ON AFFINE COLLINEATIONS IN A SPACE OF HYPERPLANES

Hitosi HIRAMATU

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The geometry of paths initiated by L. P. Eisenhart and O. Veblen has been generalized by J. Douglas to the general geometry of paths and further to the geometry of K-spreads.

In a space of (n—1)-spreads, that is to say, in a space of hyperplanes, the elements of the space are points (x^i) and hyperplane elements (u_i) attached with each point. Thus, the geometry of hyperplanes can be studied in the form dual to the general geometry of paths. This was done recently by K. Yano and the present author [8]*.

The purpose of the present paper is to study affine collineations in such a space and to generalize some of theorems by K. Yano [7]. Corresponding problems in a space of K-spreads have been studied by R. S. Clark [1], E. T. Davies [2] and Buchin Su [3, 4, 5, 6].

We shall use, throughout the paper, the same notations as those in our paper cited above.

1. In an N-dimensional space of system of hyperplanes, referred to a coordinate system (x^i), we consider an infinitesimal point transformation:

$$\bar{x}^i = x^i + \xi^i(x)dt \quad (a, b, c, \cdots\cdots, i, j, k, \cdots\cdots = 1, 2, \cdots\cdots, N), \tag{1.1}$$

where $\xi^i(x)$ is a contravariant vector field and dt an infinitesimal constant. When there is given a geometric object $\Omega(x, u)$, if we regard (1.1) as a transformation of coordinates of the space, the components $\bar{\Omega}(\bar{x}, \bar{u})$ of the object in the new coordinate system will be calculated from the old components $\Omega(x, u)$ by (1.1). Then, the Lie-derivative $X\Omega$ of $\Omega(x, u)$ with respect to $\xi^i(x)$ is defined by the equation

$$\mathrm{D}\Omega = \Omega(\bar{x}, \bar{u}) - \bar{\Omega}(\bar{x}, \bar{u}) = \mathrm{X}\Omega dt, \tag{1.2}$$

neglecting the terms in dt higher than of the first order.

From this definition, we can find the Lie-derivatives of various quantities of the space. To express these Lie-derivatives in tensor forms, we define the covariant derivatives of a scalar f and that of a tensor field $T^{i_1 \cdots i_r}_{j_1 \cdots j_s}$ by

$$f_{;j} = f_{,j} + f/^a \Gamma_{aj}, \tag{1.3}$$

$$T^{i_1 \cdots i_r}_{j_1 \cdots j_s ;k} = T^{i_1 \cdots i_r}_{j_1 \cdots j_s , k} + T^{i_1 \cdots i_r}_{j_1 \cdots j_s}/^a \Gamma_{ak} + \sum_{p=1}^{r} T^{i_1 \cdots i_{p-1} a i_{p+1} \cdots\cdots i_r}_{j_1 \cdots\cdots\cdots\cdots\cdots\cdots j_s} \Gamma^{i_p}_{ak} - \sum_{p=1}^{s} T^{i_1 \cdots\cdots\cdots\cdots\cdots\cdots i_r}_{j_1 \cdots j_{p-1} a j_{p+1} \cdots j_s} \Gamma^a_{j_p k} \tag{1.4}$$

respectively, where a comma followed by an index denotes the partial differentiation with respect to x^i and vertical stroke that with respect to u_i.

Thus, the Lie-derivatives of a scalar f, of a tensor field $T^{i_1 \cdots i_r}_{j_1 \cdots j_s}$ and of the compo-

* See the Bibliography at the end of the paper.

nents of affine connection Γ^{i}_{jk} are given by

(1.5) $$Xf=f_{;a}\xi^{a}-f/^{a}\xi^{b}_{;a}u_{b},$$

(1.6) $$XT^{i_1\cdots i_r}_{j_1\cdots j_s}=T^{i_1\cdots i_r}_{j_1\cdots j_s;a}\xi^{a}-\sum_{p=1}^{r}T^{i_1\cdots i_{p-1}ai_{p+1}\cdots i_r}_{j_1\cdots\cdots\cdots\cdots\cdots j_s}\xi^{i_p}_{;a}$$
$$+\sum_{p=1}^{s}T^{i_1\cdots\cdots\cdots\cdots\cdots i_r}_{j_1\cdots j_{p-1}aj_{p+1}\cdots j_s}\xi^{a}_{;j_p}-T^{i_1\cdots i_r}_{j_1\cdots j_s}/^{a}\xi^{b}_{;a}u_{b}$$

and

(1.7) $$X\Gamma^{i}_{jk}=\xi^{i}_{;j;k}+R^{i}_{jkl}\xi^{l}-\Gamma^{i}_{jk}/^{a}\xi^{b}_{;a}u_{b}$$

respectively. We shall remark here that the Lie-derivatives of these quantities are all components of tensors and that $Xu_j=0$.

