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Transformation groups on Heisenberg geometry

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Introduction

A geometry $(\mathcal{G}, X)$ consists of a finite dimensional Lie group $\mathcal{G}$ with finitely many components and an n-dimensional homogeneous space X from $\mathcal{G}$. A $(\mathcal{G}, X)$-structure on an n-dimensional smooth manifold M is a geometric structure locally modelled on X with coordinate changes lying in $\mathcal{G}$. A manifold equipped with a $(\mathcal{G}, X)$-structure is said to be a $(\mathcal{G}, X)$-manifold. If $X = S^n$ equipped with the group of conformal transformations $\mathcal{G} = \mathbf{Conf}(S^n)$, then the geometry $(\mathcal{G}, X)$ is called conformally flat geometry. It is well known that the sphere with one point removed is conformally equivalent to the flat euclidean space by the stereographic projection. The subgroup of $\mathbf{Conf}(S^n)$ leasving one point fixed is callled the group of similarity transformations, for which we denote $\mathbf{Sim}(\mathbf{R}^n)$. Then the pair $(\mathbf{Sim}(\mathbf{R}^n), \mathbf{R}^n)$ is said to be similarity geometry.

Spherical CR geometry is viewed as a complex version of conformally flat geometry. In fact, conformally flat geometry is identified with the boundary of the real hyperbolic space $\mathbf{H}_{\mathbf{R}}^{n+1}$ with hyperbolic group $PO(n+1, 1)$, while spherical CR geometry is identified with the boundary of complex hyperbolic space $\mathbf{H}_{\mathbf{C}}^{n+1}$ with hyperbolic group $PU(n+1, 1)$ which acts on the boundary sphere as Cauchy-Riemann transformations. As a consequence, the sphere with one point removed S^{2n+1}-$\{\infty\}$ inherits a geometry from the spherical CR geometry. Such geometry is called Heisenberg geometry. In this paper, we study the Heisenberg geometry from the viewpoint of transformation groups.

We refer to [4], [8], [9], [6], [20] for the current development of Heisenberg geometry. Especially, chains, and cross ratio.

1. Spherical CR-structure

We shall explain spherical CR-structure. Recall the projective model of Kähler hyperbolic space. Let $\mathbf{C}^{n+2}$ denote the complex vector space, equipped with the Hermitian form

$$\mathcal{B}(z, w) = -\bar{z}_1 w_1 + \bar{z}_2 w_2 + \cdots + \bar{z}_{n+2} w_{n+2}.$$

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Consider the following subspaces in $\mathbf{C}^{n+2}-\{0\}$:

$$
\begin{aligned}
V_0^{2(n+1)+1} &= \{z \in \mathbf{C}^{n+2} \mid \mathcal{B}(z,z)=0\},\\
V_{-1}^{2(n+1)+1} &= \{z \in \mathbf{C}^{n+2} \mid \mathcal{B}(z,z)=-1\},\\
V_-^{2(n+2)} &= \{z \in \mathbf{C}^{n+2} \mid \mathcal{B}(z,z)<0\}.
\end{aligned}
$$

Let $P:\mathbf{C}^{n+2}-\{0\}\longrightarrow\mathbf{CP}^{n+1}$ be the canonical projection onto the complex projective space. By definition ([2]), the complex hyperbolic space $\mathbf{H}_{\mathbf{C}}^{n+1}$ is defined to be $P(V_-^{2(n+2)})$. The group $U(n+1,1)$ is the subgroup of $\mathbf{GL}(n+2,\mathbf{C})$ whose elements preserve the form $\mathcal{B}$. Then the action of $U(n+1,1)$ on $V_-^{2(n+2)}$ induces an action of $\mathbf{H}_{\mathbf{C}}^{n+1}$. The kernel of this action is the center $\mathcal{Z}(n+1,1)$ isomorphic to the circle S^1. Let $PU(n+1,1)=U(n+1,1)/\mathcal{Z}(n+1,1)$. There is an equivariant projection

$$P:(U(n+1,1),V_-^{2(n+2)})\longrightarrow(PU(n+1,1),\mathbf{H}_{\mathbf{C}}^{n+1}).$$

Put $S^{2n+1}=P(V_0^{2(n+1)+1})$. Then the standard sphere S^{2n+1} is the boundary of the unit ball $\mathbf{H}_{\mathbf{C}}^{n+1}$ in $\mathbf{C}^{n+1}$ and the group of biholomorphic transformations of $\mathbf{H}_{\mathbf{C}}^{n+1}$ is isomorphic to the group of unitary transformations $PU(n+1,1)$. The group $PU(n+1,1)$ acts on the boundary by CR automorphisms and $\mathbf{Aut}_{CR}(S^{2n+1})=PU(n+1,1)$ ([1]). If M is a spherical CR-manifold of dimension $2n+1$, then there is a developing pair

$$(\rho,\mathbf{dev}):(\mathbf{Aut}_{CR}(\tilde{M}),\tilde{M})\longrightarrow(PU(n+1,1),S^{2n+1}).$$

For the topology of spherical CR-manifolds, we refer to [4], [20], [10].

