A THEOREM ON FIXED POINTS AND ITS APPLICATION TO THE THEORY OF DIFFERENTIAL EQUATIONS

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(Received January 20, 1952)

1. Let B be a Banach space, and K a convex compact subset of B. It is well known (Cf. [1]* [3], [5]) that any absolutely continuous mapping that maps K into itself has at least one fixed point in K. The present paper deals with the situation of dependency of fixed points on a parameter σ which, being a variable element of a second Banach space B_σ, appears in the relation of the mapping, and also one of the applications to the problem of dependency (Abhängigkeit) in the theory of ordianary differential equations.

Now, let $y=\varphi(x, \sigma)$, where $x \varepsilon B$ and $\sigma \varepsilon B_\sigma$, be an absolutely continuous mapping which maps K into itself, and let x_σ denote fixed points of this mapping corresponding to σ, and $\mathfrak{M}_\sigma$, the set of all x_σ for each $\sigma \varepsilon B_\sigma$. Thus our first purpose is to prove the

Theorem. Suppose $\varphi(x, \sigma)$ continuous in (x, σ) at the point $(x, \sigma_0) \varepsilon B \times B_\sigma$ for each $x \varepsilon K$ and a fixed $\sigma_0 \varepsilon B_\sigma$, and let $\rho(x, \mathfrak{M})$ denote the distance between a point x and a subset $\mathfrak{M}$. Then we have

$$\lim_{\sigma \to \sigma_0} \sup_{x_\sigma \varepsilon \mathfrak{M}_\sigma} \rho(x_\sigma, \mathfrak{M}_{\sigma_0})=0,$$

that is, the subset $\mathfrak{M}_\sigma$ can be contained in an arbitrarily near neighbourhood of $\mathfrak{M}_{\sigma_0}$, whenever σ is sufficiently near σ_0.

A direct consequence of this theorem is the following

Corollary. Under the same assumption of the theorem, if the subset $\mathfrak{M}_{\sigma_0}$ consists of a single point x_{σ_0},

$$\lim_{\sigma \to \sigma_0} x_\sigma = x_{\sigma_0}$$

for any selection of x_σ from $\mathfrak{M}_\sigma$.

2. **Proof of the theorem.** For the proof of the theorem we have to prove previously the following

Lemma. Under the same assumption of the theorem, if $\{\sigma_n\}$ be a sequence of points of B_σ, converging to a point $\sigma_0 \varepsilon B_\sigma$, then, for every sequence $\{x_{\sigma_n}\}$ of fixed points corresponding to $\{\sigma_n\}$, there exists a subsequence $\{x_{\sigma_{n'}}\}$, such as converges to a point

$$x_{\sigma_0} \varepsilon \mathfrak{M}_{\sigma_0}.$$

Proof. Since $\{\sigma_n\}$ converges to σ_0, $\{\varphi(x, \sigma_n)\}$ converges to $\varphi(x, \sigma)$, and that uniformly for $x \varepsilon K$, because of the compactness of K. Let $\{\varepsilon_n\}$ be a sequence of possitive numbers converging to 0, and then there exists such a sequence $\{N_n\}$ of positive integers depending on ε_n, but not on x, as

* The numbers in the brackets refer to the references at the end of the paper

$$\| \varphi(x_\sigma, \sigma_i) - \varphi(x_{\sigma_i}, \sigma_0) \| < \varepsilon_n \quad \text{for} \quad i \geqq N_n,$$

which is equivalent to

$$\| x_{\sigma_i} - \varphi(x_{\sigma_i}, \sigma_0) \| < \varepsilon_n \quad \text{for} \quad i \geqq N_n,$$

since x_{σ_i} is a fixed point corresponding to σ_i. Therefore there exists such a subsequence $\{x_{\sigma_{n'}}\}$ of $\{x_{\sigma_n}\}$, as

$$\| x_{\sigma_{n'}} - \varphi(x_{\sigma_{n'}}, \sigma_0) \| < \varepsilon_{n'}.$$

