Young diagram as follows. Let $\mu = (\mu_0, \mu_1, \cdots)$ be the Young diagram. The length of μ is $\ell(\mu) = \max\{k \mid \mu_k \neq 0\}$. Rewrite μ as $\mu = (1^{m_1} 2^{m_2} 3^{m_3} \cdots k^{m_k} \cdots)$. Put

$$c(\mu) = \frac{\ell(\mu)!}{m_1! m_2! \cdots m_k! \cdots}$$

and

$$x^\mu = x_1^{m_1} x_2^{m_2} \cdots x_k^{m_k} \cdots .$$

Then the formula (4.2) is written as

$$\phi_{ij}(x) = \sum_\mu c(\mu) x^\mu,$$

where in the summation μ runs over the set of Young diagrams of weight j with the length $\ell(\mu) = i$.

Lemma 4.2.

(1) For $i > j$, we have $\phi_{ij}(x) = 0$.

(2) For arbitrary non-negative integers i_1 and i_2, we have

$$\phi_{i_1+i_2,j}(x) = \sum_{0 \le k \le j} \phi_{i_1,k}(x) \phi_{i_2,j-k}(x). \tag{4.3}$$

Proof. Easy. □

Proposition 4.3. *Let $g \in N_G(J)$. Then g is an upper triangular matrix and is written uniquely as*

$$g = hX, \tag{4.4}$$

where $h \in J$ and $X \in N_G(J)$ such that $X = (x_{ij})_{0 \le i,j < m}$ satisfies $(x_{00}, \dots, x_{0,m-1}) = (1, 0, \dots, 0)$.

Proof. Given a $g = (g_{ij}) \in N_G(J)$. For any $A \in J$, there exists $B \in J$ such that $gAg^{-1} = B$. Comparing the components of both sides of $gA = Bg$, we see that g is upper triangular. For the above g, we can take $h = [h_0, \dots, h_{m-1}] \in J$ such that $h^{-1}g \in N_G(J)$ is an upper triangular matrix whose first row is $(1, 0, \dots, 0)$. In fact, put $h^{-1} = [c_0, c_1, \dots, c_{m-1}]$ then the first row of $h^{-1}g$ is

$$(c_0 g_{00}, c_0 g_{01} + c_1 g_{11}, \dots, c_0 g_{0,m-1} + \cdots + c_{m-1} g_{m-1,m-1}),$$

and therefore $c_0, \dots, c_{m-1}$ can be determined inductively so that this vector equals $(1, 0, \dots, 0)$. Putting $X := h^{-1}g \in N_G(J)$, we get the decomposition (4.3).

The uniqueness of the decomposition is proved as follows. Suppose that g is decomposed in two ways as

$$g = h_1 X_1 = h_2 X_2.$$

Then $X_1 X_2^{-1} = h_1^{-1} h_2$; the right hand side belongs to J, whereas first row of the left hand side is $(1, 0, \dots, 0)$; it follows that the $h_1 h_2^{-1}$ is the identity matrix and that $h_1 = h_2, X_1 = X_2$. □

Proposition 4.4. *Let $X = (x_{ij}) \in N_G(J)$ satisfy $(x_{00}, \ldots, x_{0,m-1}) = (1, 0, \ldots, 0)$. Then*

$$X = (\phi_{ij}(x))_{0 \le i,j < m}$$

for $x = (x_{11}, x_{12}, \ldots, x_{1,m-1}) \in \mathbb{C}^{m-1}$, where $\phi_{ij}(x)$ are polynomials defined by (4.1).

Proof. Let $X \in N_G(J)$ satisfy the assumption. Then for any $A = [a_0, \ldots, a_{m-1}] \in J$, there exists $B = [b_0, \ldots, b_{m-1}] \in J$ such that

$$XA = BX. \tag{4.5}$$

Since the eigenvalues of A and B coincide, we have $a_0 = b_0$ and comparing the first rows of both sides of (4.5), we get

$$(a_0, \ldots, a_{m-1}) = (b_0, \ldots, b_{m-1})X. \tag{4.6}$$

We prove the proposition by induction on m. In case $m = 1$, the assertion trivially holds. Assuming that it is true for m, we prove it for $m + 1$. Suppose X, A and B in (4.5) is of size $m + 1$. We decompose these matrices into 4 blocks as

$$X = \begin{pmatrix} X_{11} & X_{12} \\ 0 & X_{22} \end{pmatrix}, \quad A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}, \quad B = \begin{pmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{pmatrix}$$

where X_{11}, X_{12} and X_{22} are matrix of size $m \times m, m \times 1$ and 1×1, respectively, and A, B are decomposed in the same way. The equation (4.5) breaks into two conditions:

$$X_{11}[0, a_1, \ldots, a_{m-1}] = [0, b_1, \ldots, b_{m-1}]X_{11} \tag{4.7}$$

and

$$X_{11}{}^t(a_m, \ldots, a_1) = [0, b_1, \ldots, b_{m-1}]X_{12} + {}^t(b_m, \ldots, b_1)X_{22}. \tag{4.8}$$

Since A is an arbitrary element of $J(m + 1)$, and therefore $a_1, \ldots, a_{m-1}$ can be chosen arbitrary, it follows from (4.7) that $X_{11} \in N_{GL(m)}(J(m))$, and hence, by virtue of the induction assumption, we have

$$X_{11} = (\phi_{ij}(x))_{0 \le i,j < m} \tag{4.9}$$

for some $x \in \mathbb{C}^{m-1}$. The relation (4.6), in which $m - 1$ is replaced by m, is written as

$$(a_m, \ldots, a_1) = (b_0, \ldots, b_m)(\vec{X}_m, \ldots, \vec{X}_1) \tag{4.10}$$

where $\vec{X}_i$ is the i-th column vector of X. Substituting (4.10) into the left hand side of (4.8), we obtain

$$X_{11}{}^t(\vec{X}_m,\dots,\vec{X}_1)^t(b_0,\dots,b_m) = [0,b_1,\dots,b_{m-1}]X_{12} + {}^t(b_m,\dots,b_1)X_{22}. \tag{4.11}$$

