HOMOGENEOUS CONTACT TRANSFORMATIONS IN A GENERALIZED SPACE K_n

Takeo OHKUBO

(Received January 30, 1952)

1. Contact transformation. We consider the underlying n-dimensional manifofold X_n in which each set of n independent real variables $(\xi,^1\xi,^2\cdots\cdots,\xi^n)$ may be considered as the coordinates of a point, and then we associate a system of plane-elements π_α $(\alpha=1,2,\cdots\cdots,n)$ to every point in X_n. Under K_n we understand the *associated* point set. The point in X_n is called the fundamental point in K_n.

At an arbitrary fundamental point P(ξ), a linear homogeneous equation $\pi_\alpha\, d\xi_\alpha=0$ of $d\xi$'s with coefficients π's defines the (n−1)-direction at the point, and so π_α may be considered a plane-element at P(ξ).

A homogeneous contact transformation is defined by the following equations:

$$(1.1)\qquad \begin{cases} x^i = x^i(\xi^1, \cdots\cdots, \xi^n; \pi_1, \cdots\cdots, \pi_n), \\ p_i = p_i(\xi^1, \cdots\cdots, \xi^n; \pi_n, \cdots\cdots, \pi \\ p_i\, dx^i = \pi_\alpha d\xi^\alpha, \end{cases}$$

where x^i and p^i are homogeneous in the π's of degree zero and one respectivel y[1].

A neccessary and sufficient condition that equation $(1.1)_3$ holds for arbitrary values of $d\xi^\alpha$ and $d\pi^\alpha$ is

$$(1.2)\qquad p_i\, \partial x^i/\partial\xi^\alpha = \pi_\alpha, \qquad p_i\, \partial x^i/\partial\pi_\alpha = 0.$$

While, in accordance with Euler's formula, we have

$$(1.3)\qquad \pi_\alpha\, \partial x^i/\partial\pi^\alpha = 0, \qquad \pi_\alpha\, \partial p_i/\partial\pi_\alpha = p_i.$$

We also have

$$(1.4)\qquad \begin{aligned} &\partial p_i/\partial\xi^\beta\ \partial x^i/\partial\xi^\alpha - \partial p_i/\partial\xi^\alpha\ \partial x^i/\partial\xi^\beta = 0, \\ &\partial p_i/\partial\pi_\beta\ \partial x^i/\partial\xi^\alpha - \partial p_i/\partial\xi^\alpha\ \partial x^i/\partial\pi_\beta = \delta^\beta_\alpha, \\ &\partial p_i/\partial\pi_\beta\ \partial x^i/\partial\pi_\alpha - \partial p_i/\partial\pi_\alpha\ \partial x^i/\partial\pi_\beta = 0. \end{aligned}$$

The unique inverse of a contact transformation (1.1) is

$$(1.5)\qquad \begin{cases} \xi^\alpha = \xi^\alpha(x^1, \cdots\cdots, x^n; p_1, \cdots\cdots, p_n), \\ \pi_\alpha = \pi_\alpha(x^1, \cdots\cdots, x^n; p_1, \cdots\cdots, p_n), \\ p_i\, dx^i = \pi_\alpha\, d\xi^\alpha, \end{cases}$$

and analogously to (1.2) and (1.3) we have

$$(1.6)\qquad \pi_\alpha \partial\xi^\alpha/\partial x^i = p_i, \qquad \pi_\alpha \partial\xi^\alpha/\partial p_i = 0$$

and

$$(1.7)\qquad p_i\, \partial\xi^\alpha/\partial p_i = 0, \qquad p_i\, \partial\pi_\alpha/\partial p_i = \pi_\alpha.$$

We have also the following identities frequently used in this paper:

$$\partial\xi^\alpha/\partial x^i = \partial p_i/\partial\pi_\alpha, \quad \partial\pi_\alpha/\partial p_i = \partial x^i/\partial\xi^\alpha,$$
(1.8)
$$\partial\xi^\alpha/\partial p_i = -\partial x^i/\partial\pi_\alpha, \quad \partial\pi_\alpha/\partial x^i = -\partial p_i/\partial\xi^\alpha,$$

by which we derive from (1.4) the following:

$$\partial\pi_\alpha/\partial p_i\,\partial\pi_\beta/\partial x^i - \partial\pi_\alpha/\partial x^i\,\partial\pi_\beta/\partial p_i = 0,$$
(1.9)
$$\partial\pi_\alpha/\partial p_i\,\partial\xi^\beta/\partial x^i - \partial\pi_\alpha/\partial x^i\,\partial\xi^\beta/\partial p_i = \delta^\beta_\alpha,$$
$$\partial\xi^\alpha/\partial p_i\,\partial\xi^\beta/\partial x^i - \partial\xi^\alpha/\partial x^i\,\partial\xi^\beta/\partial p_i = 0.$$

Next we shall consider infinitesimal homogeneous contact transformations. Each infinitesimal homogeneous contact transformation is defined by equations of the form

(1.10) $$x^i = \xi^i + \partial C/\partial\pi_i\,\delta t, \qquad p_i = \pi_i - \partial C/\partial\xi^i\,\delta t,$$

where C is homogeneous of degree one in the π's, moreover any such function C determines an infinitesimal homogeneous contact transformation. The function C is called the characteristic function of the transformation.

If we form the differentials of (1.10), we obtain

$$dx^i = d\xi^i + \left(\frac{\partial^2 C}{\partial\pi_i\,\partial\xi^j}\,d\xi^j + \frac{\partial^2 C}{\partial\pi_i\,\partial\pi_j}\,d\pi_j\right)\delta t,$$
$$dp_i = d\pi_i - \left(\frac{\partial^2 C}{\partial\xi^i\,\partial\xi^j}\,d\xi^j + \frac{\partial^2 C}{\partial\xi^i\,\partial\pi_j}\,d\pi_j\right)\delta t.$$

These equations and (1.10) define the extended infinitesimal transformation. Hence in accordance with the general theory of continuous groups, the quantities $\pi_i\,d\xi^i$ is invariant under the finite guoup G_1 generated by the extended infinitesimal transformation, and the finite equations of G_1 are given by the integral of the equations

(1.11) $$d\xi^i/dt = \partial C/\partial\pi_i, \qquad d\pi_i/dt = -\partial C/\partial\xi^i,$$

say

(1.12) $$x^i = \varphi^i(\xi, \pi, t), \qquad p_i = \psi_i(\xi, \pi, t),$$

and their differentials. Conversely (1.12) define a one-parameter group of contact transformations.

If we transform the equations (1.11) by means of a general homogeneous contact transformation (1.1), we have

(1.13) $$dx^i/dt = \partial\bar{C}/\partial p_i, \qquad dp_i/dt = -\partial\bar{C}/\partial x^i.$$

Hence we have:

A group G_1 of homogeneous contact transformations is transformed into another group G_1 by any homogeneous contact transformation and the equations of the new group are the integrals of the equation (1.13), where $\bar{C}$ is the transform of the characteristic function of the given group.