Since $\{x_{\sigma_{n'}}\}$ is a subset of the compact set K, it has a subsequence $\{x_{\sigma_{n''}}\}$ converging to a point $x_0 \varepsilon K$, and, therefore, owing to the continuity of φ, we have

$$\lim_{n'' \to \infty} \varphi(x_{\sigma_{n''}}, \sigma_0) = \varphi(x_0, \sigma_0).$$

On the other hand

$$\| x_{\sigma_{n''}} - \varphi(x_{\sigma_{n''}}, \sigma_0) \| < \varepsilon_{n''}$$

and since ε_n converges to 0,

$$x_0 = \lim_{n'' \to \infty} \varphi(x_{\sigma_{n''}}, \sigma_0) = \varphi(x_0, \sigma_0).$$

Hence we have $x_0 \varepsilon \mathfrak{M}_{\sigma_0}$, which proves the lemma.

Proof of the theorem. Suppose the conclusion of the theorem is false. Then there exist a subsequence $\{\sigma_{n'}\}$ of the sequence $\{\sigma_n\}$ and a positive constant c, such as

$$\sup_{x_{\sigma_{n'}} \varepsilon \mathfrak{M}_{\sigma_{n'}}} \rho(x_{\sigma_{n'}}, \mathfrak{M}_{\sigma_0}) > c.$$

Therefore, there exists such a corresponding sequence $\{x_{\sigma_{n'}}\}$, as

$$\rho(x_{\sigma_{n'}}, \mathfrak{M}_{\sigma_0}) > c,$$

and then

$$\rho(x_{\sigma_{n'}}, x_{\sigma_0}) > c \quad \text{for all} \quad x_{\sigma_0} \varepsilon \mathfrak{M}_{\sigma_0},$$

which contradicts to the lemma.

3. **Application.** Let R^1 denote the space of real numbers, t its element, I an interval: $t^0 \leqq t \leqq t^1$, R^n an n-dimensional vector space, and x its element normed by

$$\| x \| = \sum_{i=1}^{n} | x_i | \quad \text{where} \quad x = (x_1, \cdots\cdots, x_n).$$

Moreover, let $f(t)$ denote a vector function, components being

$$f_1(t), \cdots\cdots, f_n(t).$$

The totality of vector functions $f(t)$, continuous in the interval I, forms, as is well-known, a function space C^n, which is a Banach space. Let $f(t)$ be denoted by $\boldsymbol{f}$, as element of C^n, and the corresponding norm, by $\| \boldsymbol{f} \|_{C^n}$ defined by

$$\| \boldsymbol{f} \|_{C^n} = \max_{t \varepsilon I} \| f(t) \|.$$

Now we consider a differental equation:

$$\frac{dx(t)}{dt} = f(x(t), t) \tag{1}$$

in the vector space R^n, $f(x,t)$ continuous in x and t, under the initial condition that when $t=\tau \varepsilon I$, $x=\xi$. As is well known, the problem of the equation (1) can be transformed into that of the integral equation:

(2) $$x(t)=\xi+\int_{\tau}^{t} f(x(t),t)\,dt.$$

If we put

(3) $$\xi+\int_{\tau}^{t} f(x(t),t)\,dt=\varphi(x(t),f,\tau,\xi)=y(t),$$

we have a mapping $y=\varphi(x, f, \tau, \xi)$ of $C^n \times C^{n+1} \times R^1 \times R^n$ into C^n.

The following equality and inequalities:

(4) $$\|\varphi(x(t), f, \tau, \xi)-\varphi(x(t), f, \tau, \bar{\xi})\|=\|\xi-\bar{\xi}\|,$$

(5) $$\|\varphi(x(t), f, \tau, \xi)-\varphi(x(t), f, \tilde{\tau}, \xi)\| \leqq \|f\|_{C^{n+1}} \cdot |\tau-\tilde{\tau}|,$$

(6) $$\|\varphi(x(t), f, \tau, \xi)-\varphi(x(t), \tilde{f}, \tau, \xi)\| \leqq \|f-\tilde{f}\|_{C^{n+1}} \cdot |t^1-t^0|$$

give the equality and inequalities respectively:

