ON SOME CHARACTERIZATIONS OF ABSTRACT EUCLIDEAN SPACES BY PROPERTIES OF ORTHOGONALITY

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1. Introducuion. Let **B** be a real- or complex-linear normed space. If there exists a real or complex inner product in **B** such that $\| x \|^2=(x, x)$ for every x in **B**, then **B** is called a real or complex abstract Euclidean space.

In a linear normed space **B**, four types of orthogonality can be considered. (Cf. [1] and [2].[1]) It was investigated by R. C. James [1] [2] [3] and M. M. Day [4] to characterize the abstract Euclidean space by the properties of these types of orthogonality in the linear normed space **B**.

In this paper, we shall show that the abstract Euclidean space may be characterized by relations between the above four types of orthogonality. The four types of orthogonality in linear normed spaces is as follows;

(1). *Definition of isosceles orthogonality:* An element x of **B** is orthogonal to an element y if and only if $\| x+y \| = \| x-y \|$.

(2). *Definition of Roberts' orthogonality:* An element x of **B** is orthogonal to an element y if and only if $\| x+ky \| = \| x-ky \|$ for all real numbers k.

(3). *Definition of Pythagorean orthogonality:* An element x of **B** is orthogonal to an element y if and only if $\| x+y \|^2= \| x \|^2+ \| y \|^2$.

(4). *Definition of Birkhoff's orthogonality:* An element x of **B** is orthogonal to an element y if and only if $\| x+ky \| \geqq \| x \|$ for all real numbers k.

The above types of orthogonaiity are equivalent for abstrabt Euclidean spaces, but are not so in a general linear normed space. Properties of the four types of orthogonality in linear normed spaces were studied by R. C. James [1] [2].

2. Characterizations by relations between isosceles orthogonality and Birkhoff's orthogonality.

Theorem 1. *If isosceles orthogonality implies Birkhoff's orthogonality in a real- or complex-linear normed space* **B**, *then* **B** *is an abstract Euclidean space.*

Proof. It will be shown that the condition given by E. R. Lorch [5] for the existence of a real inner product in a real-linear normed space is satisfied, namely:

$$\| kx+k^{-1}y \| \geqq \| x+y \|, \qquad (2.1)$$

for all real numbers $k \neq 0$ and elements x and y for which $\| x \| = \| y \|$.

Suppose that $x, y \varepsilon$ **B** and $\| x \| = \| y \|$, and set $x'= (x+y)/2$, $y'=(x-y)/2$. Then we have: $\| x'+y' \| = \| x'-y' \|$. Therefore by our condition,

$$\| x'+((k^2-1)/(k^2+1))\, y' \| \geqq \| x' \|$$

1) Numbers in brackets refer to the bibliography at the end of the paper.

for all real numbers k. From this inequality, we can derive the following inequality by using that $2|k|/(k^2+1) \geqq 1$ for all real numbers k;

$$\| k(x'+y') + k^{-1}(x'-y') \| \geqq \| 2x' \|$$

for all real numbers $k \neq 0$. Thus it is shown that the condition (2.1) is satisfied. Therefore, if **B** is a real-linear space, then **B** is a real abstract Euclidean space.

In the case that **B** is a complex-linear space, we consider the associated real-linear space **A** of **B**. (Cf. [4] pp. 334.) Then the condition (2.1) holds in **A**. Hence, by Theorem 7.2 in [4], **B** is a complex abstract Euclidean space.

Theorem 2. *If Birkhoff's orthogonality implies isosceles orthogonality in a real- or complex-linear normed space* **B**, *then* **B** *is an abstract Euclidean space.*

Proof. We prove that the condition in Theorem 1 is satisfied, namely: $\| x+y \| = \| x-y \|$ implies $\| x+ky \| \geqq \| x \|$, for all real numbers k. Suppose that (2.2): $\| x+y \| = \| x-y \|$. Since the function $f(k) = \| x+ky \|$ is continuous and non-negative in real k, it must take its minimum, say at k_0. If $\| x+k_0 y \| = \min_k \| x+ky \|$, then we have:

$$\| (x+k_0 y) + (k-k_0 y) \| \geqq \| x+k_0 y \|,$$

for all real numbers k. Hence by our condition,

$$\| x+(k_0+1)y \| = \| x+(k_0-1)y \|. \qquad (2.3)$$

If $|k_0| \geqq 1$, then $\| x+k_0 y \| \geqq \| x \|$. (Cf. Theorem 4.1 in [1].) But $\| x+k_0 y \| \leqq \| x \|$. Hence we have: $\| x+k_0 y \| = \| x \|$.