Putting

$$2\, \Pi(\xi, \pi) = C^2(\xi, \pi),$$

now we shall adopt the following as the contravariant fundamental tensor

(1.14) $$\gamma^{\alpha\beta} = \frac{\partial^2 \Pi}{\partial \pi_\alpha\, \partial \pi_\beta},$$

which introduces metric properties in our space K_n and we define the covariant fundamental tensor by means of

$$\gamma^{\alpha\beta}\, \gamma_{\beta\gamma} = \delta^\alpha_\gamma.$$

The homogeneity property of C implies the following:

(1.15) $$\begin{cases} \dfrac{\partial^2 \Pi}{\partial \pi_\alpha\, \partial \pi_\beta}\, \pi_\alpha \pi_\beta = 2\, \Pi \;\; or \;\; \gamma^{\alpha\beta}\, \pi_\alpha \pi_\beta = 2\, \Pi, \\ \dfrac{\partial^2 \Pi}{\partial \pi_\alpha\, \partial \pi_\beta}\, \pi_\beta = C \dfrac{\partial C}{\partial \pi_\alpha} \;\; or \;\; \gamma^{\alpha\beta}\, p_\beta = C \dfrac{\partial C}{\partial \pi_\alpha}, \end{cases}$$

(1.16) $$\begin{cases} l_\alpha = \dfrac{\pi_\alpha}{C} = \gamma_{\alpha\beta}\, l^\beta, \\ l^\alpha = \partial C/\partial \pi_\alpha = \gamma^{\alpha\beta}\, l_\beta. \end{cases}$$

From the above equation (1.13) we see that the magnitude of the covariant vector π_α is C and $C\partial C/\partial\pi_\alpha$ is a contravariant vector perpendicular to the plane-element π_α and moreover π_α/C and $\partial C/\partial\pi_\alpha$ are both unit vectors.

2. Linear displacements in K_n. Now we shall define the absolute differentials of vectors, when a fundamental point $P(\xi)$ is displaced to a near point $(\xi + d\xi)$ and its plane-element π_α at the point P is changed slightly, say, into $\pi_\alpha + d\pi_\alpha$, by the following equations,

(2.1) $$\begin{cases} \delta v^\alpha = dv^\alpha + C_\beta^{\;\alpha\gamma}\, v^\beta\, d\pi_\gamma + \Gamma^\alpha_{\beta\gamma}\, v^\beta\, d\xi^\gamma, \\ \delta v_\alpha = dv_\alpha - C_\alpha^{\;\beta\gamma}\, v_\beta\, d\pi_\gamma - \Gamma^\beta_{\alpha\gamma}\, v_\beta\, d\xi^\gamma. \end{cases}$$

1°. *From the standpoint of contact transformation a change* $\pi_\alpha \to \rho\pi_\alpha$ *is of no significance, so that, the functions* $\Gamma^\alpha_{\beta\lambda}(\xi, \pi)$ *and* $C_\beta^{\;\alpha\gamma}(\xi, \pi)$ *may be assumed to be homogeneous of degree zero and one in the* π*'s respectively, and the relations* $C_\alpha^{\;\beta\gamma}\, \pi_\beta = 0$ *may be also to hold good.*

2°. *Secondly we assume that our connection is a metric one. This assumption leads to the following*

(2.2) $$\partial\gamma_{\alpha\beta}/\partial\xi^\gamma = \gamma_{\tau\beta}\, \Gamma^\tau_{\alpha\gamma} + \gamma_{\tau\alpha}\, \Gamma^\tau_{\beta\gamma}, \qquad \partial\gamma_{\alpha\beta}/\partial\pi_\gamma = \gamma_{\tau\alpha}\, C_\beta^{\;\tau\gamma} + \gamma_{\tau\beta}\, C_\alpha^{\;\tau\gamma},$$

or

(2.3) $$\partial\gamma_{\alpha\beta}/\partial\xi^\gamma = \Gamma_{\alpha\beta\gamma} + \Gamma_{\beta\alpha\gamma}, \qquad \partial\gamma_{\alpha\beta}/\partial\pi_\gamma = C^{\;\gamma}_{\beta\alpha} + C^{\;\gamma}_{\alpha\beta},$$

and

(2.4) $$\partial\gamma^{\alpha\beta}/\partial\xi_\gamma = -\,\Gamma^{\beta\alpha}_{\;\;\gamma} - \Gamma^{\alpha\beta}_{\;\;\gamma}, \qquad \partial\gamma^{\alpha\beta}/\partial\pi_\gamma = -\, C^{\beta\alpha\gamma} - C^{\alpha\beta\gamma}.$$

3°. *Denote* X *and* Y *two vectors of the same linear element* (ξ, π); *let* $\bar{D}X$ *and* $\bar{D}Y$ *their absolute differentials when their contravariant components* X^i *and* Y^i *are fixed and when* *their common linear element* (ξ, π) *rotate infinitesimally about their center. Then we assume that*

$$X \cdot \bar{D}Y = Y \cdot \bar{D}X.$$

The law of symmetry implies

$$C^{\alpha\beta\gamma} = C^{\beta\alpha\gamma}.$$

Combining the above relations with $(2.4)_2$ we have

$$C^{\alpha\beta\gamma} = -\tfrac{1}{2}\,\partial\gamma^{\alpha\beta}/\partial\pi_\gamma \text{ or } C_{\alpha\beta}^{\gamma} = \tfrac{1}{2}\,\partial\gamma_{\alpha\beta}/\partial\pi_\gamma. \tag{2.5}$$

For arbitrary function $f(\xi, \pi)$ we put

$$f_{\|}^{\alpha} = \partial f/\partial\pi_\alpha\, C \tag{2.6}$$

and for arbitrary quantities, say $T_{\alpha\beta}$, we denote as follows

$$T_{\alpha\beta}\, l^\beta = T_{\alpha o}, \quad \textit{etc.} \tag{2.7}$$

Putting

$$A^{\alpha\beta\gamma} = C\, C^{\alpha\beta\gamma} = -\tfrac{1}{2}\,\gamma^{\alpha\beta}\|^{\gamma}, \tag{2.8}$$

the tensor $A^{\alpha\beta\tau}$ is symmetric and the contracted tensor $A^{o\alpha\beta}$ is a zero tensor.

The absolute differential of any vector V^α may be written as follows:

$$Dv^\alpha = dv^\alpha + v^\beta \Gamma_{\beta\gamma}^{\alpha}\, d\xi^\gamma + v^\beta C_{\beta}^{\alpha\gamma}\, d\pi_\gamma, \tag{2.9}$$

and for l^α and l_α we have respectively

$$Dl^\alpha = dl^\alpha + l^\beta \Gamma_{\beta\gamma}^{\alpha}\, d\xi^\gamma, \text{ or } dl^\alpha = Dl^\alpha - \Gamma_{o\gamma}^{\alpha}\, d\xi^\gamma \tag{2.10}$$

and

$$Dl_\alpha = dl_\alpha - l_\beta \Gamma_{\alpha\gamma}^{\beta}\, d\xi^\gamma, \quad \text{or } dl_\alpha = Dl_\alpha + \Gamma_{\alpha\gamma}^{o}\, d\xi^\gamma. \tag{2.11}$$

Substituting (2.11) into (2.9) we obtain

$$Dv^\alpha = dv^\alpha + v^\beta \overset{*}{\Gamma}{}_{\beta\gamma}^{\alpha}\, d\xi^\gamma + v^\beta A_{\beta}^{\alpha\gamma}\, Dl_\gamma, \tag{2.12}$$

where

$$\overset{*}{\Gamma}{}_{\beta\gamma}^{\alpha} = \Gamma_{\beta\gamma}^{\alpha} + A_{\beta}^{\alpha\tau}\, \Gamma_{\tau o\gamma}. \tag{2.13}$$

By means of the above relations, (2.10) and (2.11) become

$$Dl^\alpha = dl^\alpha + \overset{*}{\Gamma}{}_{o\gamma}^{\alpha}\, d\xi^\gamma \tag{2.14}$$

and

$$Dl_\alpha = dl_\alpha - \overset{*}{\Gamma}{}_{\alpha o\gamma}\, d\xi^\gamma \tag{2.15}$$

respectively.

