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On some foliations on ruled surfaces of genus one.

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§0. Introduction.

In [Sa5], we classified foliations on ruled surfaces leaving a curve invariant and having no singularities on it. There are three cases. (See Theorem 0.1 below.) The first case was investigated in [Sa6]. In this paper, we observe the second one on ruled surfaces over closed Riemann surfaces of genus one.

A foliation of dimension one can be defined in various ways. (cf. [GM1], [GM2], [Sa1], [Sa2], [Sa3] and [Sa5].) In this paper, we adopt the following one.

Let M be a complex manifold of dimension m, $\mathcal{O}_M$ the sheaf of germs of holomorphic functions on M and Θ_M the sheaf of germs of holomorphic vector fields on M.

DEFINITION 0.0.

0) A *foliation of dimension one* on M is an invertible subsheaf $\mathcal{F}$ of Θ_M satisfying the following: The analytic set

$$\{p \in M | (\Theta_M/\mathcal{F})_p \text{ is } \textit{not} \text{ a free } \mathcal{O}_{M,p}\text{-module of rank } m-1\},$$

which is called the *singular locus* of $\mathcal{F}$, is of codimension strictly greater than one.

1) Let N be a subvariety of M defined by a coherent ideal sheaf $\mathcal{I} \subset \mathcal{O}_M$. A foliation $\mathcal{F} \subset \Theta_M$ leaves N *invariant* if

$$\mathcal{F}\mathcal{I} \subset \mathcal{I}.$$

THEOREM *0.1. ([Sa5] pp.622–623.* MAIN THEOREM *2.1.)*

Let C be a closed Riemann surface of genus g, $X = \mathbf{P}(\mathcal{E}) \xrightarrow{\pi} C$ a ruled surface over C with the invariant e, a normalized locally free $\mathcal{O}_C$-module $\mathcal{E}$ and C_0 a normalized section of $X \xrightarrow{\pi} C$. Assume that a foliation $\mathcal{F} \subset \Theta_X$ on X leaves an irreducible curve $C_1 \simeq_{num} aC_0 + bf$ with $a > 0$ on X invariant and has no singularities on C_1. Then one of the following is the case.

$I - i)$ $\quad e = 0, \quad$ *$\mathcal{E}$ is decomposable* $\quad$ *and* $\quad b = 0.$

I − ii) $e = 0$, *$\mathcal{E}$ is indecomposable* *and* $b = 0$.

II) $e < 0$, $a \geq 2$ *and* $b = \frac{1}{2}ea \in \mathbf{Z}$. *(In this case, $\mathcal{E}$ is indecomposable.)*

Here f is the divisor defined by a fibre of $X \xrightarrow{\pi} C$ and '$\simeq_{num}$' represents numerical equivalence of divisors on X.

§1. The main theorem.

Let C be an elliptic curve with periods $(2\omega_1, 2\omega_2)$ and $X = \mathbf{P}(\mathcal{E})$ with $e = 0$, where $\mathcal{E}$ is an indecomposable locally free $\mathcal{O}_C$-module of rank two. Note that $g = 1$, $e = 0$ and $\mathcal{E}$ is indecomposable implies that $\mathcal{E}$ is a non-trivial extension

$$0 \to \mathcal{O}_C \to \mathcal{E} \to \mathcal{O}_C \to 0,$$

which is constructed as follows:

Let $\wp(z)$ be the Weierstrass $\wp$-function with periods $(2\omega_1, 2\omega_2)$, $\omega_3 = \omega_1 + \omega_2$ and $\alpha_i = \wp(\omega_i)$ for $i = 1$, 2 and 3. Considering the elliptic curve C as the quotient space of $\mathbf{C}$ defined by the lattice $\mathbf{Z}2\omega_1 \oplus \mathbf{Z}2\omega_2$, we denote by $[z] \in C$ the image of $z \in \mathbf{C}$. Let $a_i = [\omega_i]$ for $i = 1$, 2 and 3 and $U = C - a_3$. Taking a small enough open disk V with a coordinate x centred at 0, we identify V with an open set in C:

$$\begin{array}{ccc} V & \subset & C \\ x & \mapsto & [x + \omega_3]. \end{array}$$