We can see that the Ricci formulae for an arbitrary tensor field $T^{i_1\cdots i_r}_{j_1\cdots j_s}$:

(1.8) $$T^{i_1\cdots i_r}_{j_1\cdots j_s;k;l}-T^{i_1\cdots i_r}_{j_1\cdots j_s;l;k}=\sum_{p=1}^{r}T^{i_1\cdots i_{p-1}ai_{p+1}\cdots i_r}_{j_1\cdots\cdots\cdots\cdots\cdots j_s}R^{i_p}_{akl}$$
$$-\sum_{p=1}^{s}T^{i_1\cdots\cdots\cdots\cdots\cdots i_r}_{j_1\cdots j_{p-1}aj_{p+1}\cdots j_s}R^{a}_{j_pkl}+T^{i_1\cdots i_r}_{j_1\cdots j_s}/^{a}R_{akl},$$

(1.9) $$T^{i_1\cdots i_r}_{j_1\cdots j_s;k}/^{l}-T^{i_1\cdots i_r}_{j_1\cdots i_s}/^{l}{}_{;k}$$
$$=\sum_{p=1}^{r}T^{i_1\cdots i_{p-1}ai_{p+1}\cdots i_r}_{j_1\cdots\cdots\cdots\cdots\cdots j_s}\Gamma^{i_p}_{ak}/^{l}-\sum_{p=1}^{s}T^{i_1\cdots\cdots\cdots\cdots\cdots i_r}_{j_1\cdots j_{p-1}aj_{p+1}\cdots j_s}\Gamma^{a}_{j_pk}/^{l}$$

and the generalized Bianchi identities:

(1.10) $$(R^{i}_{jkl;m}-\Gamma^{i}_{jk}/^{a}R_{alm})+cyc\,(k,l,m)=0$$

hold good, where $cyc(k,l,m)$ denotes two terms obtained from the first by cyclic permutation of indices k,l and m.

By suitable use of the above formulae, we get the following formulae for an arbitrary tensor giving the relations between the Lie-operation and the covariant derivation or partial differentiation with respect to u_i:

(1.11) $$(XT^{i_1\cdots i_r}_{j_1\cdots j_s})_{;k}-XT^{i_1\cdots i_r}_{j_1\cdots j_s;k}=-\sum_{p=1}^{r}T^{i_1\cdots i_{p-1}ai_{p+1}\cdots i_r}_{j_1\cdots\cdots\cdots\cdots\cdots j_s}X\Gamma^{i_p}_{ak}$$
$$+\sum_{p=1}^{s}T^{i_1\cdots\cdots\cdots\cdots\cdots i_r}_{j_1\cdots j_{p-1}aj_{p+1}\cdots j_s}X\Gamma^{a}_{j_pk}-T^{i_1\cdots i_r}_{j_1\cdots j_s}/^{a}X\Gamma^{b}_{ak}u_{b},$$

(1.12) $$(XT^{i_1\cdots i_r}_{j_1\cdots j_s})/^{l}-XT^{i_1\cdots i_r}_{j_1\cdots j_s}/^{l}=0$$

and the similar formulae for the components of affine connection:

(1.13) $$(X\Gamma^{i}_{jk})_{;l}-(X\Gamma^{i}_{jl})_{;k}=XR^{i}_{jkl}-\Gamma^{i}_{jk}/^{a}X\Gamma^{b}_{al}u_{b}+\Gamma^{i}_{jl}/^{a}X\Gamma^{b}_{ak}u_{b},$$

(1.14) $$(X\Gamma^{i}_{jk})/^{l}-X\Gamma^{i}_{jk}/^{l}=0.$$

The above formulae are frequently used in the latter discussion.