4°. *Finally we assume the law of symmetry:*

$$\overset{*}{\Gamma}{}_{\alpha\beta\gamma} = \overset{*}{\Gamma}{}_{\gamma\beta\alpha}. \tag{2.16}$$

We have easily the following relations:

(2.17) $$\overset{*}{\Gamma}_{\alpha\beta\gamma} + \overset{*}{\Gamma}_{\beta\alpha\gamma} = \partial\gamma_{\alpha\beta}/\partial\xi^{\gamma} + 2A_{\alpha\beta}^{\tau}\Gamma_{\tau o\gamma},$$

(2.18) $$\overset{*}{\Gamma}_{\alpha\beta\gamma} = [\alpha\gamma, \beta] + A_{\alpha\beta}^{\tau}\Gamma_{\tau o\gamma} - A_{\alpha\gamma}^{\tau}\Gamma_{\tau o\beta},$$

and

(2.19) $$\Gamma_{\alpha\beta\gamma} = [\alpha\gamma, \beta] + A_{\gamma\beta}^{\tau}\Gamma_{\tau o\alpha} - A_{\alpha\gamma}^{\tau}\Gamma_{\tau o\beta}.$$

contracting (2.19) by l^{β}, we have

(2.20) $$\Gamma_{\alpha o\gamma} = [\alpha\gamma, 0] - A_{\alpha\gamma}^{\tau}\Gamma_{\tau oo} = [\alpha\gamma, o] - A_{\alpha\gamma}^{\tau}[\tau o, o],$$

because

(2.21) $$\Gamma_{\gamma oo} = [\gamma o, o].$$

Substituting (2.20) into (2.19) or (2.18), we have

(2.22) $$\Gamma_{\alpha\beta\gamma} = [\alpha\gamma, \beta] + A_{\gamma\beta}^{\tau}([\tau\alpha, o] - A_{\tau\alpha}^{\sigma}[\sigma o, o]) - A_{\alpha\gamma}^{\tau}([\tau\beta, o] - A_{\tau\beta}^{\sigma}[\sigma o, o]),$$

or

(2.23) $$\begin{aligned}\overset{*}{\Gamma}_{\alpha\beta\gamma} &= [\alpha\gamma, \beta] + A_{\gamma\beta}^{\tau}([\tau\alpha, o] - A_{\tau\alpha}^{\sigma}[\sigma o, o]) \\ &+ A_{\alpha\beta}^{\tau}([\tau\gamma, o] - A_{\tau\gamma}^{\sigma}[\sigma o, o]) - A_{\alpha\gamma}^{\tau}([\tau\beta, o] - A_{\tau\beta}^{\sigma}[\sigma o, o]).\end{aligned}$$

Thus we have determined the connection completely.

Consider a variable vector, for example X^{α}. Since $\partial X^{\alpha}/\partial\pi_{\gamma}\,\pi_{\gamma} = 0$, its absolute differential can be put in the form

(2.24) $$DX^{\alpha} = X^{\alpha}|_{\gamma}\, d\xi^{\gamma} + X^{\alpha}|^{\gamma}\, Dl_{\gamma},$$

where

(2.25) $$X^{\alpha}|_{\gamma} = \partial X^{\alpha}/\partial\xi^{\gamma} + X^{\alpha}||^{\beta}\,\overset{*}{\Gamma}_{\beta o\gamma} + X^{\beta}\,\overset{*\alpha}{\Gamma}_{\beta\gamma}, \quad X^{\alpha}|^{\gamma} = X^{\alpha}||^{\gamma} + X^{\beta}A_{\beta}^{\alpha\gamma}.$$

Particularly for the fundamental tensor, we have

$$\gamma_{\alpha\beta}|_{\gamma} = \gamma^{\alpha\beta}|_{\gamma} = \gamma_{\alpha\beta}|^{\gamma} = \gamma^{\alpha\beta}|^{\gamma} = 0.$$

3. Spaces T_n as transforms of spaces K_n [2]. Consider the identity

(3.1) $$P(x, p) = \Pi(\xi, \pi)$$

under a homogeneous contact transformation. We notice that the function Π is homogeneous of degree two in the π's and that the function P is homogeneous of degree two in the p's. By differentiation we have

(3.2) $$\begin{aligned}\partial P/\partial p_i &= \partial\Pi/\partial\pi_{\alpha}\,\partial\pi_{\alpha}/\partial p_i + \partial\Pi/\partial\xi_{\alpha}\,\partial\xi^{\alpha}/\partial p_i, \\ \partial P/\partial x^i &= \partial\Pi/\partial\pi_{\alpha}\,\partial\pi_{\alpha}/\partial x^i + \partial\Pi/\partial\xi_{\alpha}\,\partial\xi^{\alpha}/\partial x^i,\end{aligned}$$

or

(3.2)′ $$\begin{aligned}\partial P/\partial p_i &= \partial\Pi/\partial\pi_{\alpha}\,\partial x^i/\partial\xi_{\alpha} - \partial\Pi/\partial\xi^{\alpha}\,\partial x^i/\partial\pi_{\alpha}, \\ -\partial P/\partial x^i &= \partial\Pi/\partial\pi_{\alpha}\,\partial p_i/\partial\xi^{\alpha} - \partial\Pi/\partial\xi^{\alpha}\,\partial p_i/\partial\pi_{\alpha}.\end{aligned}$$

From homogeneity property of Π in the π's, when we define quantities ξ'^{α} by $\xi'^{\alpha} = \partial \Pi / \partial \pi_{\alpha}$, we have

(3.3) $$\xi'^{\alpha} = \partial \Pi / \partial \pi_{\alpha} = \gamma^{\alpha\beta} \pi_{\beta},$$

(3.4) $$\pi_{\alpha} = \gamma_{\alpha\beta} \xi'^{\beta},$$

and

(3.5) $$\Pi = \frac{1}{2} \gamma^{\alpha\beta} \pi_{\alpha} \pi_{\beta}, \qquad \Pi = \frac{1}{2} \gamma_{\alpha\beta} \xi'^{\alpha} \xi'^{\beta}.$$

As we can solve (3.4) with respect to π_{α}, we can write (3.5) in terms of ξ^{α} and ξ'^{α}. When it is done we denote $\frac{1}{2} \gamma_{\alpha\beta} \xi'^{\alpha} \xi'^{\beta}$ by $\phi(\xi, \xi')$ as follows:

(3.6) $$\phi(\xi, \xi') = \frac{1}{2} \gamma_{\alpha\beta} \xi'^{\alpha} \xi'^{\beta}.$$

From (3.3) and (3.6), we can introduce the following relations

(3.7) $$\gamma_{\alpha\beta} = \partial \pi_{\beta} / \partial \xi'^{\alpha},$$

(3.8) $$\partial \phi / \partial \xi'^{\alpha} = \pi_{\alpha},$$

(3.9) $$\frac{\partial^2 \phi}{\partial \xi'^{\alpha} \partial \xi'^{\beta}} = \gamma_{\alpha\beta},$$

(3.10) $$\partial \gamma_{\alpha\beta} / \partial \xi_{\gamma} = -\left[\gamma^{\alpha\varepsilon} \left\{ {\beta \atop \varepsilon\gamma} \right\} + \gamma^{\varepsilon\beta} \left\{ {\alpha \atop \varepsilon\gamma} \right\}\right], \qquad \partial \gamma_{\beta\alpha} / \partial \xi^{\gamma} = [\beta\gamma, \alpha] + [\alpha\gamma, \beta].$$

From $(3.5)_1$ and (3.10) we have

(3.11) $$\partial \Pi / \partial \xi_{\gamma} = -\xi'^{\delta} \xi'^{\beta} [\delta\gamma, \beta]$$

and from (3.6)

(3.12) $$\partial \phi / \partial \xi^{\gamma} = -\pi_{\alpha} \gamma^{\alpha\delta} \pi_{\varepsilon} \left\{ {\varepsilon \atop \delta\gamma} \right\} = \xi'^{\delta} \xi'^{\beta} [\gamma\delta, \beta].$$

Hence we have

(3.13) $$\partial \phi / \partial \xi^{\gamma} = -\partial \Pi / \partial \xi^{\gamma}.$$

When we define x'^{i} by the following

(3.14) $$x'^{i} = \partial P / \partial p_{i},$$

x'^{i} is homogeneous of degree one in the p's.

From the definition of ξ'^{α} the first equation of (3.2) is

(3.15) $$\partial P / \partial p_{i} = \partial \Pi / \partial \pi_{\alpha} \left(\partial x^{i} / \partial \xi_{\alpha} + \pi_{\sigma} \left\{ {\sigma \atop \alpha\varepsilon} \right\} \partial x^{i} / \partial \pi_{\varepsilon} \right)$$

in consequence of (3.11). When we define g^{ij} by

(3.16) $$g^{ij} = \frac{\partial^2 P}{\partial p_i \partial p_j},$$

immediately we have the following relations

(3.17) $$g^{ij} p_i p_j = \gamma^{\alpha\beta} \pi_{\alpha} \pi_{\beta} = 2P.$$

Differentiating the above equation we have

(3.18)
$$\partial P/\partial x^k = \tfrac{1}{2}\,\partial g^{ij}/\partial x^k\, p_i p_j = -\tfrac{1}{2} g^{ij}\,\partial P/\partial p_h\, p_i \left\{[jk, h] + [hk, j]\right\}$$
$$= -\,\partial P/\partial p_i\,\partial P/\partial p_j\,[jk, i],$$

in concequence of the similar equations to (3.10), where g_{ij} is the inverse of g^{ij}.

On the other hand from $(3.2)'_2$ and (3.11), we have

(3.19)
$$-\,\partial P/\partial x^k = \partial \Pi/\partial \pi_\sigma \left(\partial p_k/\partial \xi^\sigma + \pi_\varepsilon \left\{ {\varepsilon \atop \sigma\tau} \right\} \partial p_k/\partial \pi_\varepsilon\right).$$

From this result and equation (3.18), we get

(3.20)
$$\partial P/\partial p_i\,\partial P/\partial p_j\,[jk, i] = \partial \Pi/\partial \pi_\sigma \left(\partial p_k/\partial \xi^\sigma + \pi_\varepsilon \left\{ {\varepsilon \atop \sigma\tau} \right\} \partial p_k/\partial \pi_\tau\right).$$

From (3.18) and the similar equations to (3.2)':

(3.21)
$$\partial \Pi/\partial \pi_\alpha = \partial P/\partial p_i\;\partial \xi^\alpha/\partial x^i - \partial P/\partial x^i\;\partial \xi^\alpha/\partial p_i,$$
$$-\,\partial \Pi/\partial \xi^\alpha = \partial P/\partial p_i\;\partial \pi_\alpha/\partial x^i - \partial P/\partial x^i\;\partial \pi_\alpha/\partial p_i,$$

we get the following relation

(3.22)
$$\partial \Pi/\partial \pi_\alpha = \partial P/\partial p_i \left(\partial \xi^\alpha/\partial x^i + \partial P/\partial p_j\,[ik, j]\;\partial \xi^\alpha/\partial p_k\right),$$

which is the inverse of (3.15). And the inverse of (3.20) is following

(3.23)
$$\pi_\gamma\,\gamma^{\delta\gamma}\,\pi_\varepsilon\,\gamma^{\varepsilon\beta}\,[\delta\alpha, \beta] = -\,\partial P/\partial \xi^\alpha = \partial P/\partial p_j$$
$$\left(\partial \pi_\alpha/\partial x^j + \partial P/\partial p_k\,[ji, k]\;\partial \pi_\alpha/\partial p_i\right).$$

When any contact transformation is applied to our space K_n, we have spaces T_n as its transforms. Consider a non-singular transformation in K_n

(3.24)
$$\xi^\alpha = f^{-1\alpha}(\bar{\xi}), \qquad \pi_\alpha = \bar{\pi}_\beta\,\partial \bar{\xi}^\beta/\partial \xi^\alpha.$$

Then if the coordinates x^i in T_n are subject to the transformation

(3.25)
$$x^i = f^{-1i}(\bar{x})$$

and if we put

(3.26)
$$p_i\,\partial x^i/\partial \bar{x}^j = \bar{p}_j, \qquad p_i = \bar{p}_j\,\partial \bar{x}^j/\partial x^i,$$

and

(3.27)
$$\Pi(\xi, \pi) = \bar{\Pi}(\bar{\xi}, \bar{\pi}),$$

we have

$$\bar{p}_i\,\partial \bar{x}^i/\partial \xi^\alpha = \bar{\pi}_\alpha, \qquad \bar{p}_i\,\partial \bar{x}^i/\partial \bar{\pi}_\alpha = 0 \qquad \textit{and} \quad \partial \Pi/\partial \pi_\alpha = \partial \Pi/\partial \bar{\pi}_\beta\,\partial \xi^\alpha/\partial \bar{\xi}^\beta.$$

Hence $\partial \Pi/\partial \pi_\alpha$ and $\partial \bar{\Pi}/\partial \bar{\pi}_\beta$ are the components of a contravariant vector in K_n in their respective coordinates. From (3.1) and (3.27) we have

