# ON A THEOREM OF GELFAND AND NEUMARK AND THE B*-ALGEBRA

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**1. Introduction and Preliminaries.** In this paper we shall state on a new proof of Gelfand-Neumark's theorem and certain related results in the theory of normed rings.

A Banach algebra (over the field of complex numbers) is called a B*-algebra, when there is defined an involutorial anti-automorphism $x \rightarrow x^*$ such that $(\lambda x)^* = \bar{\lambda} x^*$, and $\| x^* x \| = \| x \|^2$. An element $x$ of a B*-algebra $R$ is said hermitean if $x^* = x$, and skew-hermitean if $x^* = -x$. Every element $x$ is uniquely represented as the sum $x = u + iv$, by two hermitean elements $u$ and $v$.

The spectrum of an element $x$ of a B*-algebra $R$ with the unit element e is the set of all the complex numbers for which $x - \lambda e$ does not have the inverse. The spectrum of any hermitean element consists of real numbers. If the spectrum of an hermitean element $x$ consists only of non-negative (or positive) numbers, $x$ is said non-negative (or positive). In this paper, we will consider only those B*-algebras which has the unit element.

The (real) linear space $E$ of all hermitean elements of $R$ is a Banach space with the original norm. We denote by $D$ and $D_\circ$ the sets of all hermitean, non-negative and positive elements of $R$. Gelfand and Neumark [1] (also, [3]) have shown, under the additional assumption that *every* $x^*x + e$ *have the inverse* (in this case the B*-algebra being called a C*-algebra), that $D$ and $D_\circ$ is convex. We shall show, in the following, this latter fact *without using the Gelfand-Neumark's assumption, i.e.*, that it holds for every B*-algebra with the unit element.

A (complex-valued) linear functional $f(x)$ on the B*-algebra $R$ is said positive if $f(x^*x) \geqq 0$, for every $x \varepsilon R$; we denote the totality of such functionals by $\mathfrak{P}$. On the other hand, we put $P$ the totality of (real-valued) linear functionals on the Banach space $E$ which satisfies $f(D) \geqq 0$. It can be easily seen that every $f \varepsilon P$ satisfies $f(D_\circ) > 0$. For every linear closed subspace $H$ in $E$ such that $H \frown D_\circ$ is empty, there is an $f \varepsilon P$ with $f(H) = 0$. This is a consequence of the open convexity of $D_\circ$ and the Ascoli-Mazur's theorem.

It is obvious that every functional $f \varepsilon \mathfrak{P}$ define a functional of $P$ when it is restricted on the space $E$; we shall denote this fact by $\mathfrak{P} < P$; on the other hand, if every functional of $P$ can be extended to the functional of $\mathfrak{P}$ in the sense indicated in Lemma 8 (next §), we shall denote it by $P < \mathfrak{P}$. It is not obvious in general that $\mathfrak{P} \neq$ empty and $\mathfrak{P} = P$ for an B*-algebra with the unit element. We shall find an necessary and sufficient condition for these requirements.

The following structure theorem for commutative B*-algebra is originally due to Gelfand and Neumark: *If A is any commutative B*-algebra with the unit element, then A is iso-*

*morphic and isometric to the algebra* $C(\mathfrak{M})$ *of all complex-valued, continuous functions on the compact Hausdorff space of all maximal ideals of* $A$, *such that* $\| x \| = \sup_{M \varepsilon \mathfrak{M}} | x(M) |$ *and* $x^*(M) = \overline{x(M)}$ *for every* $M \varepsilon \mathfrak{M}$. It follows at once that any hermitean element $u \varepsilon D_0$ (*or* $\varepsilon D$) may be represented as an everywhere positive (or non-negative) -valued continuous function on $\mathfrak{M}$, where $\mathfrak{M}$ denotes the compact space of all maximal ideals of any closed commutative self-adjoint B*-sub-algebra with the unit element e which contains $u$; any $u \varepsilon D$ belongs to $D_0$ if and only if $u$ is regular. Also, we see that any hermitean element $w$ is the difference of two non-negetive, hermitean elements, that is, $w = w_+ - w_-$, where $w_+ = \frac{1}{2}(\overline{w} + w)$, $w_- = \frac{1}{2}(\overline{w} - w)$ and $w = \sqrt{w^2}$; $\overline{w}, w, w_+, w_-$ commute each other and $w_- \cdot w_+ = 0$. Every $u \varepsilon D$ can be put in the form $u = u_1^2$, where $u_1 \varepsilon E$, and if $u \varepsilon D_0$, $u_1$ must be regular.