2. Heisenberg geometry

We examine the Heisenberg geometry from the viewpoint of transformation groups. First we summarize the result:

Theorem *A maximal amenable group G of $U(n+1,1)$ is isomorphic to the semidirect product $\mathcal{N}\rtimes(U(n)\times\mathbf{C}^*)$ where $\mathcal{N}$ is the Heisenberg group. It lies in the following exact sequence:*

$$1\longrightarrow\mathbf{R}\longrightarrow\mathcal{N}\longrightarrow\mathbf{C}^n\longrightarrow1.$$

For the point at infinity $\{\infty\}\in S^{2n+1}$, the space $S^{2n+1}-\{\infty\}$ is canonically identified with $\mathcal{N}$. Then $\mathbf{Aut}_{CR}(\mathcal{N})$ is the stabilizer at the point $\{\infty\}$ in $PU(n+1,1)$. Moreover, $\mathbf{Aut}_{CR}(\mathcal{N})$ is a maximal amenable subgroup of $PU(n+1,1)$. If $P:U(n+1,1)\longrightarrow PU(n+1,1)$ is the projection, then PG is isomorphic to $\mathbf{Aut}_{CR}(\mathcal{N})=\mathcal{N}\rtimes(U(n)\times\mathbf{R}^+)$.

Now we explain how to deduce the Heisenberg geometry by using Cayley-Klein projective model (cf. [2], [1], [10]).

Let

$$P : (U(n+1,1), V_-^{n+2} \cup V_0^{2(n+1)+1}) \longrightarrow (PU(n+1,1), \mathbf{H}_{\mathbf{C}}^{n+1} \cup S^{2n+1})$$

be the projection as above. Let $\{e_1, \cdots, e_{n+2}\}$ be the standard basis with respect to the Hermitian form $\mathcal{B}$, *i.e.*,

$$\mathcal{B}(e_1, e_1) = -1,\ \mathcal{B}(e_i, e_j) = \delta_{ij}\ (i, j = 2, \cdots n+2),\ \mathcal{B}(e_1, e_j) = 0\ (j = 2, \cdots n+2).$$

Since $V_0^{2(n+1)+1}$ is a cone, we can assume that the inverse image $P^{-1}(\infty)$ consists of a complex line passing through the vector $f_1 = (e_1 + e_{n+2})/\sqrt{2}$. Let G be the subgroup of $U(n+1,1)$ which leaves the vector f_1 invariant. The stabilizer $PU(n+1,1)_\infty$ at $\{\infty\}$ is the image of the subgroup G by P. As above it is known that G is a maximal amenable Lie group. Put $f_{n+2} = (e_1 - e_{n+2})/\sqrt{2}$.

Now each element g of G has the following form with respect to the basis $\{f_1, e_2, \cdots, e_{n+1}, f_{n+2}\}$,

$$g = \begin{pmatrix} \lambda & x & z \\ 0 & \mathrm{B} & y \\ 0 & 0 & \mu \end{pmatrix}$$

where $\lambda, \mu \in \mathbf{C}^*$, B is an (n,n)-matrix. x is an n-th line vector, y is an n-th column vector. As $\mathcal{B}(gz, gw) = \mathcal{B}(z, w)$ for arbitrary $z, w \in \mathbf{C}^{n+2}$, we have the following relations.

$$(1) \qquad \lambda\bar{\mu} = 1, \quad x = \lambda^t\bar{y}\mathrm{B},$$

$$(2) \quad \bar{z}\mu + \bar{\mu}z = |y|^2, \quad \mathrm{B} \in U(n).$$

Then the Heisenberg group $\mathcal{N}$ is denoted by the subgroup consisting of the following matrices;

$$\begin{pmatrix} 1 & x & z \\ 0 & \mathrm{I} & y \\ 0 & 0 & 1 \end{pmatrix}$$

for which

$$\mathrm{Re}\ z = \frac{|y|^2}{2}, \quad x = {}^t\bar{y}.$$

This follows from the relations (1), (2) that $\bar{z} + z = |y|^2$. Thus, letting $z = \frac{|y|^2}{2} + ia$, there is a one-to-one correspondence between the product $\mathbf{R} \times \mathbf{C}^n$ and this group:

$$(a, y) = \begin{pmatrix} 1 & {}^t\bar{y} & \frac{|y|^2}{2} + ia \\ 0 & \mathrm{I} & y \\ 0 & 0 & 1 \end{pmatrix}$$