If $|k_0| < 1$, then from (2.2) and (2.3) we have:

$$\| x+y \| = \| x+(k_0+1)y \| = \| x+(k_0-1)y \| = \| x-y \|.$$

Since the function $f(k) = \| x+ky \|$ is convex, from the above equalities, we have

$$\| x+k_0 y \| = \| x \|.$$

Thus $\| x+ky \| \geqq \| x+k_0 y \| = \| x \|$ for all real numbers k. Hence, by Theorem 1, **B** is an abstract Euclidean space. Q.E.D.

From Theorem 2, we have the following

Corollary 1. *A real- or complex-linear normed space* **B** *is an abstract Euclidean space if and only if Birkhoff's orthogonality implies Roberts' orthogonality.*

3. Characterizations by relations between Pythagorean orthogonaliy and Birkhoff's orthogonality.

Theorem 3. *If Pythagorean orthogonality implies Birkhoff's orthogonality in a real- or copmlex-linear normed space* **B**, *then* **B** *is an abstract Euclidean space.*

At first, we shall prove two lemmas.

Lemma1. 1. *If* $\| x+y \| = \| x \| + \| y \|$, *and if* l, m *are real numbers such that* $lm \geqq 0$, *then* $\| lx+my \| = |l| \, \| x \| + |m| \, \| y \|$.

Proof. Suppose that $\| x+y \| = \| x \| + \| y \|$. Then,

$$\| (1-k)x+ky \| = (1-k)\| x \| + k\| y \|, \text{ for } 0 \leqq k \leqq 1.$$

(Cf. Proof of Theorem 4.3 in [2]). Hence

$\| x+(k/(1-k))y \| = \| x \| + (k/(1-k)) \| y \|$, for $0 \leqq k < 1$.
Hence, $\| x+k'y \| = \| x \| + k' \| y \|$, for all $k' \geqq 0$.

From this equality, we have Lemma 1.

Lemma 2. *If* B *is a real-linear space, and if Pythagorean orthogonality implies Birkhoff's orthogonality, then* B *is strictly convex.*

Proof. Set $x' = \| y \| x + \| x \| y$, $y' = \| y \| x$. Now, there exists a real number a such that

(3.1) $\| (a+1)x'+y' \|^2 = \| x' \|^2 + \| ax'+y' \|^2$.

(Cf. Corollary 5.1 in [1]). Therefore, from our assumption,

(3.2) $\| x'+k(ax'+y') \| \geqq \| x' \|$,

for all real numbers k.

Now, we suppose $a \geqq 0$. Then from Lemma 1, we have:

$$\| (a+1)x'+y' \| = (2a+3) \| x \| \| y \|, \quad \| ax'+y' \| = (2a+1) \| x \| \| y \|.$$

Therefore from the equality (3.1), $a=-1/2$. This contradict to our assumption that $a \geqq 0$. Therefere a must be negative.

Now, we set $k=-1/a$ in (3.2). Then $|a| \leqq \| y' \| / \| x' \|$. Since $\| x' \| = 2 \| x \| \| y \|$ from Lemma 4.1, and since $\| y' \| = \| x \| \| y \|$, we have $|a| \leqq 1/2$. Moreover, we set $k=-1$ in (3.2). Then $\| (a-1)x'+y' \| \geqq \| x' \|$. Since $\| (a-1)x'+y' \| = (1-2a) \| x \| \| y \|$, $\| x' \| = 2. \| x \| \| y \|$ by Lemma, 4.1, we have $a \leqq -1/2$. On the other hand $a \geqq -1/2$. Hence $a=-1/2$. Therefore, $\| (a+1)x'+y' \| = 2 \| x \| \| y \| = \| x' \|$. Accordingly, from the equaton (3.1), we have $(-1/2)x'+y'=0$. Therefore, we have $y = (\| y \| / \| x \|) x$. Hence, **B** is srictly convex.