2. If we consider, in the space, an r-parameter continuous group G_r whose symbols are given by $X_\mu f = f_{,a}\xi^a_\mu(x)$ $(\lambda, \mu, \nu, \cdots\cdots=1, 2, \cdots\cdots, r)$, then there exist the relations

(2.1) $$(X_\mu X_\nu)f = C_{\mu\nu}{}^\lambda X_\lambda f,$$

where $C_{\mu\nu}{}^\lambda$ are constants of structure of the group. From (2.1), we have

$$\xi^a_\mu \xi^i_{\nu,a} - \xi^a_\nu \xi^i_{\mu,a} = \xi^i_{\nu;a}\xi^a_\mu - \xi^i_{\mu;a}\xi^a_\nu = C_{\mu\nu}{}^\lambda \xi^i_\lambda,$$

from which, by the definition of the Lie-derivatives, we get

(2.2) $$X_\mu \xi^i_\nu = C_{\mu\nu}{}^\lambda \xi^i_\lambda.$$

Now, consider r infinitesimal transformations of the space

(2.3) $$\bar{x}^i = x^i + \xi^i_\mu(x)dt,$$

then we have, by straightforward calculations:

(2.4) $$(X_\mu X_\nu)f = f_{;a}X_\mu \xi^a_\nu - f/^a\,(X_\mu \xi^b_\nu)_{;a}u_b,$$

(2.5) $$(X_\mu X_\nu)T^{i_1\cdots i_r}_{j_1\cdots j_s} = T^{i_1\cdots i_r}_{j_1\cdots j_s;a}X_\mu\xi^a_\nu - \sum_{p=1}^{r} T^{i_1\cdots i_{p-1}a i_{p+1}\cdots i_r}_{j_1\cdots\cdots\cdots\cdots\cdots\cdots j_s}(X_\mu\xi^{i_p}_\nu)_{;a}$$
$$+\sum_{p=1}^{s} T^{j_1\cdots\cdots\cdots\cdots\cdots\cdots i_r}_{i_1\cdots j_{p-1}aj_{p+1}\cdots j_s}(X_\mu\xi^a_\nu)_{;j_p} - T^{i_1\cdots i_r}_{j_1\cdots j_s}/^a(X_\mu\xi^b_\nu)_{;a}u_b,$$

(2.6) $$(X_\mu X_\nu)\Gamma^i_{jk} = (X_\mu\xi^i_\nu)_{;j;k} + R^i_{jkl}(X_\mu\xi^l_\nu) - \Gamma^i_{jk}/^a(X_\mu\xi^b_\nu)_{;a}u_b.$$

Thus, we have

Theorem 2.1. If we apply the operators $(X_\mu X_\nu)$ *to an arbitrary scalar, tensor and the components of affine connection, then we get their Lie-derivatives with respect to the vectors* $X_\mu\xi^i_\nu$.

If the $\xi^i_\mu(x)$ are r vectors defining an r-parameter group G_r then (2.4), (2.5) and (2.6) are reduced to

(2.7) $$(X_\mu X_\nu)f = C_{\mu\nu}{}^\lambda X_\lambda f,$$

(2.8) $$(X_\mu X_\nu)T^{i_1\cdots i_r}_{j_1\cdots j_s} = C_{\mu\nu}{}^\lambda X_\lambda T^{i_1\cdots i_r}_{j_1\cdots j_s}$$

and

(2.9) $$(X_\mu X_\nu)\Gamma^i_{jk} = C_{\mu\nu}{}^\lambda X_\lambda \Gamma^i_{jk},$$

respectively, and we have

Theorem 2.2. If r *vectors* $\xi^i_\mu(x)$ *define an r-parameter group, then the expressions similar to* (2.1) *for an arbitrary scalar, tensor and the components of affine connection hold.*

