ON THE PROJECTIVELY CONNECTED SPACES WITH HOMOGENEOUS COORDINATES WHOSE GROUPS OF HOLONOMY FIX A HYPERQUADRIC.

Yasuo NASU.

(Recieved Oct. 30, 1952)

This paper is concerned with an n-dimensional projectively connected space H_n with homogeneous coordinates whose group of holonomy fixes a non degenerate hyperquadric Q_{n-1}. For the case of ordinary projectively connected spaces, S. Sasaki, K. Yano and T. Ōtsuki have obtained intresting results.

In an n-dimensional projectively connected space H_n with homogeneous coordinates $(x^o, \cdots, x^n)$, a point x^λ is expressed by

$$x^\lambda = c^\lambda t \qquad (\lambda, \mu, \cdots = 0, 1, \cdots, n; \; c^\lambda = const.),$$

where t is a parameter. We must consider the following coordinate transformations:

$$(0.1) \quad \begin{cases} G: & \bar{x}^\lambda = \bar{x}^\lambda (x^o, \cdots, x^n), \\ F: & \bar{x}^\lambda = \rho x^\lambda, \; \rho\,(x^o, \cdots, x^n)\; x^\lambda \; (\rho \neq o), \end{cases}$$

where $\bar{x}^\lambda$ are homogeneous analytic functions of the first degree in x^λ, such that the functional determinant is different from zero for all points under consideration, and ρ is an analytic function of degree zero in x^λ. The coefficients of the projective connection $\Pi^\lambda_{\mu\nu}$ are homogeneous analytic functions of degree -1 in x^λ, and, by (0, 1), $\Pi^\lambda_{\mu\nu}$ are transformed into

$$(0.2) \quad \begin{cases} G: & \bar{\Pi}^\lambda_{\mu\nu} = \dfrac{\partial \bar{x}^\lambda}{\partial x^\alpha} \left(\dfrac{\partial x^\beta}{\partial \bar{x}^\mu} \dfrac{\partial x^\gamma}{\partial \bar{x}^\nu} \Pi^\alpha_{\beta\gamma} + \dfrac{\partial^2 x^\alpha}{\partial \bar{x}^\mu \partial \bar{x}^\nu} \right), \\ F: & \bar{\Pi}^\lambda_{\mu\nu} = \rho^{-1} \Pi^\lambda_{\mu\nu}. \end{cases}$$

We restrict ourselves to the following case:

$$\Pi^\lambda_{\mu\nu} = \Pi^\lambda_{\nu\mu} \quad , \qquad \Pi^\lambda_{\mu\nu} x^\mu = 0.$$

We also restrict ourselves to projective vectors and tensors such that the laws of transformation in (0.1) are given by:

$$G: \; \bar{u}^\lambda = \frac{\partial \bar{x}^\lambda}{\partial x^\alpha} u^\alpha \quad , \qquad F: \; \bar{u}^\lambda = \rho u^\lambda \; ,$$

$$G: \; \bar{v}_\lambda = \frac{\partial x^\alpha}{\partial \bar{x}^\lambda} v_\alpha \quad , \qquad F: \; \bar{v}_\lambda = \rho^{-1} v_\lambda \; ,$$

and

$$G:\ \overline{w}^{\lambda_1 \cdots\cdots \lambda_p}_{\cdot\ \cdots\cdots\ \cdot\ \mu_1 \cdots\cdots \mu_q} = \frac{\partial \overline{x}^{\lambda_1}}{\partial x^{\rho_1}} \cdots\cdots \frac{\partial \overline{x}^{\lambda_p}}{\partial x^{\rho_p}} \frac{\partial x^{\sigma_1}}{\partial \overline{x}^{\mu_1}} \cdots\cdots \frac{\partial x^{\sigma_q}}{\partial \overline{x}^{\mu_q}} w^{\rho_1 \cdots\cdots \rho_q}_{\cdot\ \cdots\cdots\ \cdot\ \sigma_1 \cdots\cdots \sigma_q},$$

$$F:\ \overline{w}^{\lambda_1 \cdots\cdots \lambda_p}_{\cdot\ \cdots\cdots\ \cdot\ \mu_1 \cdots\cdots \mu_q} = \rho^{p-q}\, w^{\lambda_1 \cdots\cdots \lambda_p}_{\cdot\ \cdots\cdots\ \cdot\ \mu_1 \cdots\cdots \mu_q}.$$

Hereafter we assume that the hyperquadric Q_{n-1} at a tangential point x^λ is given by

(0.3) $$Q_{n-1}:\quad G_{\lambda\mu} X^\lambda X^\mu = 0 \qquad (det\,|G_{\lambda\mu}| \neq 0,\ \ G_{\lambda\mu} = G_{\mu\lambda}),$$

where $G_{\lambda\mu}$, X^λ are a covariant projective tensor, and a contravariant projective vector respectively, and the tangential point x^λ does not lie on Q_{n-1}. Therefore we can assume $G_{\lambda\mu}\, x^\lambda\, x^\mu = -1$ without any loss of generality.