We define a vector bundle E over C, with respect to this open covering $\{U, V\}$ of C, by

$$E_{VU} = \begin{bmatrix} 1 & \frac{1}{\wp'(z)} \\ 0 & 1 \end{bmatrix}.$$

Let X be a ruled surface defined by the vector bundle E, i.e. defined by patching $U \times \mathbf{P}^1$ and $V \times \mathbf{P}^1$ together, idenifying $([z], \zeta) \in U \times \mathbf{P}^1$ and $(x, \xi) \in V \times \mathbf{P}^1$ if and only if

$$\begin{cases} [x + \omega_3] & = & [z] \\ \xi & = & \zeta + \frac{1}{\wp'(z)} \end{cases}.$$

Since $\dim \mathbf{H}^1(C, \mathcal{O}_C) = 1$, this is the only ruled surface over the fixed elliptic curve C satisfying $e = 0$ and defined by an indecomposable normalized locally free sheaf over C.

Take fibre coordinates $\eta = \frac{1}{\zeta}$ and $\rho = \frac{1}{\xi}$. The transition relation is written as follows:

$$\rho = \frac{\wp'(z)\eta}{\eta + \wp'(z)}.$$

Curves defined by $\eta = 0$ in $U \times \mathbf{P}^1$ and $\rho = 0$ in $V \times \mathbf{P}^1$ are patched together to define a normalized section C_0 of $X \xrightarrow{\pi} C$. It should be noted that C_0 is the unique section satisfying ${C_0}^2 = -e = 0$. (cf. e.g. [Gu] §5.)

On $\pi^{-1}(U) \cap \pi^{-1}(V)$,

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial z} - \frac{\wp''(z)}{\wp'(z)^2}\eta^2 \frac{\partial}{\partial \eta} \qquad \text{and} \qquad \rho^2 \frac{\partial}{\partial \rho} = \eta^2 \frac{\partial}{\partial \eta}.$$

We define a holomorphic vector field $\sigma \in \Gamma(X, \Theta_X)$ by

$$\sigma|_{\pi^{-1}(U)} = \eta^2 \frac{\partial}{\partial \eta} \qquad \text{and} \qquad \sigma|_{\pi^{-1}(V)} = \rho^2 \frac{\partial}{\partial \rho}.$$

Let θ be a holomorphic vector field

$$\frac{\partial}{\partial z} + \frac{\wp(z-\omega_3)}{\wp''(\omega_3)}\frac{\partial}{\partial \zeta} = \frac{\partial}{\partial z} - \frac{\wp(z-\omega_3)}{\wp''(\omega_3)}\eta^2 \frac{\partial}{\partial \eta} \in \Gamma(\pi^{-1}(U), \Theta_X).$$

θ extends to a vector field

$$\frac{\partial}{\partial x} + \left(\frac{\wp(x)}{\wp''(\omega_3)} - \frac{\wp''(x+\omega_3)}{\wp'(x+\omega_3)^2}\right)\frac{\partial}{\partial \xi} = \frac{\partial}{\partial x} - \left(\frac{\wp(x)}{\wp''(\omega_3)} - \frac{\wp''(x+\omega_3)}{\wp'(x+\omega_3)^2}\right)\rho^2 \frac{\partial}{\partial \rho}$$

on $\pi^{-1}(V)$. Recall the Laurent expansion of $\wp$ at 0:

$$\wp(z) = \frac{1}{z^2} + c_2 z^2 + c_4 z^4 + \cdots$$

Since

$$\wp'(\omega_3) = 0, \qquad \wp''(\omega_3) \neq 0 \qquad \text{and} \qquad \wp(x+\omega_3) = \wp(\omega_3 - x),$$

$\frac{\wp(x)}{\wp''(\omega_3)} - \frac{\wp''(x+\omega_3)}{\wp'(x+\omega_3)^2}$ is holomorphic at 0. Thus θ defines a global holomorphic vector field on X, which we denote by θ also.

Main Theorem 1.0.