(7) $$\|\varphi(x, f, \tau, \xi)-\varphi(x, f, \tau, \bar{\xi})\|_{C^n}=\|\xi-\bar{\xi}\|,$$

(8) $$\|\varphi(x, f, \tau, \xi)-\varphi(x, f, \tilde{\tau}, \xi)\|_{C^n} \leqq \|f\|_{C^{n+1}} \cdot |\tau-\tilde{\tau}|,$$

(9) $$\|\varphi(x, f, \tau, \xi)-\varphi(x, \tilde{f}, \tau, \xi)\|_{C^n} \leqq \|f-\tilde{f}\|_{C^{n+1}} \cdot |t^1-t^0|.$$

In the next place, we consider the product space $C^{n+1} \times R^1 \times R^n$, and denote its element by $\sigma=(f, \tau, \xi)$, and the function $\varphi(x, f, \tau, \xi)$ by $\varphi(x, \sigma)$.

We restrict the domain of f to $\mathfrak{D}_f$, where the norm $\|f\|_{C^{n+1}}$ is bounded by a positive number M. The equality (7) and inequalities (8), (9) give the inequality:

(10) $$\|\varphi(x, \sigma)-\varphi(x, \tilde{\sigma})\|_{C^n} \leqq \|\xi-\bar{\xi}\|+\|f\|_{C^{n+1}} \cdot |\tau-\tilde{\tau}| + |t^1-t^0| \cdot \|f-\tilde{f}\|_{C^{n+1}}$$

where $\sigma=(f, \tau, \xi)$, $\tilde{\sigma}=\tilde{f}, \tilde{\tau}, \bar{\xi})$. This inequality shows that the mapping $\varphi(x, \sigma)$ is continuous in σ, and so also uniformly whenever $f \varepsilon \mathfrak{D}_f$, and $\tau \varepsilon I$, or what is the same, $\sigma \varepsilon \mathfrak{D}_f \times I \times R^n$. That $\varphi(x,\sigma)$ is continuous in x being easily verified, we can conclude that $\varphi(x,\sigma)$ is continuous in (x, σ). Moreover the function $y(t)=\varphi(x(t)\sigma)$ is, as is easily shown, equicontinuous in t, and therefore the mapping $y=\varphi(x, \sigma)$ is compact ([4], p. 51, Th. 3. 1). Hence the mapping φ is absolutely continuous.

Thus, as is well known, the existence of solutions of the integral equation (2) and accordingly of the differential equation (1) is verified as fixed points of the mapping $\varphi(x,\sigma)$. Moreover our theorem can be transformed into the following theorem of the theory of differential equations:

A solution of the differential equation (1), where $\sigma=(f, \tau, \xi)$, can be contained in an arbitrarily near neighbourhood of the set of the solutions $x(t; \sigma_0)$, where $\sigma_0=(f_0, \tau_0, \xi_0,)$, whenever σ is sufficiently near σ_0, or in other words, every solution $x(t;\sigma,)$ can be *uniformly and arbitrarily approximated to some solution* $x(t;\sigma_0)$ *as* f *approches umiformily and sufficiently to* f_0, τ *and* ξ *sufficiently to* τ_0 *and* ξ_0 *respectively.*

Our corollary generalizes, as a special case of this theorem, ordinary theorems about the dependency (Abhängigkeit) of ordinary differential equations (Cf. [2], pp. 81—88, and pp. 142—151), and it is moreover to be noticed that the theorem just mentioned does not require the condition of uniqueness of the solution $x(t;\sigma_0)$.

Remark. A generalization of our propositions (and also existence-theorems concerned with them) to the case where the function $x(t)$ is that on R^1 to a Banach space, can hardly be expected to be verified by means of our procedure, since this procedure is based on the compactness of the mapping $\varphi(x, \sigma)$ (in x), or, what is the same, on Alzelà's theorem (Cf. [4] and [5]), the validity of which is not verified in the general case above mentioned.

References.

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