We want to show that (4.11) implies that the components $x_{1m},\dots,x_{mm}$ of X_{12} and X_{22} are determined as

$$x_{km} = \phi_{km}(x), \quad 1 \le k \le m$$

with $x = (x_{11},\dots,x_{1m})$. We note that the matrix $X_{11}\,{}^t(\vec{X}_m,\dots,\vec{X}_1)$ is written as

$$\begin{pmatrix} 0 & x_{11} & \dots & x_{1,m-2} & x_{1,m-1} & x_{1m} \\ 0 & \phi_{2m} & \dots & \phi_{m-1,m} & \phi_{mm} & 0 \\ 0 & \phi_{3m} & \dots & \phi_{mm} & 0 & \\ \vdots & \vdots & \iddots & \iddots & & \\ 0 & \phi_{mm} & 0 & & & \end{pmatrix}.$$

This can be seen by using the property (4.3) for ϕ_{ij} in Lemma 4.2 and (4.9). Then the equation (4.11) is equivalent to

$$\sum_{1\le k\le m-i} \phi_{i+k,m} b_k = \sum_{i+1\le k\le m-1} b_{k-i}x_{km} + b_{m-i}x_{mm}, \quad 1 \le i \le m-1. \tag{4.12}$$

Put $i = m-1$ in (4.12) and we get $x_{mm} = \phi_{mm}(x)$. Suppose that $x_{km} = \phi_{km}(x)$ for $i+1 \le k \le m$. Then we get $x_{im} = \phi_{im}(x)$ from (4.12). $\square$

Summing up Propositions 4.3 and 4.4, we obtain

Theorem 4.5. *Set* $W(m) := \{(\phi_{ij}(x))_{0\le i,j<m} \mid x \in \mathbb{C}^{m-1}, x_1 \ne 0\} \subset GL(m)$. *Then* $W(m)$ *is a connected subgroup of* $N_G(J)$, *and* $N_G(J)$ *is a semi-direct product of* $W(m)$ *and* J*:*

$$N_G(J) = J \rtimes W(m).$$

Proof. The result follows from the uniqueness of the decomposition of $g \in N_G(J)$ into factors $g = h \cdot X$ for $h \in J, X \in W(m)$ and Proposition 4.4. $\square$

Corollary 4.6. *For the Jordan group* J *of size* m, *we have*

(1) $\dim W(m) = m-1$,

(2) $\dim N_G(J) = 2m-1$.

Theorem 4.7. *Let λ be the partition of n and let H be the abelian group associated with the partition λ as given in Section 3. Set*

$$W = \prod_{1\leq i\leq s} \left(\prod_{p_i \text{times}} W(\lambda_i) \right) \rtimes S_i, \tag{4.13}$$

where S_i is the subgroup of $GL(p_i\lambda_i)$ consisting of permutation matrices P which, when decomposed into p^2 blocks of square matrices of size λ_i as $P = (P_{jk})_{1\leq j,k\leq p}$, each component satisfies $P_{jk} = 0$ or $P_{jk} = 1_{\lambda_i}$ and is therefore isomorphic to the symmetric group $\mathfrak{S}_{p_i}$. Then we have

$$N_G(H) = H \rtimes W.$$

Remark 4.8. Theorem 4.7 says that the group W defined by (4.13) is the analogue of the Weyl group of $GL(n)$.

Corollary 4.9. *Let λ be the partition of n as in Section 3. Then we have*
(1) $\dim W = \sum_{1\leq i\leq s} p_i(\lambda_i - 1)$,
(2) $\dim N_G(H) = n + \sum_{1\leq i\leq s} p_i(\lambda_i - 1)$.

5. Action of $N_G(H)$ on GHF of type λ

We consider in this section the action of an element of $N_G(H)$ on the general hypergeometric functions of type λ. We adopt the notations in Sections 3 and 4 for the partition λ of n and the retated subgroups of $G = GL(n)$. Let $Z = Z_{r,n}$ be the generic stratum of $M(r,n)$ with respect to the group H (see Definition 2.3) and let $Aut(Z)$ be the group of holomorphic automorphisms of Z. It is easily checked by virtue of Theorem 3.1 and Proposition 4.3 that we have $zg \in Z$ for $z \in Z$ and $g \in N_G(H)$. For $g \in N_G(H)$, define the map $\rho(g) \in Aut(Z)$ by

$$\rho(g)(z) = zg, \quad z \in Z.$$

Thus we have the anti-homomorphism

$$\rho : N_G(H) \to Aut(Z).$$

In particular we have the representation of $W \subset N_G(H)$ in the group $Aut(Z)$, which will be denoted also by ρ.

Put $M := GL(r)\backslash Z/H$. It can be seen that M is a complex affine manifold. Let $Aut(M)$ be the group of holomorphic automorphisms of M. Since we are considering the normalizer of H, the actions of $N_G(H)$ and of W on Z induces the actions of these groups on M:

$$\rho : N_G(H) \to Aut(M)$$

and

$$\rho : W \to Aut(M).$$

By Theorem 4.7, we see that

$$W = W_0 \rtimes S,$$

where

$$W_0 := \prod_{1\le i\le s} W(\lambda_i)^{p_i}, \quad S := \prod_{1\le i\le s} S_i$$

Note that W_0 is the identity component of the Lie group W and S is the finite subgroup of W isomorphic to $W/W_0 \simeq \prod_{1\le i\le s} \mathfrak{S}_{p_i}$ and $S_i \simeq \mathfrak{S}_{p_i}$.