We state the following

*Theorem 1. Let* $R$ *be any B*-algebra with the unit element e. Then the following postulates for* $R$ *are equivalent:*

(I). *For every* $x, y \varepsilon R$, $x^*x + y^*y = 0$ *implies* $x = y = 0$.

(II). *For every* $x, \varepsilon R$, $x^*x + e$ *has the inverse element.*

*Proof.* We will prove the implication (I)→(II). Since $x^*x = u_+^2 - u_-^2$, $u_+ \cdot u_- = 0$ and $u_+$, $u_- \varepsilon E$, $0 = (xu_-)^*(xu_-) + (u_-^2)^2$, which implies $xu_- = 0$, and $u_-^2 = 0$, so that $x^*x = u_+^2$ and $x^*x + e = e + u_+^2$ is regular. The proof of (II)→(I) is analogons.

*Theorem 2. The class of all two-sided ideals* $J$ *in any B*-algebra* $R$ *with the unit element satisfying the condition*

(1) $x^*x + y^*y \varepsilon J$ *implies* $x, y \varepsilon J$

*is identical with the class of all two-sided ideals* $J$ *satisfying the condition*

(2) *For every* $z \bar{\varepsilon} J$, *there is a* $f(x) \varepsilon \mathfrak{P}$ *such that* $f(z^*z) > 0$ *and* $f(x^*x) = 0$, *for every* $x \varepsilon J$.

*Proof.* We shall only notice that the residue class algebra of any B*-algebra with the unit element modulo any two-sided ideal is again a B*-algebra, and that every two-sided ideal of a B*-algebra is self-adjoint.

**2. The Positiveness Concept in the B*-algedra.** We shall prove the additivity of positiveness in any B*-algebra, *without the assumption that every* $x^*x + e$ *have the inverse.*

Let $R$ be any B*-algebra with the unit element $e$. Let $E, D, D_0$ be defined as in the above. (*Cf.* [3]).

*Lemma 1. If* $u$ *and* $v$ *are any element of* $D_0$, *then* $u + v$ *also belongs to* $D_0$.

*Proof.* For the proof, we notice that every $u \varepsilon D_0$ can be put in the form $u = u_1 + \delta e$ for some $u_1 \varepsilon D_0$ and some $\delta > 0$; that is, $u \varepsilon D_0$ if and only if $u + \lambda e$ is regular (to have the inverse) for all $\lambda \geqq -\delta$, for some $\delta > 0$. This can be easily seen by means of the structure theorem for the commutative B*-algebra.

Therefore, it can be easily seen that, in order to prove $u + v \varepsilon D_0$, for every $u, v \varepsilon D_0$, it suffices to show that, for every $u, v \varepsilon D_0$ and for every $\lambda > 0$, $u + v + \lambda e$ be regular.

As it is clear that $u \varepsilon D_0$ and $\lambda > 0$ imply $\lambda u \varepsilon D_0$, we have only to prove that, for every

$u$, $v\varepsilon D_0$, $e+u+v$ is regular, also it suffices to show that, for every $u$, $v\varepsilon D_0$, all $e+\alpha u+(1-\alpha)v$, $0\leqq\alpha\leqq1$, are regular.