3. In a space of system of hyperplanes, a point transformation in the space with the finite equations

(3.1) $$\bar{x}^i = \bar{x}^i(x^1, x^2, \cdots\cdots, x^N) \qquad \left|\frac{\partial\bar{x}}{\partial x}\right| \neq 0$$

is said to be a collineation, if it transforms each hyperplane into a hyperplane. Under this point transformation, the equations of hyperplanes

$$u_i\,dx^i = 0 \qquad du_j - \Gamma_{jk}(x, u)dx^k = 0 \qquad (u_i = \partial f/\partial x^i)$$

are transformed into

(3.2) $$\bar{u}_i\,d\bar{x}^i = 0 \qquad d\bar{u}_j - \Gamma_{jk}(\bar{x}, \bar{u})d\bar{x}^k = 0 \qquad (\bar{u}_i = \partial f/\partial \bar{x}^i).$$

On the other hand, from (3.1) regarded as a transformation of coordinates, the equations of hyperplanes are reducible to

(3.3) $$\bar{u}_i\,d\bar{x}^i = 0 \qquad d\bar{u}_j - \bar{\Gamma}_{jk}(\bar{x}, \bar{u})d\bar{x}^k = 0 \qquad (\bar{u}_i = \partial f/\partial \bar{x}^i).$$

We shall call the transformation (3.1) an affine collineation, if the equations (3.2) and (3.3) are equivalent under a suitable change of constant factor of f defining the hyperplanes. Thus, for affine collineation, we must have $\Gamma_{jk}(\bar{x}, \bar{u}) - \bar{\Gamma}_{jk}(\bar{x}, \bar{u}) = 0$, from which, differentiating these equations with respect to $\bar{u}_i$, we have

(3.4) $$\Gamma^{\,i}_{jk}(\bar{x}, \bar{u}) - \bar{\Gamma}^{\,i}_{jk}(\bar{x}, \bar{u}) = 0.$$

Conversely, as is easily seen, if this condition is satisfied, then the point transformation (3.1) defines an affine collineation.

From the above discussion, when an infinitesimal point transformation (1.1) defines an infinitesimal affine collineation, we have, from (3.4),

(3.5) $$X\Gamma^{\,i}_{jk} = \xi^i_{\,;j;k} + R^{\,i}_{jkl}\,\xi^l - \Gamma^{i}_{jk}{}/^{a}\,\xi^b_{\,;a}u_b = 0.$$

Conversely, if this condition is satisfied, the tensor equations (3.5) become $\partial\Gamma^i_{jk}/\partial x^1 = 0$ in a suitable coordinate system such that $\xi^i = \delta^i_1$, and consequently $\Gamma^i_{jk}(\bar{x}.\,\bar{u}) = \Gamma^i_{jk}(x, u)$. On the other hand, from the transformation law of Γ^i_{jk}, we have $\bar{\Gamma}^i_{jk}(\bar{x}, \bar{u}) = \Gamma^i_{jk}(x, u)$. Thus, the equations (3.4) hold. We have

Theorem 3.1. A necessary and sufficient condition that an infinitesimal transformation (1.1) be an infinitesimal affine collineation is that the Lie-derivatives of the components of the affine connection vanish.

From (1.11), we have

Theorem 3.2. In order that an infinitesimal transformation (1.1) be an infinitesimal affine collineation, it is necessary and sufficient that the Lie-operation and the covariant operation be interchangeable.

As is seen from the proof of theorem 3.1, when the space admits an infinitesimal affine collineation, there exists a coordinate system such that $\partial\Gamma^i_{jk}/\partial x^1 = 0$. If this condition is satisfied, the finite equations $\bar{x}^i = x^i + \delta^i_1\,t$ satisfy (3.4), in other words, the space admits a one-parameter group of affine collineations generated by the infinitesimal affine collineation $\bar{x}^i = x^i + \delta^i_1\,dt$.

If we take a coordinate system for which $\xi^i = x^i$, the equations (3.5) become $\Gamma^i_{jk.a}x^a + \Gamma^i_{jk} = 0$. From this condition, we can see that our space admits a one-parameter group of affine collineations $\bar{x}^i = x^i\,t$ generated by $\bar{x}^i = x^i + x^i dt$. Thus we have

Theorem 3.3. If the space admits an infinitesimal affine collineation, the space *admits also a one-parameter group of affine collineations generated by the infinitesimal affine collineation.*

Theorem 3.4. A necessary aud sufficient condition that the space admit a one-parameter group of affine collineations is that there exist a coordinate system in which $\partial\Gamma^i_{jk}/\partial x^1=0$ *or* $\Gamma^i_{jk,a}x^a+\Gamma^i_{jk}=0$.