Let $\chi(\cdot\,;\alpha)$ be the character of the universal covering group $\tilde{H}$ of H:

$$\chi(\cdot\,;\alpha) = \prod_{1\le i\le s}\prod_{1\le k\le p_i} \chi_i(\cdot\,;\alpha^{(i,k)}), \tag{5.1}$$

where $\alpha = (\alpha^{(1,1)},\dots,\alpha^{(1,p_1)},\dots,\alpha^{(s,1)},\dots,\alpha^{(s,p_s)}) \in \mathbb{C}^n$ and $\chi_i(\cdot\,;\alpha^{(i,k)})$ is the character of $\tilde{J}(\lambda_i)$ with the parameters $\alpha^{(i,k)} = (\alpha_0^{(i,k)},\dots,\alpha_{\lambda_i-1}^{(i,k)}) \in \mathbb{C}^{\lambda_i}$.

Theorem 5.1. *For $g \in W$,*

$$\chi(\iota^{-1}(\iota(h)g);\alpha) = \chi(h;\alpha^t g) \quad \text{for} \ \ h \in \tilde{H}, \tag{5.2}$$

where ι is the biholomorphic map given by (2.2). In particular, for $g \in S$, we have

$$\chi(\iota^{-1}(\iota(h)g);\alpha g) = \chi(h;\alpha) \quad \text{for} \ \ h \in \tilde{H}. \tag{5.3}$$

We immediately see the following result.

Corollary 5.2. *For $g \in W$,*

$$\chi(\iota^{-1}(tzg);\alpha) = \chi(\iota^{-1}(tz);\alpha^t g) \quad . \tag{5.4}$$

In particular, for $g \in S$, we have

$$\chi(\iota^{-1}(tzg);\alpha g) = \chi(\iota^{-1}(tz);\alpha). \tag{5.5}$$

Integrating the relations (5.4) and (5.5) on the same twisted cycle, we get the following:

Theorem 5.3. *Let $\Phi(z;\alpha)$ be the GHF of type λ. Then*
(1) for $g \in W$,

$$\Phi(zg;\alpha) = \Phi(z;\alpha^t g), \tag{5.6}$$

(2) for $g \in S$,

$$\Phi(zg;\alpha g)=\Phi(z;\alpha). \tag{5.7}$$

Following is a consequence of Theorem 5.3, (1).

Corollary 5.4. *We assume that the parameters $\alpha\in\mathbb{C}^n$ of $\Phi(z;\alpha)$ satisfy*

$$\alpha^{(i,k)}_{\lambda_i-1}\neq 0,\quad 1\le i\le s, 1\le k\le p_i.$$

Then there exists $g\in W_0$ such that the change of coordinates $z\mapsto z'=zg^{-1}$ transforms $\Phi(z;\alpha)$ into $\Phi(z';\alpha')$ with the parameters

$$\alpha'^{(i,k)}=(\alpha^{(i,k)}_0,0,\dots,0,1),\quad 1\le i\le s, 1\le k\le p_i.$$

Proof. Since W_0 acts on $\mathbb{C}^n$ blockwise as $\alpha^{(i,k)}\to\alpha^{(i,k)t}g^{(i,k)}, g^{(i,k)}\in W(\lambda_i)$, it is sufficient to prove the particular case $\lambda=(n)$. We show that, for any $\alpha=(\alpha_0,\dots,\alpha_{n-1})\in\mathbb{C}^n$ satisfying $\alpha_{n-1}\neq 0$, there exists $g\in W(n)$ such that

$$\alpha'=\alpha^t g=(\alpha_0,0,\dots,0,1).$$

Recall that $g\in W(n)$ has the form $g=(\phi_{ij}(x))_{1\le i,j<n}$ with some $x=(x_1,\dots,x_{n-1})\in\mathbb{C}^{n-1}$ such that $x_1\neq 0$. From Lemma 4.2, (1), we have

$$\begin{aligned}\alpha'_i=(\alpha^t g)_i&=\sum_{j=0}^{n-1}\alpha_j\phi_{ij}(x)\\&=\alpha_i\phi_{ii}(x)+\cdots+\alpha_{n-1}\phi_{i,n-1}(x).\end{aligned} \tag{5.8}$$

Consider (5.8) for $i=n-1$. Noting that $\phi_{n-1,n-1}=x_1^{n-1}$ and $\alpha_{n-1}\neq 1$, we can choose $x_1\neq 0$ so that $\alpha'_{n-1}=1$. Next we consider (5.8) for $i=n-2$. Note that, from (4.2), the terms $\phi_{n-2,n-2}$ and $\phi_{n-2,n-1}$ has the form

$$\phi_{n-2,n-2}=x_1^{n-2},\quad \phi_{n-2,n-1}=(n-2)x_1^{n-3}x_2.$$

Using the condition $\alpha_{n-1}\neq 0$, we can determine x_2 so that the right hand side of (5.8) becomes 0. Proceeding in inductive manner, we can choose $x_3,\dots,x_{n-1}$ so that $\alpha'_{n-3},\dots,\alpha'_1$ becomes all zero. Lastly from the condition (5.8) for $i=0$, we have $\alpha'_0=\alpha_0$ because $\phi_{0j}(x)=\delta_{0j}$ by definition. □

The rest of this section is devoted to the proof of Theorem 5.1. First we prove the theorem for $g\in W_0$. Since g acts on $\mathbb{C}^n$ block-componentwise, it is sufficient to prove the theorem for the particular case where $H=J(n)$ and χ is a character of $\tilde{J}(n)$.