We examine the group law on the product $\mathbf{R} \times \mathbf{C}^n$.

$$(a,y)\cdot(b,y') = \begin{pmatrix} 1 & {}^t\bar{y} & z \\ 0 & \mathrm{I} & y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & {}^t\bar{y}' & z' \\ 0 & \mathrm{I} & y' \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & {}^t\bar{y}' + {}^t\bar{y} & z' + {}^t\bar{y}y' + z \\ 0 & \mathrm{I} & y' + y \\ 0 & 0 & 1 \end{pmatrix}.$$

Calculate:

$$z + z' = \frac{|y|^2}{2} + \mathrm{i}a + \frac{|y'|^2}{2} + \mathrm{i}b = \frac{|y|^2 + |y'|^2}{2} + \mathrm{i}(a+b).$$

Since $\overline{< y, y' >} = \overline{{}^t\bar{y}\cdot y'} = {}^t\bar{y}'\cdot y = < y', y >$,

$$|y+y'|^2 = < y+y', y+y' > = ({}^t\bar{y} + {}^t\bar{y}')(y+y') = |y|^2 + |y'|^2 + 2\mathrm{Re} < y, y' >,$$

Thus

$$z + z' = \frac{|y+y'|^2}{2} - \mathrm{Re} < y, y' > + \mathrm{i}(a+b).$$

As above, ${}^t\bar{y}y' = < y, y' >$, we have

$$\begin{aligned} z + {}^t\bar{y}y' + z' = z + z' + < y, y' > &= \frac{|y+y'|^2}{2} + \mathrm{i}(a+b) - \mathrm{Re} < y, y' > + < y, y' > \\ &= \frac{|y+y'|^2}{2} + \mathrm{i}\mathrm{Im} < y, y' > + \mathrm{i}(a+b) \\ &= \frac{|y+y'|^2}{2} + \mathrm{i}(a + b + \mathrm{Im} < y, y' >). \end{aligned}$$

Thus by the correspondence we obtain

$$\begin{pmatrix} 1 & {}^t\bar{y}' + {}^t\bar{y} & z' + {}^t\bar{y}y' + z \\ 0 & \mathrm{I} & y' + y \\ 0 & 0 & 1 \end{pmatrix} = (a + b + \mathrm{Im} < y, y' >, y + y').$$

Hence the Heisenberg Lie group $\mathcal{N}$ is the product $\mathbf{R} \times \mathbf{C}^n$ with group law

$$(a,y)\cdot(b,y') = (a + b + \mathrm{Im} < y, y' >, y + y').$$

In fact, $\mathcal{N}$ is nilpotent because $[\mathcal{N}, \mathcal{N}] = \mathbf{R}$ which is the center consisting of the form $(a, 0)$. Note also that the above correspondence can be written as

$$(\mathrm{Im}z, y) \longrightarrow \begin{pmatrix} 1 & {}^t\bar{y} & z \\ 0 & \mathrm{I} & y \\ 0 & 0 & 1 \end{pmatrix},$$

for which $\mathrm{Re}z = \dfrac{|y|^2}{2}$.

Next, let $\mathcal{N} \rtimes (U(n) \times \mathbf{C}^*)$ be the semidirect product where the action of $U(n) \times \mathbf{C}^*$ on $\mathcal{N}$ is given by

$$(*) \qquad (A, \nu) \circ (a, y) \;=\; (|\nu|^{-2}\nu a \nu^{-1}, Ay\nu^{-1}).$$

(Note that $\nu a \nu^{-1} = a$ in the $\mathbf{C}$-case. Compare §3.)

