NON EUCLIDEAN GEOMETRY IN FINSLER SPACES

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(Received January 10, 1952)

On a projectively connected space whose group of holonomy fixes a non-degenerate hyperquadric, some interesting results have been obtained by S. Sasaki and K. Yano [1]. On the other hand, T. Otsuki has reached to the same results. by a slightly different way [2].

This paper deals with the consideration for a general space with projective connextion.

1. In a projectively connected space, we use a *repère semi-naturel* $[R_o, R_j]$. The projective connexion is defined by

$$dR_o = (\Phi_k dx^k + \Psi_k dp^k)R_o + R_k dx^k,$$

$$dR_j = (\gamma^o_{jk} dx^k + D^o_{jk} dp^k)R_o + (\gamma^i_{jk} dx^k + D^i_{jk} dp^k)R_i. \tag{1.1}$$

Then the coefficients of connexion are usually defined by the following formulae:

$$\Gamma^o_{jk} = \gamma^o_{jk}, \qquad \Gamma^i_{jk} = \gamma^i_{jk} - \delta^i_j \Phi_k,$$

$$\Gamma^i_{ok} = \gamma^i_{ok} = \delta^i_k,$$

$$C^o_{jk} = D^o_{jk}, \qquad C^i_{jk} = D^i_{jk} - \delta^i_j \Psi_k,$$

$$C^i_{ok} = D^i_{ok} = 0.$$

Generally the coefficients of connexion are the functions of the x's. and the p's, in which (x^i, p^i) and $(x^i, \rho p^i)$ $(\rho \neq 0)$ are being the same element of the manifold. Accordingly, as the connextion is independent of ρ, we can see that Φ_k, γ^o_{jk}, γ^i_{jk} together with Γ^o_{jk} and Γ^i_{jk} are the functions of degree zero in the p's, and that Ψ_k, D^o_{ik}. D^i_{jk}, together with C^o_{jk} and C^i_{jk} are the functions of degree -1 in the p's. A coordinate transformation is given by

$$\bar{x}^i = \bar{x}^i(x^1, \cdots\cdots, x^n) \quad (i, j, k \cdots\cdots = 1, 2, \cdots\cdots, n), \tag{1.2}$$

where $\bar{x}^i$ are analytical functions of the x's and the functional determinant is different from zero. The coefficients of connexion Γ^o_{jk}, Γ^i_{jk}, C^o_{jk}, C^i_{jk} are transformed by (1.2) into

$$\bar{\Gamma}^o_{jk} = \frac{\partial x^a}{\partial \bar{x}^j}\frac{\partial x^b}{\partial \bar{x}^k}\Gamma^o_{ab} + \frac{\partial x^a}{\partial \bar{x}^j}\frac{\partial^2 x^b}{\partial \bar{x}^k \partial \bar{x}^c}\frac{\partial \bar{x}^c}{\partial x^d} p^\alpha C^o_{ab},$$

$$\bar{C}^o_{jk} = \frac{\partial x^a}{\partial \bar{x}^j}\frac{\partial x^b}{\partial \bar{x}^k} C^o_{ab}, \tag{1.3}$$

$$\bar{\Gamma}^i_{jk} = \frac{\partial \bar{x}^i}{\partial x^a}\left(\frac{\partial x^b}{\partial \bar{x}^j}\frac{\partial x^c}{\partial \bar{x}^k}\Gamma^a_{bc} + \frac{\partial^2 x^a}{\partial \bar{x}^j \partial \bar{x}^k} + \frac{\partial x^b}{\partial \bar{x}^j}\frac{\partial^2 x^c}{\partial \bar{x}^k \partial \bar{x}^d}\frac{\partial \bar{x}^d}{\partial x^e} p^e C^a_{bc}\right),$$

$$\overline{C}^i_{jk} = \frac{\partial \overline{x}^i}{\partial x^a} \frac{\partial x^b}{\partial \overline{x}^j} \frac{\partial x^c}{\partial \overline{x}^k} C^a_{bc}.$$

Furthermore, we must consider a transformation of the *repère semi-naturel* $[R_o, R_j]$, namely:

(1.4) $$\overline{R}_o = R_o\ , \qquad \overline{R}_j = R_j + \lambda_j R_o \qquad (\lambda_j \text{ is a vector}),$$

which is usually called a transformation of the hyperplane at infinity. Thne Γ^0_{jk}, Γ^i_{jk}, C^0_{jk} C^i_{jk} are transformed by (1.4) into

$$\overline{\Gamma}^o_{jk} = \Gamma^o_{jk} + \frac{\partial \lambda_j}{\partial x^k} - \lambda_i \Gamma^i_{jk} - \lambda_j \lambda_k,$$

$$\overline{C}^o_{jk} = C^o_{jk} + \frac{\partial \lambda_j}{\partial p^k} - \lambda_i C^i_{jk},$$

(1.5) $$\overline{\Gamma}^i_{jk} = \Gamma^i_{jk} + \delta^i_j \lambda_k + \delta^i_k \lambda_j,$$

$$\overline{C}^i_{jk} = C^i_{jk}.$$

The proofs of (1.3). and (1.5) are given by K. Yano [3].