Lemma 5.5. *Assume $H = J(n)$. Then the identity (5.2) holds for any $g \in W(n)$.*

Proof. Take $g \in W(n)$. For $h \in J(n)$, put

$$h' = \iota^{-1}(\iota(h)g).$$

Then we have

$$\begin{aligned}\log \chi(h';\alpha) &= \sum_{0\le i<n} \alpha_i\theta_i(h') \\ &= (\theta_0(h'),\dots,\theta_{n-1}(h'))\begin{pmatrix}\alpha_0\\ \vdots \\ \alpha_{n-1}\end{pmatrix}\end{aligned}$$

Set $\vec{\theta}(h) := (\theta_0(h),\dots,\theta_{n-1}(h))$ and show that the following identity holds.

$$\vec{\theta}(h') = \vec{\theta}(h)g. \tag{5.9}$$

In fact, by the definition of the function $\theta_i(v)$, we have

$$\begin{aligned}\exp\left(\sum_i \theta_i(h')T^i\right) &\equiv h'_0 + h'_1T + \cdots + h'_{n-1}T^{n-1} \quad \text{mod. } T^n \\ &= \iota(h)g\begin{pmatrix}1\\ T\\ \vdots \\ T^{n-1}\end{pmatrix}.\end{aligned} \tag{5.10}$$

Since $g \in W(n)$ has the form $g = (\phi_{ij}(x))_{1\le i,j<n}$ with some $x = (x_1,\dots,x_{n-1}) \in \mathbb{C}^{n-1}$, from the definition $\phi_{ij}(x)$, we have

$$g\begin{pmatrix}1\\ T\\ \vdots \\ T^{n-1}\end{pmatrix} \equiv \begin{pmatrix}1\\ f(x,T)\\ \vdots \\ f(x,T)^{n-1}\end{pmatrix} \quad \text{mod. } T^n,$$

where $f(x,T) = x_1T + \cdots + x_{n-1}T^{n-1}$. Therefore the right hand side of (5.10) equals $\exp\left(\sum_i \theta_i(h)f^i\right)$ modulo T^n. It follows that

$$\vec{\theta}(h')\begin{pmatrix}1\\ T\\ \vdots \\ T^{n-1}\end{pmatrix} \equiv \vec{\theta}(h)\begin{pmatrix}1\\ f\\ \vdots \\ f^{n-1}\end{pmatrix} \equiv \vec{\theta}(h)g\begin{pmatrix}1\\ T\\ \vdots \\ T^{n-1}\end{pmatrix} \quad \text{mod. } T^n.$$

Thus we have the identity (5.9). Now the identity (5.2) is immediate. In fact,

$$\log \chi(\iota^{-1}(\iota(h)g);\alpha) = \vec{\theta}(h')^t\alpha = \vec{\theta}(h)g^t\alpha = \vec{\theta}(h)^t(\alpha^t g) = \log \chi(h; \alpha^t g)$$

by virtue of (5.9). Exponentiating this identity, we get (5.2). □

Next we want to prove (5.2) for $g \in S$. Taking account of the structure of the group $S = \prod_{1\le i\le s} S_i$, to show the identity (5.2) for $g \in S$, it is enough to show it for each S_i. Therefore it will be sufficient to consider the case that the partition λ of n has the form $\lambda = (\underbrace{m,\dots,m}_{p \text{ times}})$, namely $mp = n$, and $S \simeq \mathfrak{S}_p$ is the subgroup of $GL(n)$ consisting of permutation matrices P which, when decomposed into p^2 blocks of square matrices of size m as $P = (P_{jk})_{1\le j,k\le p}$, each component satisfies $P_{jk} = 0$ or $P_{jk} = 1_m$.

Lemma 5.6. *The identity (5.2) holds for the case $H = \prod_{p \text{ times}} J(m) \subset GL(n)$ and $g \in S$.*

Proof. Note that $g \in S$ is decomposed into blocks according to the product strucure of H, and its (i,j)-block is $\delta_{i\sigma(j)} \cdot 1_p$. Then the assertion is immediate from ${}^tg = g^{-1}$. □

Now the proof of Theorem 5.1 is already completed since any elelment of W is a product of those of W_0 and S.

6. Action of $N_G(H)$ on general hypergeometric system

It is known ([KHT1]) that GHF of type λ satisfies a system of linear differential equations which we will call the *general hypergeometric system of type* λ (GHS of type λ, for short). In this section we investigate the action of $\rho(N_G(H)) \subset Aut(Z)$ on GHS. Taking account of the structure of $N_G(H)$ given in Theorems 4.5 and 4.7, we will restrict our consideration to the case where the GHS is associated with the Jordan group $J = J(n)$, which will be called the *Airy system* in this paper because it is essentially the same equation as the Airy equation $y'' - xy = 0$ when $(r,n) = (2,4)$. The Airy system is

$$E(\alpha): \quad \begin{cases} (L_m - \alpha_m)u = 0, & 0 \le m < n, \\ (M_{ij} + \delta_{ij})u = 0, & 0 \le i,j < r, \\ \qquad \Box_{ijpq}u = 0, & 0 \le i,j < r,\ 0 \le p,q < n, \end{cases}$$

where

$$\begin{aligned} L_m &:= \sum_{0\le i<r} \sum_{m\le p<n} z_{i,p-m}\partial_{ip} \\ M_{ij} &:= \sum_{0\le p<n} z_{ip}\partial_{jp} \\ \Box_{ijpq} &:= \partial_{ip}\partial_{jq} - \partial_{iq}\partial_{jp} \end{aligned}$$

and $\partial_{ij} = \partial/\partial z_{ij}$. Take $g \in N_G(J)$ and consider the change of coodinates $z \mapsto z' = zg$. Set $\partial'_{ij} = \partial/\partial z'_{ij}$ and

$$\partial = (\partial_{ij})_{0\le i,j<n}, \quad \partial' = (\partial'_{ij})_{0\le i,j<n}.$$

These operators are related as

$$\partial = \partial' \cdot {}^t g. \tag{6.1}$$

We denote by L'_m, M'_{ij} and $\Box'_{ijpq}$ the differential operators in the variables z' obtained by replacing z by z' and ∂_{ij} by ∂'_{ij} in the corresponding operators in z. Denote by $g_*E(\alpha)$ the system of differential equations in z' obtained from $E(\alpha)$ by the transformation $z \to z' = zg$. We write down explicitely how the system $E(\alpha)'$ is related to $g_*E(\alpha)$.

Lemma 6.1. *The operator L_m can be written as*

$$L_m = trace(z\Lambda^m \cdot {}^t\partial)$$

where Λ is the $n \times n$ matrix of the form (2.1).