(3.28)
$$P(x, p) = \bar{P}(\bar{x}, \bar{p}).$$

Differentiating the above relation we have

(3.29)
$$\partial P/\partial p_i = \partial \bar{P}/\partial \bar{p}_j\,\partial x^i/\partial \bar{x}^j, \qquad \partial \bar{P}/\partial \bar{p}_i = \partial P/\partial p_j\,\partial \bar{x}^i/\partial x^j.$$

From the equations (3.24) we have

$$\partial\pi_\gamma/\partial\bar{\xi}^\delta\, \partial\xi^\gamma/\partial\bar{\xi}^\sigma + \pi_\gamma\, \partial^2\xi^\gamma/\partial\bar{\xi}^\delta\, \partial\bar{\xi}^\sigma = 0.$$

On the other hand from (2.23) we have

$$\partial^2\,\xi^\gamma/\partial\bar{\xi}^\delta\,\partial\bar{\xi}^\sigma + \Gamma^{*\gamma}_{\alpha\varepsilon}\,\partial\xi^\alpha/\partial\bar{\xi}^\delta\,\partial\xi^\varepsilon/\partial\bar{\xi}^\sigma = \bar{\Gamma}^{*\beta}_{\delta\sigma}\,\partial\xi^\gamma/\partial\bar{\xi}^\beta.$$

From above two equations and relations:

$$\partial\bar{x}^j/\partial\bar{\xi}^\sigma = \partial\bar{x}^j/\partial x^i\left(\partial x^i/\partial\xi^\alpha\,\partial\xi^\alpha/\partial\bar{\xi}^\sigma + \partial x^i/\partial\pi_\gamma\,\partial\pi^\gamma/\partial\bar{\xi}^\sigma\right),$$

$$\partial\bar{x}^j/\partial\bar{\pi}_\varepsilon = \partial\bar{x}^j/\partial x^i\,\partial x^i/\partial\pi_\gamma\,\partial\bar{\xi}^\varepsilon/\partial\xi^\gamma,$$

we have

(3.30) $$\partial\bar{x}^j/\partial\bar{\xi}^\delta + \bar{\beta}_{\delta\varepsilon}\,\partial\bar{x}^j/\partial\bar{\pi}_\varepsilon = \partial\bar{x}^j/\partial x^i\,\partial\xi^\alpha/\partial\bar{\xi}^\delta\, f^i_\alpha,$$

where $\beta_{\alpha\gamma} = \pi^\beta\,\Gamma^{*\beta}_{\alpha\gamma}$ and $f^i_\alpha = \partial x^i/\partial\xi^\alpha + \beta_{\alpha\gamma}\,\partial x^i/\partial\pi_\gamma$.

Since

$$\partial\Pi/\partial\pi_\mu\,\pi_\rho\,\gamma^{\rho\lambda}\,\Gamma^{*}_{\mu\lambda\nu} = 2\Pi\,\Gamma^{*}_{00\nu} = 2\Pi\,[0\nu,0] = \partial\Pi/\partial\pi_\mu\,\pi_\rho\left\{{\rho \atop \mu\nu}\right\},$$

from (3.30) we have

(3.31) $$\frac{\partial\bar{\Pi}}{\partial\bar{\pi}_\delta}\left(\frac{\partial\bar{\pi}^j}{\partial\bar{\xi}^\delta} + \bar{\pi}\left\{{\rho \atop \delta\varepsilon}\right\}\frac{\partial\bar{\pi}^j}{\partial\bar{\pi}_\varepsilon}\right) = \frac{\partial\Pi}{\partial\pi_\alpha}\left(\frac{\partial x^i}{\partial\xi^\alpha} + \pi_\beta\left\{{\beta \atop \alpha\gamma}\right\}\frac{\partial x^i}{\partial\pi_\gamma}\right)\frac{\partial\bar{x}^j}{\partial x^i}$$

by multiplying $\partial\bar{\Pi}/\partial\bar{\pi}_\delta$. If we multiply (3.15) by $\partial\bar{x}^j/\partial x^i$ and if we compare the result with (3.31), we have

(3.32) $$\frac{\partial P}{\partial p_i}\,\frac{\partial\bar{x}^j}{\partial x^i} = \frac{\partial\bar{\Pi}}{\partial\bar{\pi}_\delta}\left(\frac{\partial\bar{x}^j}{\partial\bar{\xi}^\delta} + \bar{\pi}_\rho\left\{{\rho \atop \delta\varepsilon}\right\}\frac{\partial\bar{x}^j}{\partial\bar{\pi}_\varepsilon}\right) = \frac{\partial\bar{P}}{\partial\bar{p}_j}.$$

4. Tensors in spaces T_n. Let $X^\alpha(\xi,\pi)$ which are homogeneous of degree zero in the π's, be the components of a contravariant vector in K_n and define functions $v^i(x,p)$ in T_n by the equations

(4.1) $$v^i = f^i_\alpha\,X^\alpha.$$

Let $X_\alpha(\xi,\pi)$ which are homogeneous of degree zero in the π's, be the components of a contravariant vector in K_n and define functions $v_i(x,p)$ in T_n by the equations

(4.2) $$v_i = X_\alpha\left(\partial\xi^\alpha/\partial x^i + b_{ij}\,\partial\xi^\alpha/\partial p_j\right),$$

where the functions b_{ij} are symmetric in its indices and are homogeneous of degree one in the p's and are to be such that

(4.3) $$\partial p_i/\partial\xi^\alpha = \partial x^j/\partial\pi_\sigma\,\beta_{\sigma\alpha}\,b_{ij} - \partial p_i/\partial\pi_\sigma\,\beta_{\sigma\alpha} + \partial x^j/\partial\xi_\alpha\,b_{ij}.$$

Then by the equation (4.3) we have

(4.4) $$f^i_\alpha\,h^\beta_i = \delta^\beta_\alpha,$$

where we put

$$h^\beta_i = \partial\xi^\beta/\partial x^i + b_{ij}\,\partial\xi^\beta/\partial p_j.$$

Therefore by means of (4.1), (4.2) and (4.4) we have

$$v_i v^i = X_\alpha X^\alpha.$$

Since

$$\partial x^i/\partial\xi^\alpha\, \partial\xi^\beta/\partial x^i = \delta^\beta_\alpha - \partial p_j/\partial\xi^\alpha\, \partial\xi^\beta/\partial p_j,$$

$$\partial x^i/\partial\pi_\gamma\, \partial\xi^\beta/\partial x^i = -\,\partial p_j/\partial\pi_\gamma\, \partial\xi^\beta/\partial p_j,$$

the equation (4.3) is written in the form

(4.5) $$\partial p_j/\partial\xi^\alpha + \beta_{\alpha\gamma}\, \partial p_j/\partial\pi_\gamma - b_{ij}\left(\partial x^i/\partial\xi^\alpha + \beta_{\alpha\gamma}\, \partial x^i/\partial\pi_\gamma\right) = 0$$

or

(4.6) $$\partial\pi_\alpha/\partial x^j + b_{ij}\, \partial\pi_\alpha/\partial p_i - \beta_{\alpha\gamma}\left(\partial\xi^\gamma/\partial x^j + b_{ij}\, \partial\xi^\gamma/\partial p_i\right) = 0.$$