We show that the subgroup G of the matrices

$$g = \begin{pmatrix} \lambda & x & z \\ 0 & \mathrm{B} & y \\ 0 & 0 & \mu \end{pmatrix}$$

is isomorphic to the semidirect product $\mathcal{N} \rtimes (U(n) \times \mathbf{C}^*)$ by the correspondence

$$g = \begin{pmatrix} \lambda & x & z \\ 0 & \mathrm{B} & y \\ 0 & 0 & \mu \end{pmatrix} \longrightarrow ((\mathrm{Im}(z\bar{\lambda}), y\lambda), (\mathrm{B}, \mu)).$$

In order to prove that, first g has the following decomposition:

$$\begin{pmatrix} \lambda & x & z \\ 0 & \mathrm{B} & y \\ 0 & 0 & \mu \end{pmatrix} = \begin{pmatrix} 1 & xB^* & z\bar{\lambda} \\ 0 & \mathrm{I} & y\bar{\lambda} \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mathrm{B} & 0 \\ 0 & 0 & \mu \end{pmatrix}.$$

Obviously,

$$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mathrm{B} & 0 \\ 0 & 0 & \mu \end{pmatrix} \in G.$$

Note that

$$\begin{aligned} xB^* = (\lambda^t\bar{y}B)B^* &= \lambda^t\bar{y} \text{ from (1)}, \\ \overline{z\bar{\lambda}} + z\bar{\lambda} &= \lambda\bar{z} + z\bar{\lambda} \\ &= (\mu\bar{\lambda})\lambda\bar{z} + z\bar{\lambda}(\lambda\bar{\mu}) \text{ from (1)} \\ &= \mu|\lambda|^2\bar{z} + z|\lambda|^2\bar{\mu} \\ &= |\lambda|^2(\mu\bar{z} + z\bar{\mu}) \\ &= |\lambda|^2|y|^2 = |y\bar{\lambda}|^2 \text{ from (2)}. \end{aligned}$$

So,

$$\begin{pmatrix} 1 & xB^* & z\bar{\lambda} \\ 0 & \mathrm{I} & y\bar{\lambda} \\ 0 & 0 & 1 \end{pmatrix} \in G$$

Then by the definition,

$$\begin{pmatrix} 1 & xB^* & z\bar{\lambda} \\ 0 & \mathrm{I} & y\bar{\lambda} \\ 0 & 0 & 1 \end{pmatrix} \longrightarrow ((\mathrm{Im}(z\bar{\lambda}), y\lambda), (\mathrm{I}, 1)).$$

$$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mathrm{B} & 0 \\ 0 & 0 & \mu \end{pmatrix} \longrightarrow ((0,0), (B, \mu)).$$

We see that the correspondence is a homomorphism for the above decomposition:

$$\begin{pmatrix} 1 & xB^* & z\bar{\lambda} \\ 0 & \mathrm{I} & y\bar{\lambda} \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mathrm{B} & 0 \\ 0 & 0 & \mu \end{pmatrix} \longrightarrow ((\mathrm{Im}(z\bar{\lambda}), y\lambda), (\mathrm{I}, 1)) \cdot ((0,0), (\mathrm{B}, \mu)).$$

In fact,

$$\begin{aligned} & ((\mathrm{Im}(z\bar{\lambda}), y\lambda), (\mathrm{I}, 1)) \cdot ((0,0)), (\mathrm{B}, \mu)) \\ = \; & ((\mathrm{Im}(z\bar{\lambda}), y\lambda) \cdot (\mathrm{I}, 1) \circ (0,0)), ((\mathrm{I}, 1)(\mathrm{B}, \mu)) \\ = \; & ((\mathrm{Im}(z\bar{\lambda}), y\lambda) \cdot (0,0)), (\mathrm{B}, \mu)) \\ = \; & ((\mathrm{Im}(z\bar{\lambda}), y\lambda), (\mathrm{B}, \mu)). \end{aligned}$$

Now,

$$\begin{pmatrix} \lambda & x & z \\ 0 & \mathrm{B} & y \\ 0 & 0 & \mu \end{pmatrix} \cdot \begin{pmatrix} \lambda' & x' & z' \\ 0 & \mathrm{B}' & y' \\ 0 & 0 & \mu' \end{pmatrix}$$

$$= \begin{pmatrix} 1 & xB^* & z\bar{\lambda} \\ 0 & \mathrm{I} & y\bar{\lambda} \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mathrm{B} & 0 \\ 0 & 0 & \mu \end{pmatrix} \cdot \begin{pmatrix} 1 & x'B'^* & z'\bar{\lambda}' \\ 0 & \mathrm{I} & y'\bar{\lambda}' \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \lambda' & 0 & 0 \\ 0 & \mathrm{B}' & 0 \\ 0 & 0 & \mu' \end{pmatrix}$$