Put $w(\alpha)=e+\alpha u+(1-\alpha)v$; as both $u$ and $v$, together with all $e+\alpha u$, $e+(1-\alpha)v$ are regular for all $\alpha$, $0\leqq\alpha\leqq1$, $w(\alpha)$ is regular if and only if

$$\overline{w(\alpha)}=u^{-1}w(\alpha)v^{-1}=(u^{-1}+\alpha e)\ (v^{-1}+(1-\alpha)e)-\alpha(1-\alpha)e \text{ is regular.}$$

By the representation theorem for $R_u$ and $R_v$ [1], we have

$$\|(u^{-1}+\alpha e)^{-1}\|=\sup_{M}(\alpha+1/u(M))^{-1}\leqq\|u\|/(\|u\|\alpha+1)$$

*and*

$$\|(v^{-1}+(1-\alpha)e)^{-1}\|\leqq\|v\|/(\|v\|(1-\alpha)+1);$$

consequently, we have

$$\|(u^{-1}+\alpha e)^{-1}\ \overline{w(\alpha)}\ (v^{-1}+(1-\alpha)e)^{-1}-e\|$$
$$\leqq\alpha(1-\alpha)\|u\|\cdot\|v\|/(1+\|u\|\alpha)(1+\|u\|(1-\alpha))<1,$$

for all $\alpha$, $0\leqq\alpha\leqq1$, which proves that $(u^{-1}+\alpha e)^{-1}\overline{w(\alpha)}\ (v^{-1}+(1-\alpha)e)^{-1}$ is regular.

Thus, $w(\alpha)$ is also regular for every $\alpha$, $0\leqq\alpha\leqq1$, and for every $u$ and $v\varepsilon D_0$, which completes the proof.

An immediate consequence of the lemma is the following

*Lemma 2. If $u$ and $v$ belong to $D$, then $u+u$ also belongs to $D$.*

*Lemma 3. Any $u\varepsilon D$ belongs to $D_0$ if and only if $u$ is an inner point of $D$.*

*Proof.* Let $u\varepsilon D_0$ and let $v\varepsilon E$ be any element such that $\|u-v\|<E$, $E$ being determined later on. We shall prove that $v\varepsilon D$.

As $u+\lambda e$ is regular for all real $\lambda\geqq-\lambda_0$, for some $\lambda_0>0$, $v+\lambda e=(u+\lambda e)+(v-u)=(u+\lambda e)\ (e+w)$, where $\|w\|\leqq\|u-v\|\cdot\|(u+\lambda e)^{-1}\|\leqq M\cdot\varepsilon$, with $M=\sup_{\lambda\geqq-\lambda_0}\|(u+\lambda e)^{-1}\|$. Such $M$ exists by the continuity of $\|(u+\lambda e)^{-1}\|$. It follows, after $\varepsilon$ being taken such that $M\cdot\varepsilon<1$, that $(u+\lambda e)\ (e+w)$ is regular for $\lambda\geqq-\lambda_0$, so that $v+\lambda e$ is positive and regular.

Conversely, if $u$ is any inner point of $D$, then $u-\lambda e\varepsilon D$ for some $\lambda>0$, so that $u=\lambda e+u_1^2\ \varepsilon D_0$.

*Theorem 3. Let $R$ be any $B^*$-algebra with the unit element $e$, and let $D_0$ and $D$ be the totality of hermitean and positive elements and of hermitean and non-negative elements of $R$. Then $D_0$ forms a convex, open cone in the real Banach space $E$ of hermitean elements of $R$, and $D$ is a convex cone. Moreover, $D_0$ is the set of all inner points of $D$.*

As this theorem is obtained easily by means of the preceeding lemmas 2, 3, we omit the proof.

*Lemma 4. $D_0\frown(-D_0)$ is empty; $D\frown(-D)=(0)$.*

*Proof.* $u+v=0$ for $u, v\varepsilon D_0$ is impossible, because $D_0$ does not contain 0, by the open-

1) Here $R_u$ may be any commutative self-adjoint subalgebra of $R$ with the unit element $e$, containing $u$,

ness of $D_0$

(ii) As $u, v \varepsilon D$ and $u=-v$ imply that $u, v$ commutes; it follows, by the representation theorem, that $u(M)=-v(M)$, for all $M \varepsilon \mathfrak{M}$; now, $u(M) \geqq 0$ and $v(M) \geqq 0$ imply $u=v=0$.