Proof of Theorem 3. (*i*). *The case that* B *is a real-linear space.* We suppose that $\| y+kx \| \geqq \| y \|$ for all real numbers k. Now, there exists a number a such that

(3.3) $\| x+(ax+y) \|^2 = \| x \|^2 + \| ax+y \|^2$.

(Corollary 5.1 in [1]). Therefore by our condition in Theorem 3, $\| (ax+y)+kx \| \geqq \| ax+y \|$ for all real numbers k. That is, y is orthogonal to x in the sense of Birkhoff, and simultaneously, $ax+y$ is also to x. Hence by Lemma 2, $a=0$. (Cf. Theorem 4.3 in [2].) Therefore from the equality (3.3), we have $\| x+y \|^2 = \| x \|^2 + \| y \|^2$. That is:

(3.4) $\| x+y \|^2 = \| x \|^2 + \| y \|^2$

whenever $\| y+kx \| \geqq \| y \|$ for all real numbers k. From our condition in Theorem 3 and (3.4), we can see that the Pythagorean orthogonality is homogeneous. Therefore **B** is an abstract Euclidean space. (Theorem 5.2 in [1].)

(*ii*). *The case that* **B** *is a complex-linear space.*

The condition in Theorem 3 holds in the associated real-linear space **A** of **B**. Therefore **A** is a real abstract Euclidean space. Hence, by Theorem 7.2 in [4], **B** is a complex abstract Euclidean space. Q. E. D.

If $\| x+ky \| = \| x-ky \|$ for all real numbers k, then

$$\| x \| \leqq (1/2) \| x+ky \| + (1/2) \| x-ky \| = \| x+ky \|$$

for all real numbers k. Therefore, from Theorem 3, we have

Corollary 2. *If Pythagorean orthogonality implies Roberts' orthogonality in a real- or com-*

plex-linear normed space, then **B** *is an abstract Euclidean space.*

Theorem 4. *If Birkhoff's orthogonality implies Pythagorean orthogonality in a real- or complex-linear normed space* **B**, *then* **B** *is an abstract Euclidean space.*

Proof. We suppose that the condition in Theorem 3 is satisfied, namely $\| x+y \|^2 = \| x \|^2 + \| y \|^2$ implies $\| x+ky \| \geqq \| x \|$ for all real numbers k. Suppose that

$$\| x+y \|^2 = \| x \|^2 + \| y \|^2. \tag{3.5}$$

Since the function $f(k) = \| x+ky \|$ is continuous in k. and non-negative, it must take its minimum, say at k_0. If $\| x+k_0 y \| = \min_k \| x+ky \|$, then $\| (x+k_0 y)+((k-k_0)/\lambda)(\lambda y) \| \geqq \| x+k_0 y \|$ for all real numbers k and $\lambda \neq 0$. Therefore, according to our condition, $\| (x+k_0 y)+\lambda y \|^2 = \| x+k_0 y \|^2 + \| \lambda y \|^2$ for all real numbers $\lambda \neq 0$. When $\lambda = 0$, this equality is also true.

Therefore

$$\| x+(k_0+\lambda)\, y \|^2 = \| x+k_0 y \|^2 + \lambda^2 \| y \|^2$$

for all real numbers λ. By setting $\lambda = -k_0,\ 1-k_0$ respectively,

$$\| x \|^2 = \| x+k_0 y \|^2 + k_0^2 \| y \|^2, \quad \| x+y \|^2 = \| x+k_0 y \|^2 + (1-k_0)^2 \| y \|^2.$$

Therefore according to the equation (3.5), we have $k_0 = 0$. Hence $\| x+ky \| \geqq \| x \|$ for all real numbers k.

Therfore. by Theorem 3, **B** is an abstract Euclidean space. Q. E. D.

Remark. It was proved by M. M. Day [4] that abstract Euclidean spaces are characterized by relations between isosceles orthogonality and Pythagorean orthogonality. Moreover, it was proved by E. R. Lorch [5] that abstract Euclidean spaces are characterized by the condition that *isosceles orthogonality implies Roberts' orthogonality.*

Furthermore, each of the following conditions are not sufficient in order that a real- or complex-linear normed space **B** should be an abstract Euclidean space:

(1). Roberts' orthogonality implies isosceles orthogonality.

(2). Roberts' orthogonality implies Birkhoff's orthogonality.

As to the case that Roberts' orthogonality implies Pythagorean orthogonality, no result is known.

Bibliography

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