Multiplying (4.5) by $\partial\Pi/\partial\pi_\alpha$ and making use of (3.15) and (3.20), we have

$$\frac{\partial P}{\partial p_i}\frac{\partial P}{\partial p_k}[ij,\ k] - b_{ij}\frac{\partial P}{\partial p_i} = 0,$$

$$i.\ e.\quad b_{ij}\frac{\partial P}{\partial p_i} = \Gamma^*_{jki}\frac{\partial P}{\partial p_i}\frac{\partial P}{\partial p_k} = \frac{\partial P}{\partial p_i}\frac{\partial P}{\partial p_k}[ij,\ k].$$

Hence we have

(4.7) $$b_{ij} = \Gamma^*_{ikj}\frac{\partial P}{\partial p_k} + C_{ij} = \beta_{ij} + C_{ij},$$

where

(4.8) $$C_{ij}\frac{\partial P}{\partial p_i} = 0, \qquad C_{ij} = C_{ji}.$$

From (4.6) we have the equation analogous to (4.3)

(4.9) $$\partial\pi_\lambda/\partial x^i = \partial\xi^\nu/\partial p_j\, b_{ij}\, \beta_{\nu\lambda} - \partial\pi_\lambda/\partial p_j\, b_{ij} + \partial\xi^\nu/\partial x^i\, \beta_{\lambda\nu}.$$

By means of (4.5) and the identities

$$\frac{\partial x^i}{\partial\xi^\alpha}\frac{\partial\xi^\alpha}{\partial x^j} + \frac{\partial x^i}{\partial\pi_\alpha}\frac{\partial\pi_\alpha}{\partial x^j} = \delta^j_i, \qquad \frac{\partial x^i}{\partial\xi^\alpha}\frac{\partial\xi^\alpha}{\partial p_j} + \frac{\partial x^i}{\partial\pi_\alpha}\frac{\partial\pi_\alpha}{\partial p_j} = 0$$

we obtain

(4.10) $$f^i_\alpha\, h^\alpha_j = \delta^i_j.$$

When a transformation of coordinates (3.24), (3.25) and (3.26) is made in K_n and T_n, the equation (4.1) multiplied by $\partial\bar{x}^j/\partial x^i$ is written by means of (3.30) in the form:

$$v^i\, \partial\bar{x}^j/\partial x^i = \bar{v}^\sigma\left(\partial\bar{x}^j/\partial\bar{\xi}^\sigma + \bar{\beta}_{\sigma\varepsilon}\, \partial\bar{x}^j/\partial\bar{\pi}_\varepsilon\right) = \bar{v}^j,$$

therefore v^i, as defined by (4.1), are components of a contravariant vector in T_n. Also we see that v^i, as defined by (4.2), are components of a covariant vector in T_n. From (3.15) we have

(4.11) $$\partial P/\partial p_i = \partial\Pi/\partial\pi_\alpha\, f^i_\alpha.$$

Differentiating (4.11), and making use of the relation

(4.12) $$\partial^2 \Pi/\partial\pi_\alpha\,\partial\xi^\varepsilon = -\left\{{\alpha \atop \delta\varepsilon}\right\}\gamma^{\tau\delta}\,\pi_\tau - \gamma^{\alpha\delta}\,\beta_{\delta\varepsilon} - A^{\alpha\ \omega}_{\ \varepsilon}\,[\omega\lambda,\,0]\,\partial\Pi/\partial\pi_\lambda,$$

which is obtained by successive differentiation of (3.5), we have, after some calculation,

$$g^{ij} = \gamma^{\alpha\beta} f^i_\alpha\, f^j_\beta + \partial\Pi/\partial\pi_\delta\big[\partial f^i_\delta/\partial\pi_\beta\,\partial x^j/\partial\xi^\beta - \big(\partial f^i_\delta/\partial\xi^\beta$$
$$- f^i_\alpha\left\{{\alpha \atop \delta\beta}\right\} - A^{\alpha\ \omega}_{\ \beta}[\omega\delta,\,o]\,f^i_\alpha\big)\,\partial x^j/\partial\pi_\beta\big].$$

But we can prove that the second term of the second member of the above equation vanishes, hence we have

(4.13) $$g^{ij} = \gamma^{\alpha\beta} f^i_\alpha\, f^j_\beta \quad or \quad g_{ij} = \gamma_{\alpha\beta}\, h^\alpha_i\, h^\beta_j.$$

Now we consider the expression for b_{ij} in (4.7). If we put

$$C_{ij} = 0,$$

we obtain from (4.7) and (2.23)

(4.14) $$b_{ij} = \beta_{ij} = \overset{*}{\Gamma}_{ikj}\,\partial P/\partial p_k = [ij,\,k]\,\partial P/\partial p_k - A^k_{ij}\,[kh,\,o]\,\partial P/\partial p_h.$$

We can verify that the covariant vector in T_n derived from a gradient vector $\partial\varphi/\partial\xi_\alpha + \beta_{\alpha\beta}\,\partial\varphi/\partial\pi_\beta$ in K_n is $\partial f/\partial x^i + b_{ij}\,\partial f/\partial p_j$ where $\varphi(\xi,\pi) = f(x,p)$. In fact, we have by differentiating the above identity with respect to ξ^α and π_α respectively

$$\partial\varphi/\partial\xi^\alpha = \partial f/\partial x^j\,\partial x^j/\partial\xi^\alpha + \partial f/\partial p_j\,\partial p_j/\partial\xi^\alpha,$$

$$\partial\varphi/\partial\pi_\alpha = \partial f/\partial x^j\,\partial x^j/\partial\pi_\alpha + \partial f/\partial p_j\,\partial p_j/\partial\pi_\alpha.$$

Substituting the expression $\partial\varphi/\partial\xi_\alpha + \beta_{\alpha\beta}\,\partial\varphi/\partial\pi_\beta$ for X_α in (4.2) and making use of the above equations, we obtain

$$v_i = \left\{\frac{\partial f}{\partial x^j}\frac{\partial x^j}{\partial\xi^\alpha} + \frac{\partial f}{\partial p_j}\frac{\partial p_j}{\partial\xi^\alpha} + \beta_{\beta\alpha}\left(\frac{\partial f}{\partial x^j}\frac{\partial x^j}{\partial\pi_\beta} + \frac{\partial f}{\partial p_j}\frac{\partial p_j}{\partial\pi_\beta}\right)\right\}\left(\frac{\partial\xi^\alpha}{\partial x^i} + b_{ik}\frac{\partial\xi^\alpha}{\partial p_k}\right)$$
$$= \frac{\partial f}{\partial x_i} + b_{ik}\frac{\partial f}{\partial p_k} + \frac{\partial f}{\partial x_j}\left\{-\frac{\partial x^j}{\partial\pi_\alpha}\frac{\partial\pi_\alpha}{\partial x_i} + b_{ik}\frac{\partial\pi_\alpha}{\partial p_j}\frac{\partial\pi_\alpha}{\partial p_k} + \frac{\partial x^j}{\partial\pi_\beta}\left(\frac{\partial\pi_\beta}{\partial x^i} + b_{ik}\frac{\partial\pi_\beta}{\partial p_k}\right)\right\}$$
$$+ \frac{\partial f}{\partial p_j}\left\{-\frac{\partial\pi_\alpha}{\partial x^i}\frac{\partial p_j}{\partial x^\alpha} - b_{ik}\frac{\partial p_j}{\partial\pi_\alpha}\frac{\partial\pi_\alpha}{\partial p_k} + \frac{\partial p_j}{\partial\pi_\beta}\left(\frac{\partial\pi_\beta}{\partial x^i} + b_{ik}\frac{\partial\pi_\beta}{\partial p_k}\right)\right\}.$$