Proof. We have

$$\begin{aligned} L_m &= \sum_{0\le i<r} (z_{i0}, \dots, z_{i,n-1-m})^t(\partial_{im}, \dots, \partial_{i,n-1}) \\ &= \sum_{0\le i<r} (\underbrace{0, \dots, 0}_{m \text{ times}}, z_{i0}, \dots, z_{i,n-1-m})^t(\partial_{i0}, \dots, \partial_{i,n-1}) \\ &= \sum_{0\le i<r} (z_{i0}, \dots, z_{i,n-1})\Lambda^{m\,t}(\partial_{i0}, \dots, \partial_{i,n-1}) \\ &= trace(z\Lambda^m \cdot {}^t\partial). \end{aligned}$$

□

Proposition 6.2. *Let $g \in W(n)$ which is given by $g^{-1} = (\phi_{i,j}(x))_{0\le i,j<n}$. Then the systems $E(\alpha)'$ and $g_*E(\alpha)$ are related as*

$$\begin{aligned} g_*L_m &= \sum_k \phi_{mk}(a)L'_k, \quad 0 \le m < n, \\ g_*M_{ij} &= M'_{ij}, \quad 0 \le i,j < r, \\ g_*\Box_{ijpq} &= \sum_{0\le k<l<n} \Delta_{pqkl}\Box'_{ijkl}, \quad 0 \le i,j < r,\ 0 \le p,q < n, \end{aligned}$$

where $a = (a_1, \dots, a_{n-1})$ is determined by the condition that $f(a,T) = \sum_{i\ge1} a_iT^i$ is the inverse function of $f(x,T)$ mod T^n, and

$$\Delta_{pqkl} = \phi_{pk}(a)\phi_{ql}(a) - \phi_{qk}(a)\phi_{pl}(a).$$

Proof. We have

$$g_* L_m = trace(z' g^{-1} \Lambda^m g^t \partial') = trace(z'(g^{-1}\Lambda g)^{m\,t}\partial')$$

Since $g \in N_G(J)$ and $\Lambda \in J$, we have $g^{-1}\Lambda g \in J$. Then there exists $a_1, \dots, a_{n-1}$ such that

$$g^{-1}\Lambda g = \sum_{1 \le i < n} a_i \Lambda^i. \tag{6.2}$$

Therefore we get

$$g_* L_m = \sum_k \phi_{mk}(a) L'_k.$$

We show that $f(a,T) = a_1 T + \cdots + a_{n-1}T^{n-1}$ is the inverse function of $f(x,T)$ mod T^n. Rewrite (6.2) as

$$g^{-1}\Lambda = (\sum_{1 \le i < n} a_i \Lambda^i) g^{-1}.$$

Multiplying the vector ${}^t(1, T, \dots, T^{n-1})$ to the both sides of the above relation from the right, we get

$$g^{-1}\Lambda {}^t(1, T, \dots, T^{n-1}) = (\sum_{1 \le i < n} a_i \Lambda^i) g^{-1t}(1, T, \dots, T^{n-1})$$

which is equivalent to

$${}^t(1, f, f^2, \dots, f^{n-1}) T \equiv (\sum_{1 \le i < n} a_i \Lambda^i) {}^t(1, f, f^2, \dots, f^{n-1}) \mod T^n.$$

Comparing the first entries of the both sides, we obtain

$$T \equiv \sum_{1 \le i < n} a_i f^i \quad mod\, T^n.$$

Hence we conclude that $f(a,T)$ is the inverse function of $f(x,T)$ mod. T^n. This show the first assertion. Next we cosider the operator $g_* M_{ij}$. We have

$$g_* M_{ij} = \sum_{0 \le p < n} z_{ip}\partial_{jp} = \sum_{0 \le p \le n} z_{ip} \sum_{0 \le l \le n} g_{pl}\partial'_{jl} = M'_{ij}.$$

The assertion for $g_* \Box_{ijpq}$ is easily proved using the relation (6.1). □

Remark 6.3. The third relation can be understood as follows. Define the lexicographic order on $\mathbb{Z} \times \mathbb{Z}$, namely, $(p,q) < (k,l)$ if and only if $p < k$, or $p = k$ and $q < l$. For fixed (i,j), define the column vector $\vec{\Box}_{ij} := (\Box_{ijpq})$ arraying the entry in the order so that the index (p,q)

increases. Then the matrix $\Delta = (\Delta_{pqkl})$ is interpreted as an upper triangular matrix of size $n(n-1)/2$ whose diagonal element is $\Delta_{pqpq} = \phi_{pp}(a)\phi_{qq}(a) \neq 0$ and

$$g_* \vec{\square}_{ij} = \Delta \cdot \vec{\square}'_{ij}.$$

7. Examples

In this section we study the general hypergeometric functions of type λ on $Z_{2,4}$ and on $Z_{2,5}$ for various partitions λ of 4 and 5, respectively. There are two things to be done. One is to give the relation between the classical special functions of hypergeometric type of one and two variables and the GHF on $Z_{2,4}$ and on $Z_{2,5}$, respectively. Another one is to make clear what the theorems in the preceeding sections imply for the classical special functions.

Put $M_{r,n} := GL(r)\backslash Z_{r,n}/H_\lambda$, where $Z_{r,n}$ is the generic stratum of $M(r,n)$ with respect to H_λ.

7.1. GHF on $M_{2,4}$.

The Young diagrams of weight 4 are listed as

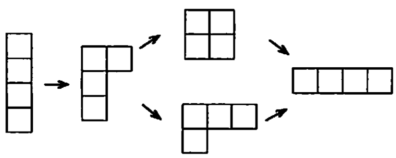

Figure 2. Young diagrams of weight 4.

The arrows between the diagrams indicate the process of confluence among the GHF of type λ for various λ, cf. [KHT2]. The parameters α of the GHF will be indexed as $\alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3)$. Let us list up the normal form of $z \in Z_{2,4}$ by the action of $GL(2) \times H_\lambda$ which provides an affine coordinate of $M_{2,4}$ and the normal form of the parameters α by the action of $W_0 = \prod_i W(\lambda_i)$ (see Corollary 5.4).