Under these conditions we shall investigate the structure of the projectively connected spaces with homogeneous coordinates.

1. We consider the following n equations:

(1.1) $$\xi^i = \xi^i\,(x^o, x^1, \cdots\cdots x^n) \qquad (i, j, k, \cdots = 1, 2, \cdots, n),$$

where ξ^i are homogeneous analytic functions of degree zero in x^λ and we assume that the matrix has rank n.

Then we put:

(1.2) $$E^i_{\cdot\lambda} = \frac{\partial \xi^i}{\partial x^\lambda}.$$

Furthermore we must consider a hyperplane:

(1.3) $$p_\lambda\, x^\lambda = 0,$$

which does not contain the tangential point x^λ and is used as a plane at infinity. This projective covariant vector p_λ enables us to define the inverse of $(E^i_{\cdot\lambda})$. We define the quantities $E^{\cdot\lambda}_i$, $E^{\cdot\lambda}_o$, $E^o_{\cdot\lambda}$ by means of the equations

(1.4) $$E^o_{\cdot\lambda} = p_\lambda, \qquad E^{\cdot\lambda}_o = x^\lambda, \qquad E^{\cdot\lambda}_i E^j_{\cdot\lambda} = \delta^j_i,$$

$$E^i_{\cdot\lambda} E^{\cdot\mu}_i = \delta^\mu_\lambda - x^\mu p_\lambda, \qquad E^{\cdot\lambda}_i p_\lambda = 0, \qquad E^i_{\cdot\lambda} x^\lambda = 0.$$

Then we define Γ^a_{bk} $(a, b, c \cdots = 0, 1, \cdots, n)$ as follows:

(1.5) $$\Gamma^a_{bk} = E^a_{\cdot\lambda} E^{\cdot\mu}_b E^{\cdot\nu}_k \Pi^\lambda_{\mu\nu} - E^{\cdot\mu}_b E^{\cdot\nu}_k \frac{\partial}{\partial x^\nu} E^a_{\cdot\mu}.$$

Γ^a_{bk} are analytic functions of degree zero in x^λ, so that we can express as the functions in ξ^i.

Then we get, by putting $a=0,\ b=0;\ a=0,\ b=j;\ a=i,\ b=0;\ a=i,\ b=j$ in (1.5), the following equations:

$$\Gamma^o_{ok} = 0, \qquad \Gamma^i_{ok} = \delta^i_k ,$$

(1.6)
$$\Gamma^o_{jk} = - E^{\cdot\mu}_j E^{\cdot\nu}_k \left(\frac{\partial p^\mu}{\partial x^\nu} - p_\lambda \Pi^\lambda_{\mu\nu}\right),$$

$$\Gamma^i_{jk} = E^i_{\cdot\lambda} E^{\cdot\mu}_j E^{\cdot\nu}_k \Pi^\lambda_{\mu\nu} - E^{\cdot\mu}_j E^{\cdot\nu}_k \frac{\partial^2 \xi^i}{\partial x^\mu \partial x^\nu}.$$

We can easily prove that Γ^i_{jk} are the coefficients of the affine connection, and Γ^o_{jk} are tensor components.

If we define H_{ab} by

$$H_{ab} = E^{\cdot\lambda}_a E^{\cdot\mu}_b G_{\lambda\mu},$$

we can find that $\det|H_{ab}| \neq 0$ in virtue of $\det|G_{\lambda\mu}| \neq 0$.

