ON RELATIONS AMONG VARIOUS CONNECTIONS IN FINSLERIAN SPACE

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1. Introduction.

In Finslerian space F_n Berwald's[1] and Cartan's[2] connections are most famous. Recently. H. Rund[3] determined another linear connection in Finslerian space F_n introbucing four convensions, as is shown below.

Cartan's and Rund's connections are metric, but Berwald's is not. Cartan gave the relation between the coefficients of his and Berwald's connections. The connection determined by Rund's convensions A, B, C and D is not unique but its coefficients are of the form (11').

We shall show that if Berwald's connection is metrised, it is coincident with Cartan's, during a supporting element is displaced parallelly to itself.

In Finslerian space F_n two metric connections being introduced, we shall also show that there are some relations among their coefficients. Now we adjoint a tangential Minkowskian space $T_n(P)$ to every point $P(x^i)$ in Finslerian space F_n. Rund observed two points $A(x^i)$, $B(x^i+dx^i)$ and a vector field $X^i(x)$ in F_n. These points A, B decide an unique geodesic E, which passes the both points. He seeked such a vector $X^i+d^*X^i$ in $T^n(B)$ that would satisfy the following conditions and defined a covariant differentiation by $D=d-d^*$:

A. *The scalar products of the vectors* X^i *and* $X^i+d^*X^i$ *with respect to the tangent vectors in* A *and* B *on* E *are identical.*

B. $dX^i - dX^i = 0$, *when* $X^i = x'^i$

C. d^*X^i *are linear in* X^k.

D. $Dg_{ij}(x, x') = 0$, *where* $x'^i ds = dx^i$ *express the direction of displacement* AB *along the geodesic* E.

We shall also determine a metric connection with some uncertainties introducing three convensions analogous to those introduced by Rund. In our case the convension which corresponds to the convension C in Rund's connection is needles, because it is reduced from

1) See [1]. Numbers in brackets refer to the bibliography at the end of the paper.
2) See [2].
3) See [3], [4].
4) See [2]. VIII.

the convensions I and II. His results are different from ours.

2. Metrisation of Berwald's connection.

As is well known, we have the following relations between the coefficients G^i_{jk} of Berwald's connection and those of Cartan's connection which enter in the expressions of the absolute differential of a vector when its supporting element displaces parallelly to itself:

$$G_{ihj} = \Gamma^*_{ihj} + A_{ihj|o} . \tag{1}$$

Berwald's connection being not metric, if we metrise it we obtain the corresponding coefficients as follows:

$$\begin{aligned}
\hat{\omega}^i_j &= \omega^i_j + \tfrac{1}{2}\,\delta g_{jr}\, g^{ir} \\
&= G^i_{jk}\, dx^k + \tfrac{1}{2}(dg_{jr} - G^l_{jk}\, g_{lr}\, dx^k - G^l_{rk}\, g_{jl}\, dx^k)\, g^{ir} \\
&= \tfrac{1}{2}(\Gamma^{*i}_{jk} + A^i_{jk|o})\, dx^k + \tfrac{1}{2}\frac{\partial g_{jr}}{\partial x'^k}\, g^{ir}\, dx'^k + \tfrac{1}{2}\frac{\partial g_{jr}}{\partial x^k}\, g^{ir}\, dx^k \\
&\qquad - \tfrac{1}{2}\, g^{ir}(\Gamma^*_{rjk} + A_{rjk|o})\, dx^k \\
&= C^i_{jk}\, dx'^k + \Gamma^i_{jk}\, dx^k ,
\end{aligned}$$

on account of $\Gamma^{*i}_{jk} = \Gamma^i_{jk} - C^i_{jr}\, G^r_k$, where we put $\omega^i_j = G^i_{jk}\, dx^k$ and denote an absolute differentiation in Berwald's sense by δ.

Thus we see that the coefficients $\hat{\omega}^i_j$ coincide with those of Cartan's connection.

3. Relations between two metric connections.

Suppose that in Finslerian space F_n two kinds of metric connections are defined in the following forms:

$$DX^i = dx + \omega^i_j X^j = dX^i + (C^i_{jk}\, dx'^k + \Gamma^i_{jk}\, dx^k)\, X^j \tag{2}$$

and

$$\hat{D}X^i = dx^i + \hat{\omega}^i_j X^j = dx^i + (\hat{C}^i_{jk}\, dx'^k + \hat{\Gamma}^i_{jk}\, dx^k)\, X^j \tag{3}$$