The relations of GHF on $Z_{2,4}$ with the classical special functions of one variable are given by

Proposition 7.1. *Let λ, z and α be as in Table 1. Then the general hypergeometric function*

TABLE 1

λ	Normal form of z	Normal form of α
$1,1,1,1$	$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & -x \end{pmatrix}$	$\alpha_0, \alpha_1, \alpha_2, \alpha_3$
$2,1,1$	$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & x & 1 & -1 \end{pmatrix}$	$\alpha_0, -1, \alpha_2, \alpha_3$
$2,2$	$\begin{pmatrix} 1 & 0 & 0 & -x \\ 0 & 1 & 1 & 0 \end{pmatrix}$	$\alpha_0, 1, \alpha_2, 1$
$3,1$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & x & 1 \end{pmatrix}$	$\alpha_0, 0, 1, \alpha_3$
4	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -x \end{pmatrix}$	$-2, 0, 0, 1$

$\Phi_\lambda(z;\alpha)$ on $M_{2,4}$ are expressed as

$$\begin{aligned}
\Phi_{(1,1,1,1)}(z;\alpha) &= \int_{\gamma_1} u^{\alpha_1}(1-u)^{\alpha_2}(1-x_1u)^{\alpha_3}du \\
&= \frac{\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)}{\Gamma(\alpha_1+\alpha_2+2)}\,{}_2F_1(\alpha_1+1,-\alpha_3,\alpha_1+\alpha_2+2;x)
\end{aligned}$$

(Gauss hypergeometric function) ,

$$\begin{aligned}
\Phi_{(2,1,1)}(z;\alpha) &= \int_{\gamma_2} e^{xu}u^{\alpha_2}(1-u)^{\alpha_3}du \\
&= \frac{\Gamma(\alpha_2+1)\Gamma(\alpha_3+1)}{\Gamma(\alpha_2+\alpha_3+2)}\,{}_1F_1(\alpha_2+1,\alpha_2+\alpha_3+2;x)
\end{aligned}$$

(Kummer's confluent hypergeometric function),

$$\Phi_{(2,2)}(z;\alpha) = \int_{\gamma_3} e^{u-x/u}u^{\alpha_2}du = 2\pi i(-1)^{\alpha_2+1}e^{\pi i\alpha_2}x^{\frac{\alpha_2+1}{2}}J_{-\alpha_2-1}(2\sqrt{x})$$

(Bessel function),

$$\Phi_{(3,1)}(z;\alpha) = \int_{\gamma_4} e^{-u^2/2+xu}u^{\alpha_3}du = \frac{2\pi i}{\Gamma(-\alpha_3)}H_{-\alpha_3-1}(x)$$

(generalized Hermite function),

$$\Phi_{(4)}(z;\alpha) = \int_{\gamma_5} e^{u^3/3-xu}du = -2\pi i Ai(x)$$

(Airy function),

where the paths γ_i of integration are given in Figure 3.

Theorem 5.3 explains the group theoretic aspect of the transformation formulas for classical special functions (See also Introduction). In fact, we can show the following

Proposition 7.2. *For $\lambda = (2,1,1)$, the group $S \subset W$ is isomorphic to $\mathfrak{S}_2$ with the generator*

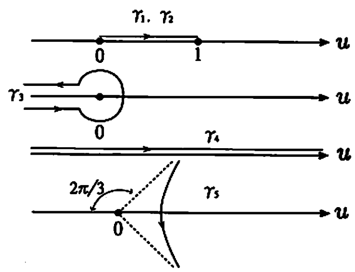

FIGURE 3. Path of integration γ_i, $i = 1, \dots, 5$.

$g = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & & 1 \\ & & 1 & \end{pmatrix}$. *Then the relation (5.7) gives*

$$ {}_1F_1(a, c; x) = e^x {}_1F_1(c - a, c; -x), $$

which is known as Kummer's first transformation formula for the confluent hypergeometric function.

Proof. Put $\Phi(z, \alpha) = \Phi_{(2,1,1)}(z, \alpha)$. By Theorem 5.3, we have

$$ \Phi(zg, \alpha g) = \Phi(z, \alpha). $$

Put

$$ z' = zg = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & x & -1 & 1 \end{pmatrix}, \quad \alpha' = \alpha g = (\alpha_0, -1, \alpha_3, \alpha_2). $$

For $\Phi(z', \alpha')$, the integral is taken along the path starting from 0 and terminating at 1. Note that z' is normalized by the action of $GL(2) \times H$ as

$$ z' = \begin{pmatrix} 1 & 1 \\ & -1 \end{pmatrix} z'' h, \quad z'' = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & -x & 1 & -1 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & -x & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}. $$

Furthermore notice that the integral which defines $\Phi(z''; \alpha')$ is taken along the path which goes from 1 to 0. Therefore we have

$$ \Phi(z'; \alpha') = -\Phi(z''; \alpha')\chi(h) = -e^x \Phi(z''; \alpha'). \tag{7.1} $$

Expressing $\Phi(z''; \alpha')$ in terms of Kummer's confluent hypergeometric function, we get from

(7.1)

$$ {}_1F_1(\alpha_2+1,\alpha_2+\alpha_3+2;x) = e^x {}_1F_1(\alpha_3+1,\alpha_2+\alpha_3+2;-x). $$