The covariant differentials ΔH_{ab} with respect to Γ^a_{bk} are related to the covariant differentials $DG_{\lambda\mu}$ with the following equations

(1.7)
$$\Delta H_{ab} = E^{\cdot\lambda}_a E^{\cdot\mu}_b (DG_{\lambda\mu} - 2 G_{\lambda\mu} p_\rho dx^\rho),$$

where $\Delta H_{ab} = dH_{ab} - \Gamma^c_{ak} H_{bc} d\xi^k - \Gamma^c_{bk} H_{ac} d\xi^k$, *and* $DG_{\lambda\mu} = dG_{\lambda\mu} - \Pi^\beta_{\lambda\alpha} G_{\beta\mu} dx^\alpha - \Pi^\beta_{\mu\alpha} G_{\lambda\beta} dx^\alpha$.

If the group of holonomy fixes Q_{n-1}, then $D(G_{\lambda\mu} X^\lambda X^\mu)$ must be proportional to $G_{\lambda\mu} X^\lambda X^\mu$ in virtue of the relation $DX^\lambda = 0$. But as X^λ is an arbitrary vector, we can write these results into

(1.8)
$$DG_{\lambda\mu} = (\tau_\rho dx^\rho) G_{\lambda\mu}.$$

Hence we get by putting (1.8) in (1.7) the following relation:

$$\Delta H_{ab} = E^{\cdot\lambda}_a E^{\cdot\mu}_b (\varphi_\rho dx^\rho) DG_{\lambda\mu},$$

where $\varphi_\rho = \tau_\rho - 2p_\rho$. Hence we get:

$$\left(\frac{\partial H_{ab}}{\partial \xi^k} - \Gamma^c_{ak} H_{cb} - \Gamma^c_{bk} H_{ac}\right) E^k_{\cdot\rho} = H_{ab} \varphi_\rho .$$

Therefore

(1.9)
$$\frac{\partial H_{ab}}{\partial \xi^k} - \Gamma^c_{ak} H_{cb} - \Gamma^c_{bk} H_{ac} = \varphi_k H_{ab},$$

where $\varphi_k = E^{\cdot\lambda}_k \varphi_\lambda$. If we write (1.9) in full detail, then we get

$$(1.10)\quad \begin{aligned} a=0,\ b=0 &: \quad \varphi_k = 2H_k \qquad (H_k = H_{ok} = H_{ko}),\\ a=0,\ b=j &: \quad H_{j;k} + \Gamma^o_{jk} - H_{jk} = 2H_j H_k\ ,\\ a=i,\ b=j &: \quad H_{ij;k} - H_i\Gamma^o_{jk} - H_j\Gamma^o_{ik} = 2H_k H_{ij}\ , \end{aligned}$$

where $H_{j;k} = \dfrac{\partial H_j}{\partial \xi^k} - \Gamma^i_{jk} H_i$, $H_{ij;k} = \dfrac{\partial H_{ij}}{\partial \xi^k} - \Gamma^l_{ik} H_{jl} - \Gamma^l_{jk} H_{il}$.

We know, from the definition of φ_ρ, $\varphi_\rho x^\rho = 0$, hence if we replce p_λ with $\bar{p}_\lambda = p_\lambda + \frac{1}{2}\varphi_\lambda$, then we get from the definitions of $E_a^{\cdot\lambda}$, $E_b^{\cdot\lambda}$, and Γ^a_{bk} the following relations

$$\bar{E}^i_{\cdot\lambda} = E^i_{\cdot\lambda} \quad , \quad \bar{E}_o^{\cdot\lambda} = E_o^{\cdot\lambda},$$

$$\bar{E}_j^{\cdot\mu} = E_j^{\cdot\mu} - x^\mu H_j,$$

and

$$(1.11)\quad \begin{cases} \bar{\Gamma}^i_{jk} = \Gamma^i_{jk} - H_j\delta^i_k - H_k\delta^i_j\ ,\\ \bar{\Gamma}^o_{jk} = \Gamma^o_{jk} - \dfrac{\partial H_j}{\partial \xi^k} + H_i\Gamma^i_{jk} - H_j H_k\ . \end{cases}$$

Therefore we get from (1.10), (1.11) the following results.

$$(1.12)\quad \begin{aligned} \bar{\Gamma}^o_{jk} &= g_{jk}\ ,\\ \bar{\Gamma}^i_{jk} &= \left\{ {i \atop jk} \right\}\ , \end{aligned}$$

where $g_{ij} = H_{ij} - H_i H_j$ and $\left\{ {i \atop jk} \right\}$ are the Christoffel's symbols. We can easily see det $|g_{ij}| \neq 0$ from det $|G_{\lambda\mu}| \neq 0$ (or det $|H_{ab}| \neq 0$)

Q_{n-1} is also written by putting $Z^a = E^a_{\cdot\lambda} X^\lambda$ as $H_{ab} Z^a Z^b = 0$, and hence we obtain

$$(Z^o)^2 = g_{ij} Z^i Z^j$$

as an equation of Q_{n-1} .