$$= \begin{pmatrix} 1 & xB^* & z\bar{\lambda} \\ 0 & \mathrm{I} & y\bar{\lambda} \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & \lambda x'B'^*B^{-1} & \lambda z'\bar{\lambda}'\mu^{-1} \\ 0 & \mathrm{I} & By'\bar{\lambda}'\mu^{-1} \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mathrm{B} & 0 \\ 0 & 0 & \mu \end{pmatrix} \cdot \begin{pmatrix} \lambda' & 0 & 0 \\ 0 & \mathrm{B}' & 0 \\ 0 & 0 & \mu' \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \lambda x' B'^* B^{-1} + xB^* & \lambda z'\bar{\lambda}'\mu^{-1} + xy'\bar{\lambda}'\mu^{-1} + z\bar{\lambda} \\ 0 & \mathrm{I} & By'\bar{\lambda}'\mu^{-1} + y\bar{\lambda} \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \lambda\lambda' & 0 & 0 \\ 0 & \mathrm{BB}' & 0 \\ 0 & 0 & \mu\mu' \end{pmatrix},$$

which is mapped by the above remark;

$$\longrightarrow ((\mathrm{Im}(\lambda z'\bar{\lambda}'\mu^{-1} + xy'\bar{\lambda}'\mu^{-1} + z\bar{\lambda}), \mathrm{B}y'\bar{\lambda}'\mu^{-1} + y\bar{\lambda}), (\mathrm{I}, 1)) \cdot ((0,0), (\mathrm{BB}', \mu\mu')).$$

On the other hand, $\lambda\bar{\mu} = 1$ implies $\lambda = |\mu|^{-2}\mu$. Since $\alpha \cdot \mathrm{Im}\beta \cdot \alpha^{-1} = \mathrm{Im}(\alpha\beta\alpha^{-1})$, and

$$< y\bar{\lambda}, By'\bar{\lambda}'\mu^{-1} >= (\lambda^t\bar{y})\mathrm{B}y'\bar{\lambda}'\mu^{-1} = xy'\bar{\lambda}'\mu^{-1} \text{ from (1)},$$

the above formula becomes:

$$\begin{aligned} = \; & ((\mathrm{Im}(\lambda z'\bar{\lambda}'\mu^{-1}) + \mathrm{Im}(xy'\bar{\lambda}'\mu^{-1}) + \mathrm{Im}(z\bar{\lambda}), \mathrm{B}y'\bar{\lambda}'\mu^{-1} + y\bar{\lambda}) \cdot (\mathrm{I}, 1) \circ (0,0)), \\ & (\mathrm{I}, 1)(\mathrm{BB}', \mu\mu')) \\ = \; & ((|\mu|^{-2}\mathrm{Im}(\mu z'\bar{\lambda}'\mu^{-1}) + \mathrm{Im} < y\bar{\lambda}, \mathrm{B}y'\bar{\lambda}'\mu^{-1} > + \mathrm{Im}(z\bar{\lambda}), \mathrm{B}y'\bar{\lambda}'\mu^{-1} + y\bar{\lambda}) \cdot (0,0)), \\ & (\mathrm{BB}', \mu\mu')) \\ = \; & (|\mu|^{-2}\mu \cdot \mathrm{Im}(z'\bar{\lambda}') \cdot \mu^{-1} + \mathrm{Im} < y\bar{\lambda}, \mathrm{B}y'\bar{\lambda}'\mu^{-1} > + \mathrm{Im}(z\bar{\lambda}), \mathrm{B}y'\bar{\lambda}'\mu^{-1} + y\bar{\lambda}), \\ & (\mathrm{BB}', \mu\mu')). \end{aligned}$$

By the action $(*)$ of $U(n) \times \mathbf{C}^*$,

$$\begin{aligned} & ((\mathrm{Im}(z\bar{\lambda}), y\bar{\lambda}), (\mathrm{B}, \mu)) \cdot ((\mathrm{Im}(z'\bar{\lambda}'), y'\bar{\lambda}'), (\mathrm{B}', \mu')) \\ = \; & ((\mathrm{Im}(z\bar{\lambda}), y\bar{\lambda}) \cdot ((\mathrm{B}, \mu) \circ (\mathrm{Im}(z'\bar{\lambda}'), y'\bar{\lambda}')), (\mathrm{B}, \mu) \cdot (\mathrm{B}', \mu')) \\ = \; & ((\mathrm{Im}(z\bar{\lambda}), y\bar{\lambda}) \cdot (|\mu|^{-2}\mu \cdot \mathrm{Im}(z'\bar{\lambda}') \cdot \mu^{-1}, \mathrm{B}y'\bar{\lambda}'\mu^{-1}), (\mathrm{BB}', \mu\mu')) \\ = \; & ((\mathrm{Im}(z\bar{\lambda}) + |\mu|^{-2}\mu \cdot \mathrm{Im}(z'\bar{\lambda}') \cdot \mu^{-1} + \mathrm{Im} < y\bar{\lambda}, \mathrm{B}y'\bar{\lambda}'\mu^{-1} >, y\bar{\lambda} + \mathrm{B}y'\bar{\lambda}'\mu^{-1}), \\ & (\mathrm{BB}', \mu\mu')). \end{aligned}$$