*Lemma 5.* $x^*x+xx^*=0$ *implies* $x=0$.

*Proof.* Put $x=u+iv$, where $u, v$ are hermitean, then $x^*x+xx^*=2\ (u^2+v^2)$, from which we have $u=v=0$ and $x=0$, by the preceeding lemma.

*Lemma 6.* $x^*x \neq -e$.

*Proof.* Let $x=u+iv$, $u, v \varepsilon E$, and suppose $x^*x=-e$. As $x^*x+xx^*=2(u^2+v^2)$, we have $xx^*= e+2(u^2+v^2)$ and so $xx^*$ is regular ($\varepsilon D^\circ$) (by Lemma 1). On the other hand, $(xx^*)^2 = x\ (x^*x)x^* = -xx^*$, which is impossible, because both $xx^*$ and $xx^*+e$ are regular. This shows that $x^*x \neq -e$.

*Lemma 7. If $u$ is hermitean and $x^*x=-u^2$ for some $x \varepsilon R$, then $u$ is not regular.*

*Proof.* If otherwise, we shall have $(xu^{-1})^*(xu^{-1})=-e$; this is impossible by the reason of the preceeding lemma.

By this lemma, any element $x^*x$ which has the non-positive spectrum cannot belong to $-D_0$, but only to $-(D \frown D_0^C)$. ($D_0^C$ denotes the complement of $D_0$).

*Lemma 8. Let $R$ be any $B^*$-algebra with the unit element $e$, and let $E$ be lhe Banach space of all hermitean elements in $R$. If $f(u)$ is any real-valued linear functional, defined on $E$, such that $f(D) \geqq 0$, then the functional $F(x)$ on $R$ defined by*

$$F(x)=f(u)+if(v), \text{ for } x=u+iv \varepsilon R,\ u, v \varepsilon E,$$

*is a complex-valued linear functional on $R$ such that $|F(x)| \leqq F(e) \|x\|$;*
*$F(x^*x) \geqq 0$ for every $x \varepsilon R$ if and only if $f(x^*x) \geqq 0$, for every $x \varepsilon R$.*

*Proof.* $F(\lambda x)=\lambda F(x)$ and $F(x+z)=F(x)+F(y)$ are obvious. Since $|f(u)| \leqq f(e) \|u\|$, $|f(u)|^2 \leqq f(e) f(u^2)$ and $2\ (u^2+v^2)=x^*x+xx^*$, we have $|F(x)| = \{ |f(u)|^2 + |f(v)|^2 \}^{\frac{1}{2}}$ $\leqq f(e)^{\frac{1}{2}} (f(u^2+v^2))^{\frac{1}{2}} \leqq f(e) \|u^2+v^2\|^{\frac{1}{2}} = f(e) \|\tfrac{1}{2}(x^*x+xx^*)\|^{\frac{1}{2}} \leqq f(e) \|x\| = F(e) \|x\|$.

The last statement is obvious.

*Lemma 9. Let $G$ be a real Banach space, and $H$ a linear closed subspace of $G$. Suppose that there is defined a set $K$ in $G$ such that*

1. *$x \varepsilon K$ and $\lambda > 0$ imply $\lambda x \varepsilon K$,*
2. *$x$ and $y \varepsilon K$ implies $x+y \varepsilon K$,*
3. *$x \varepsilon K$ implies $-x \bar{\varepsilon} K$,*
4. *$K$ is an open set,*
5. *$K \frown H$ is empty.*

*Then the factor space $G-H$ is a real Banach space in its usual definition of the norm, in which the set $K-H$ ($\subset G-H$) satisfies the same postulates 1.—4.*

*Proof.* Let $U, V$ be the elements of $K-H$, *i, e.*, the cosets $U=u+H$, $V=v+H$ with $u$, $v \varepsilon K$. The validity of 1, 2 and 3 for $K-H$ is obvious. Take any $U \varepsilon K-H$, then we can take, by 4, an $\varepsilon > 0$ such that the sphere $\{x / \|u-x\| < \varepsilon,\ x \varepsilon G\}$ is contained in $K$. Consider the sphere $\{X/\|U-X\|<\frac{\varepsilon}{2}\}$ in $G-H$. As $\|U-X\|=\inf_{h\varepsilon H}\|u-x+h\|$, there is an $h\varepsilon H$ such that $\|u-x+h\|<\varepsilon$, so that we have $v=x-h\varepsilon K$ and $X=x+H=(v+h)+H=v+H$ and $X\varepsilon K-H$. This proves 4. for $K-H$.