Hereafter we assume that $g_{ij}\ Z^i\ Z^j$ is positive definite or negative definite.

Now we can formulate the above mentioned facts as follows:

Theorem I. When the group of holonomy of a projectively connected space with homegeneous coordinates fixes a non degenerate hyperquadric Q_{n-1}, the coefficients of the connection Γ^j_{ik} are induced from $\Pi^\lambda_{\mu\mu}$ as the Christoffel's symbols with respect to g_{ij} which are derived from $G_{\lambda\mu}$.

2. When the paths in the space are given by

$$(2.1)\quad x^\lambda = x^\lambda(u^0, u^1),$$

these equations are satisfied with the following differential equations:

(2.2) $$\frac{\partial^2 x^\lambda}{\partial u^\alpha \partial u^\beta} + \Pi^\lambda_{\mu\nu} \frac{\partial x^\mu}{\partial x^\alpha} \frac{\partial x^\nu}{\partial x^\beta} = \Gamma^\gamma_{\alpha\beta} \frac{\partial x^\lambda}{\partial u^\gamma} \quad (\alpha, \beta, \gamma, \cdots \quad = 0, 1).$$

We can consider that $x^\lambda(u^0, u^1)$ are homogeneous analytic functions of first degree in u^α. These differential equations of paths were defined by D. van Dantig. We can easily find that under the transformation of the parameter u^α, $\Gamma^\gamma_{\alpha\beta}$ are transformed like coefficients of projective connexion. Moreover we can see from (2.2) $\Gamma^\gamma_{\alpha\beta} = \Gamma^\gamma_{\beta\alpha}$, $\Gamma^\gamma_{\alpha\beta} u^\alpha = 0$.

According to J. Hantjes, under a suitable transformation of the coefficients of the projective connexion

$$\overline{\Pi}^\lambda_{\mu\nu} = \Pi^\lambda_{\mu\nu} + \delta^\lambda_\mu \varphi_\nu + \delta^\lambda_\nu \varphi_\mu + \varphi_{\mu\nu} x^\lambda,$$

where $\varphi_\mu x^\mu = 0$, $\varphi^\mu + \varphi_{\nu\mu} x^\nu = 0$, it is possible to make the contracted curvature tensor $\overline{\Pi}_{\nu\mu}$ with respect to $\overline{\Pi}^\lambda_{\mu\nu}$ identically zero. Moreover he proved that the curvature tensor $R^\alpha_{\cdot\beta\gamma\delta}$ with respect to $\Gamma^\gamma_{\alpha\beta}$ is identically zero. Therefore the differential equations (2.2) are reduced to the following form:

(2.3) $$\frac{\partial^2 x^\lambda}{\partial u^\alpha \partial u^\beta} + \Pi^\lambda_{\mu\nu} \frac{\partial x^\mu}{\partial u^\alpha} \frac{\partial x^\nu}{\partial u^\beta} = 0.$$

We know by simple calculation that (2.3) are reduced to the following differential equations:

(2.3)′ $$\frac{d^2 x^\lambda}{dp^2} + \Pi^\lambda_{\mu\nu}(1, p) \frac{dx^\mu}{dp} \frac{dx^\nu}{dp} = 0,$$

where $p = {u^1}/{u^0}$. Furthermore, by (1.1), we can transform (2.3)′ into

$$\frac{d^2 \xi^i}{dp^2} + \Gamma^i_{jk} \frac{d\xi^j}{dp} \frac{d\xi^k}{dp} = -2\left(p_\lambda \frac{dx^\lambda}{ds}\right) \frac{d\xi^i}{dp}.$$

Therefore if we transform the parameter p into s by the equation

(2.4) $$2 p_\lambda \frac{dx^\lambda}{dp} = \frac{\frac{d^2 p}{ds^2}}{\frac{dp}{ds}},$$

we get

(2.5) $$\frac{d^2 \xi^i}{ds^2} + \Gamma^i_{jk} \frac{d\xi^j}{ds} \frac{d\xi^k}{ds} = 0.$$