Let C be an elliptic curve, $X \xrightarrow{\pi} C$ the ruled surface over C satisfying $e = 0$ and defined by the normalized indecomposable locally free sheaf over C and C_0 the normalized section of $X \xrightarrow{\pi} C$, which are described as above. Assume a foliation $\mathcal{F} \subset \Theta_X$ on X leaves an *irreducible curve C_1 on X invariant, has no singularities on C_1 and is not the ruling. Then the curve C_1 is C_0 and the foliation $\mathcal{F}$ is defined by a global holomorphic vector field*

$$\theta + k\sigma \in \Gamma(X, \Theta_X) \quad \textit{with} \quad k \in \mathbf{C},$$

where θ and σ are holomorphic vector fields on X defined as above.

§2. Proof of the main theorem.

It follows from Theorem 0.1 that $C_1 \simeq_{num} nC_0$ for a certain $0 < n \in \mathbf{Z}$. First, we claim the following proposition, which we prove in §3 bellow.

PROPOSITION 2.0.
If an effective divisor D on X satisfies $D \simeq_{num} nC_0$ then D is nC_0.

It follows that C_1, which is redeced, is C_0 and that $\mathcal{F}$ leaves C_0 invariant and has no singularities on C_0. It holds that

$$\mathcal{F} \subset \mathrm{Der}_X(\log C_0),$$

where $\mathrm{Der}_X(\log C_0)$ is the sheaf of germs of logarithmic vector fields with respect to C_0. (cf. [Sai] pp.267–268, (1.4) Definition.) Since the coherent ideal $\mathcal{I}$ defining C_0 satisfies

$$\mathcal{I}|_{\pi^{-1}(U)} = \mathcal{O}_{\pi^{-1}(U)}\eta \qquad \text{and} \qquad \mathcal{I}|_{\pi^{-1}(V)} = \mathcal{O}_{\pi^{-1}(V)}\rho$$

$\mathrm{Der}_X(\log C_0)$ is as follows:

PROPOSITION 2.1.

$$\mathrm{Der}_X(\log C_0)|_{\pi^{-1}(U)} = \mathcal{O}_{\pi^{-1}(U)}\sigma + \mathcal{O}_{\pi^{-1}(U)}\eta^{-1}\sigma$$

and

$$\mathrm{Der}_X(\log C_0)|_{\pi^{-1}(V)} = \mathcal{O}_{\pi^{-1}(V)}\sigma + \mathcal{O}_{\pi^{-1}(V)}\rho^{-1}\sigma.$$

Note that any holomorphic line bundle over X is meromorphically trivial. Thus the foliation $\mathcal{F}$, which is an invertible subsheaf of Θ_X, has a non-trivial global meromorphic section, which can be regarded as a global meromorphic vector field.

We can take such a vector field of the form $\theta + h\sigma$ with a global meromorphic function h on X since $\mathcal{F}$ is not the ruling. We claim that the meromorphic function h is constant: $h \in \mathbf{C}$.

Let (h) and (σ) be the divisors on X defined by h and σ, respevtively. (h) is written as

$$(h) = (h)_+ - (h)_-,$$

where $(h)_+$ and $(h)_-$ are the zero and the polar divisors of h. Note that the numerically equivalent classes of divisors on X form an abelian group

$$\mathbf{Z}C_0 \oplus \mathbf{Z}f,$$

where f is a fibre of $X \xrightarrow{\pi} C$ and that

$${C_0}^2 = -e = 0,\ C_0 \cdot f = 1 \text{ and } f^2 = 0.$$

Since the foliation leaves C_0 invariant and

$$(\sigma) = 2C_0,$$

one of the following two must be the case.

a) $(h)_- - C_0 \not\geq 0$ and $(h)_- \cdot C_0 = 0$ or

b) $(h)_- - C_0 \geq 0$, $(h)_- - 2C_0 \not\geq 0$ and $((h)_- - C_0) \cdot C_0 = 0$.