The following theorem is a modification of the Ascoli-Mazur's Theorem.

*Theorem 4. Let G, H be a real Banach space and its any linear closed subspace. If there is defined a set K satisfying the following conditions:*

1. *$x\varepsilon K$ and $\lambda>0$ imply $\lambda x\varepsilon K$,*
2. *$x$ and $y\varepsilon K$ imply $x+y\ \varepsilon K$,*
3. *$x\varepsilon K$ implies $-x\bar{\varepsilon}K$,*
4. *K is an open set,*
5. *$K\frown H$ is empty.*

*Then there exists a linear functional $f(x)$ on G satisfying $f(K)>0$ and $f(H)=0$.*

**3. The Positive Functional on the B*-algebra.**

*Theorem 5. Let R, E and $D_o$ be defined as above and let H be any linear closed subspace of E. suchthat $H\frown D_o$ is empty. Then there is an $f(x)\varepsilon P$ such that $f(H)=0\,(f(D_o)>0)$.*

*Cor. 1. For every $v\varepsilon D\frown D_o{}^C$, there is a $f(x)\varepsilon P$ with $f(v)=0$.*

*Cor. 2. For every $v\varepsilon D\frown D_o{}^C$, there is a $f(x)\varepsilon P$ with $f(v)>0$.*

*Proof.* Put $u=e+\alpha v, (-\infty<\alpha<\infty)$, then there is an $\alpha_1<0$ such that $u_1=e+\alpha_1 v\varepsilon D\frown D_o{}^C$. If we put $H$ the linear closed subspace of $E$ spanned by the vector $u_1$, then $H\frown D_o=$ empty is obvious, so that, we obtain a $f(x)\varepsilon P$ with $f(H)=0$, which is the required one.

*Lemma 10. If R satisfies the condition:*

(III). *$x^*x+y^*y+\cdots+z^*z\neq -e$, for every finite set of elements $x,y,\ldots,z$, then, there is a linear functional $f(x)\varepsilon\mathfrak{P}$, that is, $\mathfrak{P}$ is not empty.*

*Proof.* At first, we consider all those elements $w$ with $w=x^*x=-u^2$, where $x\varepsilon R$ and $u\varepsilon E$. Let $G$ be the totality of all positive linear combinations of any finite number of such elements. It is obvious that $\lambda G\subset G$ for every $\lambda>0$ and that every element of $G$ is identical with a certain $-u$, $u\varepsilon D$ (Lemma 2), where $u$ cannot belong to $D_o$. For otherwise, we have a relation

$$x^*x+y^*y+\cdots+z^*z=-u,\ u \text{ being regular,}$$

which imply the relation

$$x_1{}^*x_1+y_1{}^*y_1+\cdots+z_1{}^*z_1=-e,$$

which contradicts to (III).

Thus, any $w\varepsilon G$ does not belong to $-D_o$. Also the linear closed subspace spanned by $G$ does not contain any point of $D_o$. For, if $\bar{G}=G-G$ does contain an $u\varepsilon D_o$, then there is a relation $v_1-v_2=u$ with $v_1, v_2\varepsilon D\frown D_o{}^C$, so that we must have $v_1=u+v_2\ \varepsilon D_o$, which is a cotradicton.

Therefore $\bar{G}\frown D_o{}^C$ is empty. Applying Theorem 5, we find a linear functional $f(x)\varepsilon P$ such that $f(\bar{G})=0$.