Therefore, the correspondence is a homomorphism:

$$\begin{pmatrix} \lambda & x & z \\ 0 & \mathrm{B} & y \\ 0 & 0 & \mu \end{pmatrix} \cdot \begin{pmatrix} \lambda' & x' & z' \\ 0 & \mathrm{B}' & y' \\ 0 & 0 & \mu' \end{pmatrix} \longrightarrow ((\mathrm{Im}(z\bar{\lambda}), y\bar{\lambda}), (\mathrm{B}, \mu)) \cdot ((\mathrm{Im}(z'\bar{\lambda}'), y'\bar{\lambda}'), (\mathrm{B}', \mu')).$$

Recall that $\mathbf{Aut}_{CR}(\mathcal{N}) = PG$, where $S^1 \to U(n+1,1) \xrightarrow{P} PU(n+1,1)$ is the projection as before. Then, PG is isomorphic to $\mathcal{N} \rtimes (U(n) \times \mathbf{R}^+)$, for which the action of $U(n) \times \mathbf{R}^+$

on $\mathcal{N}$ is given by

$$(A,t)\circ(a,y)=(t^{-2}a,t^{-1}Ay).$$

As a final remark, we show that $P(G)$ acts transitively on $S^{2n+1}-\{\infty\}$. As

$$S^{2n+1}=P(V_0^{2(n+1)+1})=\{[z_1,z_2,\cdots,z_{n+2}]\mid -|z_1|^2+|z_2|^2+\cdots+|z_{n+2}|^2=0\}$$

by the definition, it implies that with respect to the basis $\{f_1,e_2,\cdots,e_{n+1},f_{n+2}\}$

$$S^{2n+1}=\{[z,,y_1,\cdots,y_n,\mu]\mid -(\bar{z}\mu+\bar{\mu}z)+|y|^2=0\}.$$

Moreover, $P(f_1)=\infty=[1,0,\cdots,0]$. So,

$$S^{2n+1}-\{\infty\}=\{[z,y_1,\cdots,y_n,\mu]\mid -(\bar{z}\mu+\bar{\mu}z)+|y|^2=0,\ \mu\neq 0\}.$$

Choose the point $x_0=[0,\cdots,0,1]=P(f_{n+2})\in S^{2n+1}-\{\infty\}$. As $\mu\neq 0$, setting

$$\lambda=\frac{1}{\bar{\mu}},\quad x=\lambda{}^t\bar{y}\,\mathrm{I},$$

the element

$$g=\begin{pmatrix}\lambda & x & z\\ 0 & \mathrm{I} & y\\ 0 & 0 & \mu\end{pmatrix}$$

belongs to G and

$$(z,,y_1,\cdots,y_n,\mu)=g\cdot(0,\cdots,0,1)=g\cdot f_{n+2},$$

so that $P(g)\cdot x_0\in S^{2n+1}-\{\infty\}$. Since $P:G\to PG$ maps $\mathcal{N}$ onto itself, it is easy to check that

$$\mathcal{N}\cdot x_0=P(G)\cdot x_0=S^{2n+1}-\{\infty\}$$

with stabilizer $P(G)_{x_0}$ isomorphic to

$$P(\begin{pmatrix}\lambda & 0 & 0\\ 0 & \mathrm{B} & 0\\ 0 & 0 & \mu\end{pmatrix})\approx U(n)\times\mathbf{R}^+.$$

3. Remark

We can also consider the quaternionic Heisenberg geometry. In general, our geometry $(\mathcal{G},X)$ is lying on the boundary of rank one symmetric space with noncompact factor.