The following theorem can also be easily proved:

Theorem 3.5. If $X_\mu f$ *are the symbols of r infinitesimal affine collineations, then* $C^\lambda X_\lambda f$ *is also that of an infinitesimal affine collineation, where* C^λ *are arbitrary constants not all zero.*

From theorem 2.1 and theorem 3.3, we have

Theorem 3.6. If $\xi^i_\mu(x)$ *are the vectors defining r one-parameter groups of affine collineations, then so are* $X_\mu\xi^i_\nu$.

When $X_\mu f$ are the generators of an r-parameter group G_r, the transformations of this group consist of the transformations of one-parameter group generated by the infinitesimal transformation with the symbol $C^\lambda X_\lambda f$ (C^λ: constants) and of the products of such transformations. Therefore, from theorem 3.5, we have

Theorem 3.7. If each of r generators of an r-parameter group G_r *is a generator of a one-parameter group of affine collineations, every transformation of* G_r *is also an affine collineations.*

From Lie's second fundamental theorem and theorem 3.7, we have

Theorem 3.8. If $X_\mu f$ *are generators of a complete set of one-parameter groups of affine collineations, they are generators of an r-parameter group of affine collineations.*

4. We shall find the integrability conditions of the equations

$$X\Gamma^i_{jk}=\xi^i_{;j;k}+R^i_{jkl}\xi^l-\Gamma^i_{jk}/^a\,\xi^b_{;a}u_b=0. \tag{4.1}$$

(4.1) are written in the standard form:

$$\xi^i_{;j}=\eta^i_j,\quad \xi^i/^k=0,\quad \eta^i_{j;k}=-R^i_{jkl}\xi^l+\Gamma^i_{jk}/^a\eta^b_a u_b,\quad \eta^i_j/^k=\xi^a\,\Gamma^i_{ja}/^k \tag{4.2}$$

with $N(N+1)$ unknown functions ξ^i and η^i_j, independent variables being x^i and u_j; the integrability conditions can be obtained in the usual way. But, using the formulae already obtained, we can find these conditions as follows:

If we substitute (4.1) into (1.13) and (1.14), then we get the conditions

$$XR^i_{jkl}=0 \tag{4.3}$$

and

$$X\Gamma^i_{jk}/^l=0 \tag{4.4}$$

respectively.

Our conditions are (4.3), (4.4) and those obtainable from them by repeated covariant differentiation and partial differentiation with respect to u, For this purpose, we shall use freely (1.11) and (1.12), that is to say, the fact that, under the condition (4.1), the Lie-operation is interchangeable with both covariant differentiation and partial differentiation with respect to u_j.

We first consider the condition (4.4). By repeated differentiation with respect to u, we have a set of conditions

(4.5) $$X\Gamma^{i}_{jk}/^{l_1}/^{l_2}/\cdots\cdots/^{l_s} = 0 \quad (s=1, 2, \cdots\cdots),$$

but, from the relations $Xu_j=0$ and the homogeneity properties of the left hand sides of the above equations with respect to u, the final equations of (4.5) include all the preceding ones. Thus, we can rewrite (4.5) as

(4.6) $$X\Gamma^{i}_{jk}/^{l_1}/^{l_2}/\cdots\cdots/^{l_s} = 0.$$

Differentiating (4.6) covariantly, we have a set of conditions

(4.7) $$X\Gamma^{i}_{jk}/^{l_1}/^{l_2}\cdots\cdots/^{l_s}{}_{;m_1;m_2;\cdots\cdots;m_t} = 0 \quad (t=1, 2, \cdots\cdots).$$

We consider the partial derivatives of (4.7) for $t=1$ with respect to u_n. If we apply the formula (1.9) to the tensor $\Gamma^{i}_{jk}/^{l_1}/^{l_2}/\cdots\cdots/^{l_s}$ and next apply the Lie-operation to the resulting equations, we can see that

$$(X\Gamma^{i}_{jk}/^{l_1}/^{l_2}/\cdots\cdots/^{l_s}{}_{;m_1})/^{n} = X\Gamma^{i}_{jk}/^{l_1}/^{l_2}/\cdots\cdots/^{l_s}{}_{;m_1}/^{n}$$

are expressible in terms of $X\Gamma^{i}_{jk}/^{n}$, $X\Gamma^{i}_{jk}/^{l_1}/^{l_2}/\cdots\cdots^{l_s}$ and $X\Gamma^{i}_{jk}/^{l_1}/^{l_2}\cdots\cdots/^{l_s}/^{n}{}_{;m_1}$ which is the same type as the left hand side of (4.7) for $t=1$. Thus, the partial derivatives of (4.7) for $t=1$ with respect to u_n give no new conditions. This argument can be repeated for $t=2, 3, \cdots\cdots$.