In the general space with the projective connexion (1.1), if the group of holonomy fixes na non degenerate hyperquadric at a tangential point x^i with the equation

(1.6) $$Q_{n-1}: \qquad a_{\lambda\mu} X^\mu X^\lambda = 0 \qquad (det\,|a_{\lambda\mu}| \neq 0,\ a_{\lambda\mu} = a_{\lambda\mu})$$

where $X_{\mu\lambda}$ being a projective tensor and X_λ a projective vector, then we must have the condition

$$D\,(a_{\lambda\mu} X^\lambda X^\mu) = \tau\, a_{\lambda\mu} X^\lambda X^\mu,$$

where we denote by D the covariant differential and by τ a scalar factor. Since $DX_\lambda = 0$ for an arbitrary projective vector we have the following equations:

$$(Da_{\lambda\mu}) X^\lambda X^\mu = \tau\, a_{\lambda\mu} X^\lambda X^\mu.$$

Therefore , a necessary and sufficient condition in order that the group of holonomy fixes the hyperquadric Q_{n-1} is as follows:

(1.7) $$Da_{\lambda\mu} = \tau a_{\lambda\mu} \quad (\tau = \tau_k dx^k + \tau'_k dp^k).$$

This is also expressed in the following form:

$$\frac{\partial a_{\lambda\mu}}{\partial x^k} - \gamma^\nu_{\mu k} a_{\lambda\nu} - \gamma^\nu_{\lambda k} a_{\mu\nu} = \tau_k a_{\lambda\mu},$$

(1.8) $$\frac{\partial a_{\lambda\mu}}{\partial p^k} - D^\nu_{\mu k} a_{\nu\lambda} - D^\nu_{\lambda k} a_{\mu\nu} = \tau'_k a_{\lambda\mu},$$

When $a_{oo} \neq 0$, we can put, without loss of generality,

(1.9) $$a_{oo} = \varepsilon\ (= \pm 1).$$

Hence putting $\lambda=0$, $\mu=0$ in (1.8) we get the following relations

$$\tau_k+2\Phi_k=-2\varepsilon a_k, \quad (a_k=a_{ok}=a_{ko}),$$
(1.10)
$$\tau'_k=-2\Psi_k.$$

Furthermore, putting $\lambda=0$, $\mu=i$ in (1.8), we get the following relations:

$$\frac{\partial a_j}{\partial x^k}-\Phi_k a_j-\gamma^o_{jk}\varepsilon-a_{jk}-\gamma^l_{jk}a_l=\tau'_k a_j,$$

$$\frac{\partial a_j}{\partial p^k}-\Psi_k a_j-C^o_{jk}\varepsilon-C^l_{jk}a_l=\tau'_k a_j.$$

The above relations and (1.9) show us that

$$\Gamma^o_{jk}+\frac{\partial}{\partial x^k}(-\varepsilon a_j)-(-\varepsilon a_l)\Gamma^l_{jk}-(-\varepsilon a_j)(-\varepsilon a_k)=-\varepsilon g_{jk},$$
(1.11)
$$C^o_{jk}+\frac{\partial}{\partial p^k}(-\varepsilon a_j)-(-\varepsilon a_l)\Gamma^l_{jk}=0,$$

where $g_{ij}=a_{ij}-\varepsilon a_i a_j$.