Let $\mathbf{F}$ stand for the noncommutative field of quaternions $\mathbf{F}$. Let $\mathbf{F}^{n+2}$ denote the $\mathbf{R}$-vector space, equipped with the Hermitian pairing over $\mathbf{F}$ as before;

$$\mathcal{B}(z,w) = -\bar{z}_1 w_1 + \bar{z}_2 w_2 + \cdots + \bar{z}_{n+2} w_{n+2}.$$

Let $P : \mathbf{F}^{n+2} - \{0\} \longrightarrow \mathbf{FP}^{n+1}$ be the canonical projection onto the quaternionic projective space. Consider the $4(n+2)$-subspace $V_-^{4(n+2)} = \{z \in \mathbf{F}^{n+2} | \ b(z,z) < 0\}$. Then the quaternion hyperbolic space $\mathbf{H}_{\mathbf{F}}^{n+1}$ of dimension $4n+4$, is defined to be $P(V_-^{4(n+2)})$.
The group $\mathbf{O}(n+1,1;\mathbf{F})$ is the subgroup of $\mathbf{GL}(n+2,\mathbf{F})$ whose elements preserve the Hermitian form $\mathcal{B}$. It is called the Lorentz group of type $(n+1,1)$. The group $\mathbf{O}(n+1,1;\mathbf{F})$ leaves $V_-^{4(n+2)}$ invariant and thus it induces an action on $\mathbf{H}_{\mathbf{F}}^{n+1}$. The kernel of this action is the center $\mathcal{Z}(n+1,1;\mathbf{F})$ which is isomorphic to $\{\pm 1\}$. The quotient group $\mathbf{O}(n+1,1;\mathbf{F})/\mathcal{Z}(n+1,1;\mathbf{F})$ is denoted by $P\mathbf{Sp}(n+1,1)$ Then it is known that the geometry $(P\mathbf{Sp}(n+1,1), \mathbf{H}_{\mathbf{F}}^{n+1})$ is a complete simply connected Riemannian manifold of $-1 \leqq$ sectional curvature $\leqq -\frac{1}{4}$ (cf. [14],[15]).

The projective compactification of $\mathbf{H}_{\mathbf{F}}^{n+1}$ is obtained by taking the closure $\bar{\mathbf{H}}_{\mathbf{F}}^{n+1}$ of $\mathbf{H}_{\mathbf{F}}^{n+1}$ in $\mathbf{FP}^{n+1}$. Moreover if we put a $(4(n+2)-1)$-dimensional subspace $V_0^{4(n+2)-1} = \{z \in \mathbf{F}^{n+2} | \ b(z,z) = 0\}$, then it follows that $\bar{\mathbf{H}}_{\mathbf{F}}^{n+1} = \mathbf{H}_{\mathbf{F}}^{n+1} \cup P(V_0^{4(n+2)-1})$. The boundary of $\mathbf{H}_{\mathbf{F}}^{n+1}$ is the standard sphere of dimension $4n+3$. Put $P(V_0^{4(n+2)-1}) = S^{4n+3}$. The group of isometries $P\mathbf{Sp}(n+1,1)$ extends to a smooth action on the boundary sphere S^{4n+3}. The geometry $(P\mathbf{Sp}(n+1,1), S^{4n+3})$ is said to be pseudo-quaternionic flat geometry.
As usual, we write $\mathbf{Aut}_{\mathbf{FSp}}(S^{4n+3}) = P\mathbf{Sp}(n+1,1)$.

Put $S^{4n+3} - \{\infty\} = \mathcal{M}$. Denote by $\mathbf{Aut}_{\mathbf{FSp}}(\mathcal{M})$ the subgroup of $\mathbf{Aut}_{\mathbf{FSp}}(S^{4n+3})$ which stabilizes the point at infinity $\{\infty\}$. Then the geometry $(\mathbf{Aut}_{\mathbf{FSp}}(\mathcal{M}), \mathcal{M})$ is called quaternionic Heisenberg geometry.

A maximal amenable group G of $\mathbf{Sp}(n+1,1)$ is isomorphic to the semidirect product $\mathcal{M} \rtimes (Sp(n) \times \mathbf{F}^*)$ where $\mathcal{M}$ is the quaternion Heisenberg group. It lies in the following exact sequence: $1 \longrightarrow \mathbf{R}^3 \longrightarrow \mathcal{M} \longrightarrow \mathbf{F}^n \longrightarrow 1$. For the point $\{\infty\}$ of S^{4n+3}, as we identify $S^{4n+3} - \{\infty\}$ with $\mathcal{M}$, $\mathbf{Aut}_{\mathbf{Fsp}}(\mathcal{M})$ is the stabilizer in $P\mathbf{Sp}(n+1,1)$ of the point $\{\infty\}$. Then $\mathbf{Aut}_{\mathbf{Fsp}}(\mathcal{M})$ is a maximal amenable subgroup of $P\mathbf{Sp}(n+1,1)$.