Next, we consider (4.3). The equations $XR^{i}_{jkl}/^{m} = 0$ obtained from (4.3) by the partial differentiation with respect to u_m give no new conditions. For, if we apply the Lie-operation to both sides of the identities

$$R^{i}_{jkl}/^{m} = \Gamma^{i}_{jk}/^{m}{}_{;l} - \Gamma^{i}_{jl}/^{m}{}_{;k},$$

then we can see that $(XR^{i}_{jkl})/^{m} = XR^{i}_{jkl}/^{m}$ are the consequences of (4.7).

By repeated covariant derivation, we have a set of new conditions

(4.8) $$XR^{i}_{jkl;m_1;m_2;\cdots\cdots;m_t} = 0 \quad (t=1, 2, \cdots\cdots).$$

By the similar method as in the precedent case, the partial derivatives of (4.8) for $t=1$ with respect to u_n give no new conditions. For, if we apply the formula (1.9) to the tensor R^{i}_{jkl} and next apply the Lie-operation to the resulting expressions, it can be seen that $(XR^{i}_{jkl;m_1})/^{n} = XR^{i}_{jkl;m_1}/^{n}$ are expressible in terms of $X\Gamma^{i}_{jk}/^{n}$, XR^{i}_{jkl} and $XR^{i}_{jkl}/^{n}{}_{;m_1}$, but the expressions $RX^{i}_{jkl}/^{n}{}_{;m_1} = 0$ are the consequences of (4.7) for t=2.

Thus, our integrability conditions are that a set of equations (4.3), (4.6), (4.7) and (4.8) is algebraically consistent in ξ^i and $\xi^i_{;j}$. We shall remark here that the solutions ξ^i of (4.1) or (4.2) do not contain the variables u_i.

Theorem 4.1. A necessary and sufficient condition that the space of hyperplanes admit an infinitesimal affine collineation is that

$$(4.9)\quad \begin{cases} X\Gamma^{i}_{jk}/^{l_1}/^{l_2}/\cdots\cdots/^{l_s}{}_{;m_1;m_2;\cdots\cdots;m_p} = 0 & (p=0,1,2,\cdots\cdots) \\ XR^{i}_{jkl;m_1;m_2;\cdots\cdots;m_t} = 0 & (t=0,1,2.\cdots\cdots) \end{cases}$$

be algebraically consistent in ξ^i *and* $\xi^i_{;j}$.

The left hand sides of the integrability conditions are linear homogeneous in ξ^i and $\xi^i_{;j}$, so that we have

Theorem 4.2. A necessary and sufficient condition that the space admit r linearly independent infinitesimal affine collineations is that there exist $N(N+1)-r$ *independent equations in (4.9).*

Furthermore, an arbitrary solution of (4.1) is a linear combination (with constant coefficients) of such r vectors $\xi^i_\lambda(x)$. From theorems 2.1 and 3.1, $X_\mu\xi^i_\nu$ are solutions of (4.1), so that they are linear combineations of $\xi^i_\nu(x)$ with constant coefficients. Thus, from theorem 3.8, we have

Theorem 4.3. A necessary and sufficient condition that the space admit an r-parameter group of affine collineations is that there exist $N(N+1)-r$ *independent solutions in (4.9).*

If the equations (4.1) are completely integrable, then $X\Gamma^i_{jk}/^l=0$ and $XR^i_{jkl}=0$ must be identities in ξ_i and $\xi^i_{;j}$, and consequently $\Gamma^i_{jk}/^l=0$ and $R^i_{jkl}=0$. Thus, we have

Theorem 4.4. In order that the space admit a group of infinitesimal affine collineations of maximum order, it is necessary and sufficient that the space be affinely flat.

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