Similarly, putting $\lambda=i$, $\mu=j$, we obtain from (1.10) and (1.11) the following results:

$$\frac{\partial g_{ij}}{\partial x^k}-\overline{\Gamma}^l_{ik}g_{jl}-\overline{\Gamma}^l_{jk}g_{il}=0,$$
(1.12)
$$\frac{\partial g_{ij}}{\partial p^k}-\overline{C}^l_{ik}g_{jl}-\overline{C}^l_{jk}g_{il}=0.$$

where $\overline{\Gamma}^i_{jk}=\gamma^i_{jk}-\delta^i_j\Phi_k+(-\varepsilon a_j)\delta^i_k+(-\varepsilon a_k)\delta^i_j$ and $\overline{C}^i_{jk}=D^i_{jk}-\delta^i_j\Psi_k$. Now we can put $g_{il}\overline{C}^l_{jk}=\overline{C}_{jik}$, and assume the followig condition:

(i) *$\overline{C}_{jik}$ are symmetric with respect to i, j, k.*

Then we get from the second equation of (1.12),

$$\frac{\partial g_{ij}}{\partial p^k}=\overline{C}_{ijk},$$

so that the differential equations

(1.13)
$$\frac{\partial^2\overline{F}}{\partial p^i\partial p^j}=g_{ij}$$

are completely integrable by the reason of (i)

If a solution of (1.13) satisfies the following conditions

(ii) $\overline{F}$ $(x^i, p^i)>0$ *or* <0 *for every* $p^1, p^2, \cdots\cdots, p^n$ *not all zero,*

(iii) $\overline{F}$ *is a homogeneous function of degree* 2 *in the* p*'s,*

then, following E. Cartan [4] we can introduce the connexion of the Finsler space whose fundamental metric function is given by $F=k\sqrt{\overline{F}(x^i, p^i)}$ (k being a constant) and the coefficients of connexion are given by the following formula:

$$\bar{\Gamma}^{i}_{jk} = \left\{ {i \atop jk} \right\} - g^{im} \left(\bar{C}_{jkr} \frac{\partial G^r}{\partial p^m} - \bar{C}_{mjr} \frac{\partial G^r}{\partial p^k} \right),$$

where $\left\{ {i \atop jk} \right\}$ is the Christoffel symbol and G^r are given by

$$G_h = g_{hr} G^r = \tfrac{1}{2} \left(\frac{\partial^2 F}{\partial p^h \partial x^k} p^k - \frac{\partial F}{\partial x^h} \right).$$

By these arguments, we have the following

Theorem. In a general space with projective connexion, when the group of holonomy fixes any non-degenerate hyperquadric, the coefficients of the projective connexion are given by those of a Finsler space.

2. In this paragraph, following S. Sasaki [5], we show that a metric can be defined for a general space with projective connexion whose group of holonomy fixes a hyperquadric $a_{\mu\lambda} X_\lambda X_\mu = 0$.

We consider an arbitrary continuous curve $x^i = x^i(t)$. The arc-length ds of this curve is defined by

$$\cos \frac{ds}{k} = \frac{1}{\sqrt{a_{\lambda\mu} u^\mu u^\lambda}}, \qquad (k \neq 0)$$

where k is an arbitrary constant and u_λ are defined as follows:

$$u^o = \varepsilon \left(1 - a_{oj}\left(x, \frac{dx}{dt}\right) dx^j\right), \quad u^k = dx^k,$$

where $dx^k = \frac{dx^k}{dt} dt$.

From this definition of the metric, we can easily find by simple calculation the following result:

$$\cos \frac{ds}{k} = \frac{1}{\sqrt{\varepsilon + g_{ij}\left(x, \frac{dx}{dt}\right) dx^i dx^j}}. \tag{2.1}$$

If $g_{ij}\left(x, \frac{dx}{dt}\right)$ is positive definite, we can put $\varepsilon = 1$ and get, by expanding (2.1) in power series,

$$ds^2 = k^2 g_{ij}\left(x, \frac{dx}{dt}\right) dx^i dx^j.$$

If $g_{ij}\left(x, \frac{dx}{dt}\right)$ is negative definite, then by putting $\varepsilon = -1$. the imaginary distance is defined by (2.1).

3. We have discussed the projective connexion of the general space whose group of holonomy fixes the hyperquadric Q_{n-1}. It is obvious that the Q_{n-1} is expressible by

$$\varepsilon (X^o)^2 + g_{ij} X^i X^j = 0.$$

The projective connexion is then expressed by the following formulae:

$$dR_0 = R_i\, dx^i,$$

$$dR_j = g_{jk} dx^k R_0 + (\overline{\Gamma}^i_{jk} dx^k + \overline{C}^i_{jk} dp^k) R_i.$$

S. Sasaki formerly studied the relation between the metric of a projectively (or conformally) connected space whose group of holonomy fixes a non-degenerate hyperquadric and the non-euclidean geometry with a hyperquadric as the absolute figure [6].

We can prove that there is the same relation, in other words, the Klein's representation is applicable to the hyperquadric (as the absolute figure) and to the metric which we have introduced in this paper.

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