Let $\mathbf{Z}_2 \to \mathbf{Sp}(n+1,1) \xrightarrow{P} P\mathbf{Sp}(n+1,1)$ be the projection. Since G is as above, PG is isomorphic to $\mathbf{Aut}_{\mathbf{Fsp}}(\mathcal{M}) = \mathcal{M} \rtimes (Sp(n) \cdot Sp(1) \times \mathbf{R}^+)$.
Let

$$P : (\mathbf{Sp}(n+1,1), V_-^{n+2} \cup V_0^{4n+7}) \longrightarrow (P\mathbf{Sp}(n+1,1), \mathbf{H}_{\mathbf{F}}^{n+1} \cup S^{4n+3})$$

be the projection as before. For the point at infinity $\{\infty\} \in S^{4n+3}$, the stabilizer $PU(n+1,1)_\infty$ at $\{\infty\}$ is the image of the subgroup G by P. As above it is known that

G is a maximal amenable Lie group. Each $g \in G$ consists of

$$g = \begin{pmatrix} \lambda & x & z \\ 0 & \mathrm{B} & y \\ 0 & 0 & \mu \end{pmatrix}$$

where $\lambda, \mu \in \mathbf{F}^*$, B is an (n,n)-matrix. x is an n-th line vector, y is an n-th column vector. As $\mathcal{B}(gz, gw) = \mathcal{B}(z, w)$ for arbitrary $z, w \in \mathbf{F}^{n+2}$, we have the following relations.

$$(1) \qquad \lambda\bar{\mu} = 1, \quad x = \lambda^t\bar{y}B,$$

$$(2) \quad \bar{z}\mu + \bar{\mu}z = |y|^2, \quad B \in Sp(n).$$

Then the Heisenberg group $\mathcal{M}$ is denoted by the subgroup consisting of the following matrices;

$$\begin{pmatrix} 1 & x & z \\ 0 & \mathrm{I} & y \\ 0 & 0 & 1 \end{pmatrix}$$

for which

$$\mathrm{Re}\ z = \frac{|y|^2}{2}, \quad x = {}^t\bar{y}.$$

This follows from the relations (1), (2) that $\bar{z} + z = |y|^2$.

Thus, putting $z = \dfrac{|y|^2}{2} + \mathrm{i}a + \mathrm{j}b + \mathrm{k}c$, there is a one-to-one correspondense between the product $\mathbf{R}^3 \times \mathbf{F}^n$ and this group:

$$((a,b,c),y) = \begin{pmatrix} 1 & {}^t\bar{y} & \frac{|y|^2}{2} + \mathrm{i}a + \mathrm{j}b + \mathrm{k}c \\ 0 & \mathrm{I} & y \\ 0 & 0 & 1 \end{pmatrix}$$

The group law on the product $\mathbf{R}^3 \times \mathbf{F}^n$ is obtained similarly; thus the Heisenberg Lie group $\mathcal{M}$ is the product $\mathbf{R}^3 \times \mathbf{F}^n$ with group law

$$(a, y) \cdot (b, y') = (a + b + \mathrm{Im} < y, y' >, y + y').$$

$\mathcal{M}$ is nilpotent because $[\mathcal{M}, \mathcal{M}] = \mathbf{R}^3$ which is the center consisting of the form $(a, 0)$. As above, $\mathcal{M} \rtimes (Sp(n) \times \mathbf{F}^*)$ is the semidirect product for which the action of $Sp(n) \times \mathbf{F}^*$ on $\mathcal{M}$ is given by

$$(*) \qquad (A, \nu) \circ (a, y) = (|\nu|^{-2}\nu a \nu^{-1}, Ay\nu^{-1}).$$

Finally, since $\mathbf{Aut}_{\mathbf{F}\mathbf{Sp}}(\mathcal{M}) = PG$, where $\mathbf{Z}/2 \rightarrow \mathbf{Sp}(n+1,1) \xrightarrow{P} P\mathbf{Sp}(n+1,1)$ is the projection, PG is isomorphic to $\mathcal{M} \rtimes (Sp(n) \cdot Sp(1) \times \mathbf{R}^+)$, for which the action of $Sp(n) \cdot Sp(1) \times \mathbf{R}^+$

on $\mathcal{M}$ is given as follows: if $\nu = (g,t)$, then

$$(A,(g,t)) \circ (a,y) = (t^{-2}gag^{-1}, t^{-1}Ayg^{-1}).$$

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