Next, we shall prove that the thus obtained $f(x) \varepsilon P$ is a functional of $\mathfrak{P}$, *i. e.* $f(x^*x) \geqq 0$, for every $x \varepsilon R$. Every $x^*x$ is expressed in the form $x^*x=u-v$, $u$, $v \varepsilon D$ and $u.v=0$.

Case (i). $u=0$. We have $x^*x=-v$, so that $f(x^*x)=0$ by the definition.

Case (ii). $v=0$. We have $x^*x=u$. so that $x^*x \varepsilon D$ and $f(x^*x) \geqq 0$.

Case (iii). $u, v \neq 0$. It is easy to see, by means of the structure theorem, that there is, for every $k>0$, a positive element $q_k$ which commutes with $x^*x$, $u$ and $v$, and such that $q_k x^*x\, q_k=-v\, q_k^2$. Then we have $f(v(1-q_k^2)) \leqq \frac{1}{k}$, and $f(v)=f(v(1-q_k^2))+f(v\, q_k^2)=f(v(1-q_k^2)) \leqq \frac{1}{k}$, since $f(vq_k^2)=0$ by the definition of $f(x)$. So we have $f(v)=0$, and $f(x^*x)=f(u) \geqq 0$.

*Theorem* 6. *For the set* $\mathfrak{P}$ *of all positive linear functionals on any* $B^*$*-algebra* $R$ *with the unit element* $e$ *to be not empty, it is necessary and sufficient that in* $R$ *the condition* (III) *is satisfied.*

*Proof.* The sufficiency of (III) are shown in Lemma 10. The necessity is evident.

*Theorem* 7. *The necessary and sufficient condition for* $P=\mathfrak{P}$ *to be valid for any* $B^*$*-algebra* $R$ *with the unit element* $e$ *is the following condition:*

(IV) $x^*x+y^*y+\cdots+z^*z=0$ *implies* $x=y=\cdots=z=0$,

*for every* $x, y, \cdots, z \varepsilon R$.

*Proof.* Suppose that (IV) is satisfied in $R$. It is clear that the condition (II) is valid, and that every $x^*x=u$, for some $u \varepsilon D$. Thus, $f(x) \varepsilon P$ implies $f(x^*x)=f(u) \geqq 0$, which shows $P<\mathfrak{P}$ As $\mathfrak{P}<P$ is evident, we have $\mathfrak{P}=P$.

Conversely, suppose that $P<\mathfrak{P}$ and there exists a relation $x^*x+y^*y+\cdots+z^*z=0$, where $x \neq 0$, say. Put $x^*x=u-v$, where $u, v \varepsilon D$ and $u \cdot v=0$. After two-sided multiplication by some adequate elements, we obtain another relation $x_1^*x_1+y_1^*y_1+\cdots+z_1^*z_1=0$, where $x_1^*x_1=-v_1$, $v_1 \varepsilon D \frown D_0^C$.

By Theorem 5, Cor. 2, we have an $f(x) \varepsilon P$ with $f(v_1)>0$. Since $P<\mathfrak{P}$ implies $f(x_1^*x_1) \geqq 0$, we have a contradiction. Thus $v_1$, and $v$, must be equal to 0. In the same manner for $y, \cdots, z$, we obtain $x=y=\cdots=z=0$. This proves the necessity of (IV).[2]

## Bibliography.


1. Gelfand and Neumark, On the imbedding of normed rings into the ring of operators in Hilbert spaces, Rec. Mat. N. S., vol. 12 (1943).
2. I. Kaplansky, Normed Algebra, Duke Math. Journ., vol. 16 (1949).
3. M. Fukamiya, On B*-algebras, Proc. Japan Academy, vol. 27 (1951).
4. M. Nakamura, Notes on Banach Spaces (XII): A Remark on a Theorem of Gelfand and Neumark, Tohokn Math. J., (2) vol. 2 (1950).


2) I. Kaplansky have shown that every primitive B*-algebra with a minimal right ideal is a C*-algebra. This follows from our Theorem 7. (See Lemma 4.2 in [2]). Cf. [4], also.