But the last two terms of the last member vanish identically, because the relations

$$\partial\pi_\alpha/\partial p_j\,\partial\xi^\alpha/\partial p_k + \partial x^j/\partial\pi_\beta\,\partial\pi_\beta/\partial p_k =$$
$$- \partial x^j/\partial\xi^\alpha\,\partial x^k/\partial\pi_\alpha + \partial x^j/\partial\pi_\alpha\,\partial x^k/\partial\xi^\alpha = 0,$$
$$- \partial\pi^\alpha/\partial x^j\,\partial p_i/\partial\pi_\alpha + \partial p_j/\partial\pi_\alpha\,\partial\pi_\alpha/\partial x^i =$$
$$\partial p_j/\partial\xi^\alpha\,\partial p_i/\partial\pi_\alpha - \partial p_j/\partial\pi_\alpha\,\partial p_i/\partial\xi^\alpha = 0,$$

hold, so that we have

$$v_i = \partial f/\partial x^i + b_{ij}\,\partial f/\partial p_j.$$

5. **Contact frame.** Differential forms $d\pi_\alpha - \beta_{\alpha\gamma}\,d\xi^\gamma$ and $dp_i - b_{ij}\,dx^j$ are the

components of a covariant vector in respective spaces K_n and T_n. In fact we have

$$d\pi_\alpha - \beta_{\alpha\gamma}\, d\xi^\gamma = \left(\partial\pi_\alpha/\partial x^i - \beta_{\alpha\gamma}\, \partial\xi^\gamma/\partial x^i\right) dx^i$$

$$+ \left(\partial x^i/\partial\xi^\alpha - \beta_{\alpha\gamma}\, \partial\xi^\gamma/\partial p_i\right) dp_i = f^i_\alpha\left(dp_i - b_{ij}\, dx^j\right)$$

by making use of (4.5) and (1.8). Hence we put

(5.1) $$\delta\pi_\alpha = d\pi_\alpha - \beta_{\alpha\gamma}\, d\xi^\gamma .$$

While differential $d\xi^\alpha$ and dx^i are transformed in the manner:

(5.2) $$f^i_\alpha d\xi^\alpha = dx^i - \partial x^i/\partial\pi_\alpha\, \delta\pi_\alpha .$$

We introduce here a "**contact frame**" [3][4] defined by the functions $\Gamma^{\alpha\beta}(\xi, \pi)$, homogeneous of degree -1 in the π's and symmetric in the superier indices, which are transformed in the following manner:

(5.3) $$-\partial x^j/\partial\pi_\alpha + \Gamma^{\alpha\beta} f^i_\beta = \Gamma^{ij} h^\alpha_i .$$

Then the quantities defined by

(5.4) $$\delta\xi^\alpha = d\xi^\alpha + \Gamma^{\alpha\beta}\, \delta\pi^\beta = d\xi^\alpha + \Gamma^{\alpha\gamma}\, d\pi_\gamma - \Gamma^{\alpha\gamma}\, \beta_{\gamma\sigma}\, d\xi^\sigma ,$$

are transformed as follows

(5.5) $$\delta\xi^\alpha f^i_\alpha = \delta x^i ,$$

where

$$\delta x^i = dx^i + \Gamma^{ij}\, \delta p_j = dx^i + \Gamma^{ij}\, dp_j - \Gamma^{ij}\, b_{jk}\, dx^k .$$

In our theory, these Pfaffian forms (5.1) and (5.4) play the role of differentials of ordinary coordinates. If we resolve (5.1) and (5.4) with respect to $d\xi^\alpha$ and $d\pi_\alpha$, we obtain

(5.6) $$\begin{cases} d\xi^\alpha = \delta\xi^\alpha - \Gamma^{\alpha\gamma}\, \delta\pi^\gamma , \\ d\pi_\alpha = \delta\pi_\alpha + \beta_{\alpha\gamma}\, \delta\xi_\gamma - \beta_{\alpha\gamma}\, \Gamma^{\gamma\sigma}\, \delta\pi_\sigma . \end{cases}$$

Now we can define the covariant derivatives of a vector as follows·

(5.7) $$DX_\alpha = dX^\alpha + C^{\alpha\gamma}_\beta\, X^\beta\, d\pi_\gamma + \Gamma^\alpha_{\beta\gamma}\, X^\beta\, d\xi^\gamma$$

$$= \nabla_\sigma X^\alpha\, \delta\xi^\sigma + \nabla^\sigma X^\alpha\, \delta\pi_\sigma ,$$

where we put

(5.8) $$\nabla_\sigma X^\alpha = \partial X^\alpha/\partial\xi^\sigma + \Gamma^\alpha_{\beta\sigma}\, X^\beta + \beta_{\gamma\sigma}\left(\partial X^\alpha/\partial\pi_\gamma + C^{\alpha\gamma}_\beta\, X^\beta\right),$$

(5.9) $$\nabla^\sigma X^\alpha = \partial X^\alpha/\partial\pi_\sigma + C^{\alpha\sigma}_\beta\, X^\beta - \Gamma_{\gamma\sigma}\left(\partial X^\alpha/\partial\xi^\gamma\right.$$

$$\left. + \Gamma^\alpha_{\beta\gamma}\, X^\beta\right) - \beta_{\gamma\varepsilon}\, \Gamma^{\varepsilon\sigma}\left(\partial X^\alpha/\partial\pi_\gamma + C^{\alpha\gamma}_\beta\, X^\beta\right).$$

In particular for π_α, we have

$$\nabla_\sigma\pi_\alpha = 0 \qquad \textit{and} \qquad \nabla^\sigma\pi_\alpha = \delta^\sigma_\alpha .$$

Now we can immediately obtain various curvature tensors and torsion tensors in usual manner, but here we write the only following relation for a scalar function $f(\xi, \pi)$

$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) f = S_{\mu\nu\tau} \nabla^\tau f + S_{\mu\nu\tau} \Gamma^{\alpha\tau} \nabla_\alpha f,$$

where

(5.10) $$S_{\mu\nu\tau} = \partial\beta_{\tau\nu}/\partial\xi^\mu - \partial\beta_{\tau\mu}/\partial\xi^\nu + \beta_{\varepsilon\mu}\,\partial\beta_{\tau\nu}/\partial p_\varepsilon - \beta_{\varepsilon\nu}\,\partial\beta_{\tau\mu}/\partial p_\varepsilon .$$

6. Absolute differential in T_n. In the space K_n with linear connections $\left(\Gamma^\alpha_{\beta\gamma}, C^{\alpha\gamma}_\beta\right)$ the expression $DX^\alpha = dX^\alpha + \Gamma^\alpha_{\beta\gamma} X^\beta d\xi^\gamma + C^{\alpha\gamma}_\beta X^\beta d\pi^\gamma$ is an absolute differential of the vector X^α. We denote by