The differences

$$DX^i - \hat{D}X^i = X^j\left[(C^i_{jk} - \hat{C}^i_{jk})\, dx'^k + (\Gamma^i_{jk} - \hat{\Gamma}^i_{jk})\, dx^k\right]$$

are the components of a vector and so under a transformation of the coordinates the relations

(4) $$(C^i_{jk} - \hat{C}^i_{jk}) \frac{\partial x^k}{\partial \bar{x}^\alpha} \frac{\partial^2 \bar{x}^\alpha}{\partial x^l \partial x^m} x'^l + \frac{\partial x^i}{\partial \bar{x}^\alpha} (\bar{\Gamma}^\alpha_{\beta\gamma} - \hat{\bar{\Gamma}}^\alpha_{\beta\gamma}) \frac{\partial \bar{x}^\beta}{\partial x^j} \frac{\partial \bar{x}^\gamma}{\partial x^m} = \Gamma^i_{jm} - \hat{\Gamma}^i_{jm}$$

hold. By hypofhesis, as the both connections are metric, the relations

(5) $$C_{ijk} + C_{jik} = \frac{\partial g_{ij}}{\partial x'^k}, \quad \hat{C}_{ijk} + \hat{C}_{jik} = \frac{\partial g_{ij}}{\partial x^k}$$

hold, therefore C_{ijk} and $\hat{C}_{ijk}$ have the same symmetric parts with respect to the first two indices. Accordingly the differences $C_{ijk} - \hat{C}_{ijk}$ are the components of a skew symmetric tensor, which we denote by b_{ijk} , i. e.

(6) $$C_{ijk} - \hat{C}_{ijk} = b_{ijk} \text{ or } \hat{C}_{ijk} = C_{ijk} - b_{ijk}, \quad b_{(ij)k} = 0.$$

1°. when $C_{ijk} = C_{jik}$ and $\hat{C}_{jik} = \hat{C}_{ijk}$, we have $C_{ijk} = \hat{C}_{ijk}$. In this case the expressions $\Gamma^i_{jk} - \hat{\Gamma}^i_{jk}$ are the components of a tensor, as is seen from (4). Now we put

(7) $$\Gamma_{ijk} - \hat{\Gamma}_{ijk} = a_{ijk} ,$$

then the relations

(8) $$a_{ijk} = - a_{jik}$$

hold, because the both connections are metric by hypothesis.

2°. Suppose that one of the connections (2) and (3), say (2), is Cartan's and the other is general metric one. According to Cartan's symbols[5] we have the following transformation law:

(9) $$x'^m \frac{\partial^2 \bar{x}^\alpha}{\partial x^k \partial x^m} + G^\alpha_\beta \frac{\partial \bar{x}^\beta}{\partial x^k} = G^i_k \frac{\partial \bar{x}^\alpha}{\partial x^i}.$$

If we put the above relations (9) into (4), we get

$$\left(\Gamma^\alpha_{\beta\gamma} - \hat{\Gamma}^\alpha_{\beta\gamma} - b^\alpha_{\beta\tau} G^\tau_\gamma\right) \frac{\partial x^i}{\partial \bar{x}^\alpha} \frac{\partial \bar{x}^\beta}{\partial x^j} \frac{\partial \bar{x}^\gamma}{\partial x^k} = \Gamma^i_{jk} - \hat{\Gamma}^i_{jk} - b^i_{jl} G^l_k .$$

Putting

(10) $$T^i_{jk} = \Gamma^i_{jk} - \hat{\Gamma}^i_{jk} - b^i_{jl} G^l_k ,$$

we see that the expressions T^i_{jk} are the compressions of a tensor. Moreover the relations $T_{(ij)k} = 0$ hold. More generally we may use in place of G^i_j in (9) any expressions which are transformed by the transformation law as (9).

5) See [2].

From (10) and (4) we obtain the following:

$$
(11)\qquad \begin{cases} \hat{\Gamma}_{ijk} = \Gamma_{ijk} - b_{ijl}\, G^l_k - T_{ijk}\,, \quad T_{(ij)k} = 0, \\ \hat{C}_{ijk} = C_{ijk} - b_{ijk}\,, \quad b_{(ij)k} = 0, \end{cases}
$$

which are the relations among the coefficients of the two metric connections.