□

7.2. GHF on $M_{2,5}$.

A list of special functions of two variables of hypergeometric type can be found in [E] and is known as Horn's list. In the list, there are functions whose associated holonomic systems are of rank = 3. Except the function G_1, they are given by the series

$$
\begin{aligned}
F_1(a,b,b',c;x,y) &= \sum_{m,n\ge 0} \frac{(a)_{m+n}(b)_m(b')_n}{(c)_{m+n}m!n!}x^m y^n \\
G_2(a,a',b,b';x,y) &= \sum_{m,n\ge 0} \frac{(a)_m(a')_n(b)_{n-m}(b')_{m-n}}{m!n!}x^m y^n \\
\Phi_1(a,b,c;x,y) &= \sum_{m,n\ge 0} \frac{(a)_{m+n}(b)_m}{(c)_{m+n}m!n!}x^m y^n \\
\Phi_2(b,b',c;x,y) &= \sum_{m,n\ge 0} \frac{(b)_m(b')_n}{(c)_{m+n}m!n!}x^m y^n \\
\Phi_3(b,c;x,y) &= \sum_{m,n\ge 0} \frac{(b)_m}{(c)_{m+n}m!n!}x^m y^n \\
\Gamma_1(a,b,b';x,y) &= \sum_{m,n\ge 0} \frac{(a)_m(b)_{n-m}(b')_{m-n}}{m!n!}x^m y^n \\
\Gamma_2(b,b';x,y) &= \sum_{m,n\ge 0} \frac{(b)_{n-m}(b')_{m-n}}{m!n!}x^m y^n.
\end{aligned}
$$

The holonomic systems on $\mathbb{P}^1 \times \mathbb{P}^1$ for the first two functions are of Fuchsian type and those for the rest have irregular singularity in addition to regular one. The latters are obtained from the first two by so-called the process of confluence:

$$
\begin{aligned}
\Phi_1(a,b,c;x,y) &= \lim_{\epsilon\to 0} F_1(a,b,\frac{1}{\epsilon},c;x,\epsilon y), \\
\Phi_2(b,b',c;x,y) &= \lim_{\epsilon\to 0} F_1(\frac{1}{\epsilon},b,b',c;\epsilon x,\epsilon y), \\
\Phi_3(b,c;x,y) &= \lim_{\epsilon\to 0} F_1(\frac{1}{\epsilon},b,\frac{1}{\epsilon},c;\epsilon x,\epsilon^2 y), \\
\Gamma_1(a,b,b';x,y) &= \lim_{\epsilon\to 0} G_2(a,\frac{1}{\epsilon},b,b';x,\epsilon y), \\
\Gamma_2(a,b,b';x,y) &= \lim_{\epsilon\to 0} \Gamma_1(\frac{1}{\epsilon},b,b';\epsilon x,y), \\
&= \lim_{\epsilon\to 0} G_2(\frac{1}{\epsilon},\frac{1}{\epsilon},b,b';\epsilon x,\epsilon y).
\end{aligned}
$$

Let us relate these functions to the GHF's on $M_{2,5}$ with various Young diagram of weight 5. The Young diagrams of weight 5 are listed as

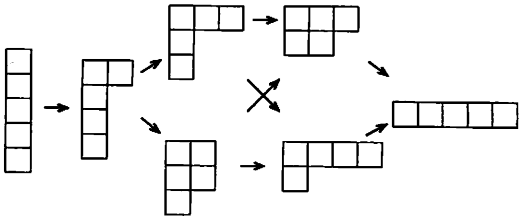

FIGURE 4. Young diagrams of weight 5.

The meaning of the arrows in the figure is the same as in §§7.1. The normal form $\mathbf{z}$ of $z \in Z_{2,5}$ and that of parameters $\alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4)$ are gathered in Table 2. Note that there is other possibility of choice of normal form $\mathbf{z}$ and accordingly the GHF of type λ takes different form in appearance. The detailed study of this kind will be taken up in [KKT].

TABLE 2

λ	Normal form of z	Normal form of α
1, 1, 1, 1, 1	$\mathbf{z}_1 = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & x & y \end{pmatrix}$	$\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4$
	$\mathbf{z}_2 = \begin{pmatrix} 1 & 0 & 1 & x & 1 \\ 0 & 1 & 1 & 1 & y \end{pmatrix}$	
2, 1, 1, 1	$\mathbf{z}_3 = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & y & 1 & 1 & x \end{pmatrix}$	$\alpha_0, -1, \alpha_2, \alpha_3, \alpha_4$
	$\mathbf{z}_4 = \begin{pmatrix} 1 & 0 & 0 & x & y \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix}$	
	$\mathbf{z}_5 = \begin{pmatrix} 1 & 0 & 0 & x & 1 \\ 0 & y & 1 & 1 & 1 \end{pmatrix}$	
2,2,1	$\mathbf{z}_6 = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & y & 1 & 0 & x \end{pmatrix}$	$\alpha_0, -1, \alpha_2, -1, \alpha_4$
	$\mathbf{z}_7 = \begin{pmatrix} 1 & 0 & 0 & x & 1 \\ 0 & y & 1 & 0 & 1 \end{pmatrix}$	
3,1,1	$\mathbf{z}_8 = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & x & y & 1 & 1 \end{pmatrix}$	$\alpha_0, 0, 1, \alpha_3, \alpha_4$
3,2	$\mathbf{z}_9 = \begin{pmatrix} 1 & 0 & 0 & 0 & y \\ 0 & 1 & x & 1 & 0 \end{pmatrix}$	$\alpha_0, 0, 1, \alpha_3, 1$
4,1	$\mathbf{z}_{10} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & x & y & 1 \end{pmatrix}$	$\alpha_0, 0, 0, 1, \alpha_4$
5	$\mathbf{z}_{11} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & x & y \end{pmatrix}$	$-2, 0, 0, 0, 1$