(6.1) $$DY^i = dY^i + Y^j L^i_{jk}\, dp_k + Y^j D^{ik}_j\, dp_k$$

the absolute differential of the vector Y^i in T_n derived from X^α in K_n by equation (4.1). From $Y_i DY^i = X_\alpha DX^\alpha$, we have

$$dY^i + Y^j L^i_{jk}\, dx^k + Y^j D^{ik}_j\, dp_k = \left(dX^\alpha + \Gamma^\alpha_{\beta\gamma} X^\beta d\xi^\gamma + C^{\alpha\gamma}_\beta X^\beta d\pi_\gamma\right) f^i_\alpha .$$

Since $dX^i = df^i_\alpha X^\alpha + dX^\alpha f^i_\alpha$ for arbitrary vector, we must have

(6.2) $$L^i_{jk}\, dx^k + D^{ik}_j\, dp_k = -\, df^i_\beta h^\beta_j + \left[\Gamma^\alpha_{\beta\gamma} d\xi^\gamma + C^{\alpha\gamma}_\beta d\pi_\gamma\right] f^i_\alpha h^\beta_j .$$

On the other hand, since π_α and p_i are covariant vector in K_n and T_n respectively, we have

(6.3) $$dp_k - L^j_{ki} p_j\, dx^i - D^{ji}_k p_j\, dp_i = \left(d\pi_\alpha - \Gamma^\beta_{\alpha\gamma} \pi_\beta\, d\xi^\gamma\right) h^\alpha_k$$

$$= \delta\pi_\alpha h^\alpha_k = \delta p_k = dp_k - b_{ki}\, dx^i .$$

hence, we obtain

(6.4) $$D^{ji}_k p_j = 0 \qquad or \qquad D_{kji} p^j = 0.$$

(6.5) $$L^j_{ki} p_j = b_{ki} .$$

By covariant derivation of the fundamental tensor, we have

$$dg^{ij} + L^i_{lk} g^{lj}\, dx^k + L^j_{lk} g^{il}\, dx^k + D^{ik}_l g_{lj}\, dp_k$$
$$+ D^{jk}_l g^{il}\, dp_k = Dg^{\alpha\beta} f^i_\alpha f^j_\beta = 0.$$

Hence, we have

(6.6) $$\partial g^{ij}/\partial x^k = L^{ji}{}_k + L^{ij}_k ,$$

(6.7) $$\partial g^{ij}/\partial p_k = D^{jik} + D^{ijk} .$$

From (6.4), (6.7) and relation $p_j\, \partial g^{ij}/\partial p_k = 0$, we have

(6.8) $$D^{jik} p_j = 0.$$

By equation (6.4), (6.3) becomes

(6.9) $$dp_k - L^l_{ki} p_l\, dx^i = (d\pi_\alpha - \beta_{\alpha\gamma}\, d\xi^\gamma)\, h^\alpha_k .$$

If we put $\delta\pi_\alpha = 0$, from (6.9) we have $dp_k = L^{\;l}_{ki}\, p_l\, dx^i$, hence (6.2) becomes

$$\left(L_{j\;k}^{\;i} + D_j^{ih} L_{h\;k}^{\;l} p_l\right) dx^k =$$
$$- h_j^\beta \left[\left(\partial f_\beta^i / \partial \xi^\alpha + \partial f_\beta^i / \partial \pi_\gamma\, \beta_{\gamma\alpha}\right) d\xi^\alpha - \overset{*}{\Gamma}{}^{\alpha}_{\beta\gamma}\, d\xi^\gamma f_\alpha^i\right].$$

Therefore, we have

(6.10) $$\overset{*}{L}{}^{i}_{jk} f^k_\alpha f^j_\beta = -\left(\partial f^i_\beta / \partial \xi^\alpha + \partial f^i_\beta / \partial \pi_\gamma\, \beta_{\gamma\alpha}\right) + \overset{*}{\Gamma}{}^{\gamma}_{\beta\alpha} f^i_\gamma,$$

where

(6.11) $$\overset{*}{L}{}^{i}_{jk} = L^i_{jk} + D_j^{ih} L_{h\;k}^{\;l}\, p_l .$$

When we alternate the indices α and β in (6.10) and substruct the result from (6.10), we have

(6.12) $$\left(\overset{*}{L}{}^{i}_{jk} - \overset{*}{L}{}^{i}_{kj}\right) f^k_\alpha f^j_\beta = - S_{\alpha\beta\tau}\, \partial x^i / \partial \pi_\tau ,$$

because $\overset{*}{\Gamma}{}^{\gamma}_{\beta\alpha}$ is symmetric in indices β and α.

If the torsion tensor $S_{\alpha\beta\tau}$ vanishes, from (6.12) the quantity $\overset{*}{L}{}^{i}_{jk}$ is symmetric in lower indices, i.e.

(6.13) $$\overset{*}{L}{}^{i}_{jk} = \overset{*}{L}{}^{i}_{kj}.$$

Assume that the quantity D^{ijk} is symmetric in indices i and j, from (6.7) we have

(6.14) $$D^{ijk} = \tfrac{1}{2}\, \partial g^{ij} / \partial p_k ,$$

then, in such a case, the conditions (6.4) and (6.8) upon the quantities D^{ijk} are satisfied. In this way we have, from (6.11),

(6.15) $$\overset{*}{L}{}^{i}_{jk} = L^i_{jk} + A_j^{ih} L_{hok},$$

where

$$A_j^{ih} = \sqrt{2P}\, D_j^{ih} \qquad \textit{and} \qquad L_{hok} = L_{hik}\, l^i .$$

By alternating the lower indices j, k in $\overset{*}{L}_{ijk}$ and by summing the result to $\overset{*}{L}_{ijk}$, we have

$$\overset{*}{L}_{jik} + \overset{*}{L}_{ijk} = L_{jik} + L_{ijk} + 2A_{ij}^{\;h} L_{hok} = \partial g^{ij} / \partial x^k + 2A_{ij}^{\;h} L_{hok},$$

because

$$Dg_{ij} = dg_{ij} - (L_{ijk} + L_{jik})\, dx^k - 2C_{ij}^{\;k}\, dp_k = Dg_{\alpha\beta}\, h_i^\alpha h_j^\beta = 0.$$

Hence we have

$$\partial g_{ij} / \partial x^k = L_{ijk} + L_{jik},$$

accordingly from (2.13), (2.16) and (2.17), we conclude

(6.16) $$L^i_{jk} = \Gamma^i_{jk}.$$

In this case the transformed space T_n is also the space K_n with the connections C's and Γ's.

Bibliography.

1. L. P. Eisenhart: Continuous groups of transformations, 1933.
2. L. P. Eisenhart: Finsler spaces derived from Riemann spaces by contact transformations. Annals of Math. vol. 49, 227–254.
3. Y. Mutô: On the connections in the Manifold admitting homogeneous contact transformations. Proc. Physico-Math. Soc. Japan. vol. 20, 451–457.
4. Y. Mutô and K. Yano: Sur les transformations de contact et les espaces de Finsler. Tôhoku Math. Journ. vol. 45, 295– 307.