4. Another metric connection.

Now we consider a vector field $X^i\ (x, x')$ and two points $A\ (x^i)$ and $B\ (x^i + dx^i)$ in F_n, which an unique geodesic passes. We search for the vector $X^i + d^* X^i$ which satisfies the following convensions:

I. *The scalar products of the vectors* X^i *and* $X^i + d^* X^i$ *with respect to the tangent vectors in* A *and* B *on* E *are identical.*

II. $d^* X^i$ *are Paffian forms of* dx^k *and* dx'^k *linear in* X^k.

III. $Dg_{ij}\ (x, x') = 0$, *where we put* $DX^i = dX^i - d^* X^i$,

We define the absolute differential of X^i by DX^i. From I we have

$$
g_{ij}(x, x')\, x'^i X^j = g_{ij}(x + dx,\ x' + dx')(x'^i + dx'^i)(X^j + d^* X_j)
$$

or

$$
(12)\qquad \frac{\partial g_{ij}}{\partial x^k}\, x'^i X^j dx^k + g_{ij}(x, x')\, x'^i d^* X^j + g_{ij} X^j dx'^i = 0.
$$

By II for any vector X^i, $d^* X^i$ are expressed in the forms:

$$
(13)\qquad d^* X^i = -\, P^i_{jk}\ (x, x')\, dx'^k X^j - Q^i_{jk}\ (x, x') X^j dx^k\,,
$$

so that we have

$$
(14)\qquad DX^i = dX^i - d^* X^i = dx^i + (P^i_{jk}\, dx'^k + Q^i_{jk}\, dx^k)\, X^j.
$$

Considering that our connection is metric from III, if we put (13) into (12) we obtain:

$$
Q_{ilk}\, x'^i X^l dx^k - g_{ij}\, x'^i P^j_{lk}\, dx'^k X^l + g_{il}\, dx'^i X^l = 0,
$$

where $Q_{ilk} = g_{lh} Q^h_{ik}$. As the vector X^i is arbitrary we get

$$
(15)\qquad g_{il}\, dx'^i - P_{lik}\, dx'^k x'^i + Q_{ilk}\, x'^i dx^k = 0.
$$

On the other hand from III we have

$$
x'^i P_{ilk} + x'^i P_{lik} = 0
$$

or

$$
(16)\qquad -\, P_{lik}\ x'^i = P_{ilk}\ x'_i\,,
$$

where $P_{ilk} = g_{hl} P^h_{ik}$. When we put (16) into (15) and multiply the resulting equation by g^{mj} and contract with respect to the indices l and m, we have

(17) $$dx'^j + P^j_{ik}\, x'^i\, dx'^k + Q^j_{ik}\, x'^i\, dx^k = 0.$$

The left side of the above equation (17) is coincident with

$$dx'^j - \overset{*}{d}\, x'^j \equiv Dx'^j ,$$

therefore the equation (17) is nothing but the convension B in Rund's case, that is to say, the convension B in Rund's case follows from our three convensions I, II, and III.

In particular when the relation $dx'^i + G^i_j\, dx^j = 0$ holds, we get from (15)

(18) $$- G^i_j + g^{li} P_{lmk}\, G^k_j\, x'^m + Q^i_{mj} x'^m = 0.$$

When the equations (11) or

(11') $$Q_{ijk} = \Gamma_{ijk} - b_{ijl}\, G^l_k - T_{ijk} , \qquad P_{ijk} = C_{ijk} - b_{ijk}$$

are subsfituted in (18), we get

(19) $$T_{ilj}\, x'^i = 0,$$

owing to the relations $b_{lik} + b_{ilk} = 0$ and $\Gamma_{ilj}\, x'^i = G^r_j\, g_{rl}$. From (19) and the skew symmetricity of T_{ilj} with respect to first two indices, we get immediately

(20) $$T_{lij}\, x'^l = 0.$$

Therefore we have:

Given two arbitrary tensors b_{ijk} *and* T_{ijk} *satisfying the relations* $b_{(ij)k} = 0$, $T_{(ij)k} = 0$ *and* $T_{ijk}\, x'^i = 0$, *a metric connection in* F_n *is determined by* (11').

The coefficients Q^{*i}_{jk} which enter in the expressions of the absolute differential of a vector, when its supporting element is displaced parallelly to itself, are as follow

$$\begin{aligned} Q^{*j}_{ik} &= - P^j_{il}\, G^l_k + Q^j_{ik} \\ &= - (C^j_{il} - b^j_{il})\, G^l_k + \Gamma^j_{ik} - b^j_{il}\, G^l_k - T^j_{ik} \\ &= - C^j_{il}\, G_k + \Gamma^j_{ik} - T^j_{ik} = \Gamma^{*j}_{ik} - T^j_{ik} , \end{aligned}$$

where C^j_{il}, Γ^j_{ik} and Γ^{*j}_{ik} are Cartan's symbols.

Thus we see that Q^*_{ijk} are only different from Cartan's symbols Γ^*_{ijk} by a skew symmetric tensor T_{ijk} ,

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