Proposition 7.3. *Let $\lambda, \mathbf{z}_i$ and α be given as in Table 2. The GHF of type λ on $M_{2,5}$ are related with the classical hypergeometric functions of two variables as*

$$\begin{aligned}
\Phi_{(1,1,1,1,1)}(\mathbf{z}_1;\alpha) &= \int_{\gamma_1} u^{\alpha_1}(1+u)^{\alpha_2}(1+xu)^{\alpha_3}(1+yu)^{\alpha_4}du \\
&= A_1 \cdot F_1(\alpha_1+1, -\alpha_3, -\alpha_4, \alpha_1+\alpha_2+2; -x, -y) \\
\Phi_{(1,1,1,1,1)}(\mathbf{z}_2;\alpha) &= \int_{\gamma_2} u^{\alpha_1}(1+u)^{\alpha_2}(u+x)^{\alpha_3}(1+yu)^{\alpha_4}du \\
&= A_2 \cdot G_2(-\alpha_3, -\alpha_4, \alpha_1+\alpha_3+1, -\alpha_1-\alpha_2-\alpha_3-1; -x, -y) \\
\Phi_{(2,1,1,1)}(\mathbf{z}_3;\alpha) &= \int_{\gamma_3} e^{-yu}u^{\alpha_2}(1+u)^{\alpha_3}(1+xu)^{\alpha_4}du \\
&= A_3 \cdot \Phi_1(\alpha_2+1, -\alpha_4, \alpha_2+\alpha_3+2; x, y) \\
\Phi_{(2,1,1,1)}(\mathbf{z}_4;\alpha) &= \int_{\gamma_4} u^{\alpha_0}e^{-1/u}(1+xu)^{\alpha_3}(1+yu)^{\alpha_4}du \\
&= A_4 \cdot \Phi_2(-\alpha_3, -\alpha_4, \alpha_0+2; x, y) \\
\Phi_{(2,1,1,1)}(\mathbf{z}_5;\alpha) &= \int_{\gamma_5} e^{yu}u^{\alpha_2}(x+u)^{\alpha_3}(1+u)^{\alpha_4}du \\
&= A_5 \cdot \Gamma_1(-\alpha_3, -\alpha_2-\alpha_3-\alpha_4-1, \alpha_2+\alpha_3+1; -x, -y) \\
\Phi_{(2,2,1)}(\mathbf{z}_6;\alpha) &= \int_{\gamma_6} e^{-yu}u^{\alpha_2}e^{-1/u}(1+xu)^{\alpha_4}du \\
&= A_6 \cdot \Phi_3(-\alpha_4, \alpha_2+2; x, y) \\
\Phi_{(2,2,1)}(\mathbf{z}_7;\alpha) &= \int_{\gamma_7} e^{-yu}u^{\alpha_2}e^{-x/u}(1+u)^{\alpha_4}du \\
&= A_7 \cdot \Gamma_2(\alpha_2+1, -\alpha_2-\alpha_4-1; x, y)
\end{aligned}$$

where the paths of integration γ_i $(i = 1, \ldots, 7)$ are given in Figure 5 and A_i denotes the constants given by

$$\begin{aligned}
A_1 &= (-1)^{\alpha_1}\frac{\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)}{\Gamma(\alpha_1+\alpha_2+2)}, \\
A_2 &= (-1)^{\alpha_1+\alpha_3}\frac{\Gamma(\alpha_1+\alpha_3+1)\Gamma(\alpha_2+1)}{\Gamma(\alpha_1+\alpha_2+\alpha_3+2)}, \\
A_3 &= (-1)^{\alpha_2}\frac{\Gamma(\alpha_2+1)\Gamma(\alpha_3+1)}{\Gamma(\alpha_2+\alpha_3+2)}, \\
A_4 &= \frac{-2\pi i}{e^{\pi i\alpha_0}\Gamma(\alpha_0+2)}, \\
A_5 &= (-1)^{\alpha_2+\alpha_3}\frac{\Gamma(\alpha_2+\alpha_3+1)\Gamma(\alpha_4+1)}{\Gamma(\alpha_2+\alpha_3+\alpha_4+2)}, \\
A_6 &= \frac{-2\pi i}{e^{\pi i\alpha_2}\Gamma(\alpha_2+2)}, \\
A_7 &= (-1)^{\alpha_2}\frac{\Gamma(\alpha_2+1)\Gamma(\alpha_4+1)}{\Gamma(\alpha_2+\alpha_4+2)}.
\end{aligned}$$

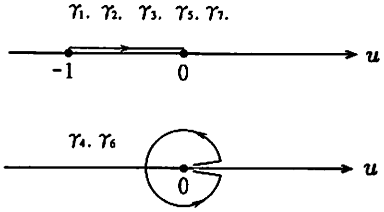

FIGURE 5. Path of integration γ_i, $i = 1, \dots, 7$.

We can give the similar results as in Proposition 7.2 for the classical hypergoemetric functions $F_1, G_2, \Phi_1, \Phi_2, \Phi_3, \Gamma_1$ and Γ_2. The explicit formulas will be given in [KKT].

For the other partitions, $\lambda = (3,1,1), (3,2), (4,1)$ and (5), we have the following functions.

$$\Phi_{(3,1,1)}(\mathbf{z}_8; \alpha) = \int_\gamma \exp(-\frac{1}{2}x^2u^2 + yu)u^{\alpha_3}(1+u)^{\alpha_4}du$$
$$\Phi_{(3,2)}(\mathbf{z}_9; \alpha) = \int_\gamma \exp(-\frac{1}{2}u^2 + xu + \frac{y}{u})u^{\alpha_3}du$$
$$\Phi_{(4,1)}(\mathbf{z}_{10}; \alpha) = \int_\gamma \exp(\frac{1}{3}u^3 - xu^2 + yu)u^{\alpha_4}du$$
$$\Phi_{(5)}(\mathbf{z}_{11}; \alpha) = \int_\gamma \exp(-\frac{1}{4}u^4 - xu^2 + yu)du,$$

where γ are the cycles of the homology associated with the above integrals given in Figure 5.

Remark 7.4. The GHF Φ_λ in the affine coordinates on $M_{2,5}$ listed above have already appeared as particular solutions for the completely integrable Hamiltonian systems called *Garnier system* and its "systems of confluent type ", [K, OK]. It is to be noted that these Hamiltonian systems arose in an entirely different context, namely, from the theory of monodromy preserving deformation of the second order linear differential equations on $\mathbb{P}^1$ and that they were indexed by the partitions of 5.

The detailed study of the general hypergeometric functions of type λ $\Phi_\lambda(z;\alpha)$ on $M_{2,5}$ such as series expansions at the singular locus, asymptotic expansions, transformation formulae and the explicit form of Pfaffian system on $\mathbb{P}^1 \times \mathbb{P}^1$, etc. will be given in the forthcoming paper [KKT].

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