Let

$$(h)_+ \sim (h)_- \simeq_{num} nC_0 + mf,$$

where '$\sim$' represents linear equivalence of divisors on X. In both of the cases a) and b), $m = 0$. It follows from Propositon 2.0 that $(h)_+ = (h)_- = nC_0$. Thus $(h)_+ = (h)_- = 0$ and the function h is constant, which completes the proof. ∎

§3. Proof of Proposition 2.0.

Suppose that the divisor D would *not* be nC_0. We may assume $D - aC_0 \not\geq 0$ for any $0 < a \in \mathbf{Z}$.

Consider the holomorphic vector bundle $E \xrightarrow{\pi_E} X$ described in §1 and take the following fibre coordinates of E:

$$(\lambda_U, \mu_U) \text{ on } \pi_E{}^{-1}(U) \qquad \text{and} \qquad (\lambda_V, \mu_V) \text{ on } \pi_E{}^{-1}(V),$$

which satisfy the transition relation

$$\begin{pmatrix} \lambda_V \\ \mu_V \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{\wp'(z)} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda_U \\ \mu_U \end{pmatrix} \qquad \text{on} \quad \pi_E{}^{-1}(U) \cap \pi_E{}^{-1}(V).$$

C_0 is defined by $\mu_U = 0$ on $\pi_E{}^{-1}(U)$ and by $\mu_V = 0$ on $\pi_E{}^{-1}(V)$, respectively. Since $D \cdot C_0 = 0$, the divisor D should be defined by the following homogeneous polynomials of degree n in λ_U and μ_U and in λ_V and μ_V.

$$P_U = \lambda_U{}^n + \sum_{j=1}^{n} p_{U,j} \lambda_U{}^{n-j} \mu_U{}^j \qquad \text{on} \quad \pi_E{}^{-1}(U)$$

and

$$P_V = \lambda_V{}^n + \sum_{k=1}^{n} p_{V,k} \lambda_V{}^{n-k} \mu_V{}^k \qquad \text{on} \quad \pi_E{}^{-1}(V),$$

where

$$p_{U,j} \in \mathcal{O}_C(U) \subset \mathcal{O}_E(\pi_E{}^{-1}(U)) \qquad \text{for} \quad j = 1, \cdots, n$$

and

$$p_{V,k} \in \mathcal{O}_C(V) \subset \mathcal{O}_E(\pi_E{}^{-1}(V)) \qquad \text{for} \quad k = 1, \cdots, n.$$

Set $p_{U,0} = 1 \in \mathcal{O}_C(U)$ and $p_{V,0} = 1 \in \mathcal{O}_C(V)$, respectively. From the above transition relation, P_V would satisfy on $\pi_E{}^{-1}(U) \cap \pi_E{}^{-1}(V)$

$$P_V = \lambda_U{}^n + \sum_{j=1}^{n} \left(\sum_{k=0}^{j} p_{V,k} \frac{(n-k)!}{(n-j)!(j-k)!} \wp'(z)^{-(j-k)} \right) \lambda_U{}^{n-j} \mu_U{}^j$$

on $\pi_E{}^{-1}(U) \cap \pi_E{}^{-1}(V)$. Since these polynomials would define the same zero loci on $\pi_E{}^{-1}(U) \cap \pi_E{}^{-1}(V)$, there should exist a never vanishing holomorphic function $\alpha \in \mathcal{O}_E(\pi_E{}^{-1}(U) \cap \pi_E{}^{-1}(V))$ such that $P_V = \alpha P_U$. Since both P_U and P_V are homogeneous polynomials of degree n in λ_U and μ_U and the coefficients of $\lambda_U{}^n$ of both equals to 1, it should hold that

$$P_U|_{\pi_E{}^{-1}(U) \cap \pi_E{}^{-1}(V)} = P_V|_{\pi_E{}^{-1}(U) \cap \pi_E{}^{-1}(V)}.$$

Thus

$$p_{U,1} = p_{V,1} + \frac{n}{\wp'(z)}$$

and $p_{U,1} \in \mathcal{O}_C(U)$ would extend to a global meromorphic function $p_{U,1} \in \mathcal{M}_C(C)$. Then $p_{U,1}$ would have a pole of order one at $[\omega_3]$ and be holomorphic on $C - [\omega_3]$, which is a contradiction. ∎

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