We can see from (2.5) that s is an affine parameter. Moreover we get from (1.6) the following equations;

$$- E^{j}_{\cdot\mu} E^{k}_{\cdot\nu} \Gamma^{o}_{jk} = \frac{\partial p_\mu}{\partial x^\nu} - p_\lambda \Pi^{\lambda}_{\mu\nu} + p_\mu p_\nu .$$

Therefore the above equations are reduced, by differentiating (2.4) with respect to s, to the following equation:

$$\{p, s\} = -2 \Gamma^{o}_{jk} \frac{d\xi^j}{ds} \frac{d\xi^k}{ds}, \tag{2.7}$$

where $\{p, s\}$ is the Schwarzian derivative. From this we conclude that p is a projective parameter.

On the other hand, we can find that the curvature tensor $\Pi^{\lambda}_{\cdot\mu\nu\omega}$ with respect to $\Pi^{\lambda}_{\mu\nu}$ is related by the following relation to the curvature tensor $\bar{R}^{i}_{\cdot jkh}$ with respect to $\bar{\Gamma}^{i}_{jk}$.

$$E^{i}_{\cdot\lambda} E^{\cdot\mu}_{j} E^{\cdot\nu}_{k} E^{\cdot\omega}_{h} \Pi^{\lambda}_{\cdot\mu\nu\omega} = \bar{R}^{i}_{\cdot jkh} + \bar{\Gamma}^{o}_{jk} \delta^{i}_{h} - \bar{\Gamma}^{o}_{jh} \delta^{i}_{k} - \delta^{i}_{j} (\bar{\Gamma}^{o}_{kh} - \bar{\Gamma}^{o}_{hk}) \tag{2.8}$$

Since $\Pi_{\mu\nu}\ (= \Pi^{\lambda}_{\cdot\mu\gamma\lambda}) = 0$, we get from (2.8) the following relation:

$$\Gamma^{o}_{jk} = -\frac{1}{n-1} R_{jk} \quad (R_{jk} = R^{i}_{\cdot jki}).$$

Hence from (1.12) the following relation holds good.

$$R_{jk} = -(n-1) g_{jk}. \tag{2.9}$$

We can formulate the above results as follows.

Theorem 2. *When the group holonomy of a projectively connectid space with homogeneous coordinate fixes a non-degenerate hyperquadric, this space is a projectively connected space with corresponding paths including an Einstein space with non vanishing scalar curvature.*

3. The differential equations of a path in the Riemann space with the fundamental tensor g_{ij} are given by

$$\frac{d^2 \xi^i}{d\bar{s}^2} + \bar{\Gamma}^{i}_{jk} \frac{d\xi^i}{d\bar{s}} \frac{d\xi^k}{d\bar{s}} = 0,$$

where the parameter $\bar{s}$ is the arc-lengh of the path, and is an affine parameter.

This facts imply that there is a relation between $\bar{s}$ and s in (2.5) such that

$$\bar{s} = as + b \quad (a \neq 0), \tag{3.1}$$

where a, b are constants.

Hence we know from (2.7), (2.9), (3.1) the following result:

$$\{p, s\} = K \tag{3.2}$$

where K is a constant and is negative or positive in accordance with g_{ij} being a positive definite or a negative definite tensor.

i) $K<0$. If we put $K=-2k^2$, then we find the following solution from (3.2)

$$p=\frac{m_1 e^{ks}+m_2 e^{-ks}}{n_1 e^{ks}+n_2 e^{-ks}}, \tag{3.3}$$

where m_1, m_2, n_1, n_2 are arbitrary constants with $m_1 n_2 - m_2 n_1 \neq 0$.

Hence we get from (3.3)

$$s=\frac{1}{2k} log \left(\frac{p-p_1}{p_2-p} : \frac{p_0-p_1}{p_2-p_0}\right), \tag{3.4}$$

where $p_1=\frac{m_2}{n_2}$, $p_2=\frac{m_1}{n_1}$ and p_0 are constants such that $\frac{p_0-p_1}{p_2-p_0}=\frac{n_1}{n_2}$.

But by the assumption the tangential point x^λ does not lie on Q_{n-1}. Accordingly we can not apply the Klein's representation of non Euclidean geometry.

2) $K>0$. By putting $K=2k^2$ we get similarily

$$s=\frac{1}{2ik} log \left(\frac{p-p_1}{p_2-p} : \frac{p_0-p_1}{p_2-p_0}\right) \qquad (i=\sqrt{-1}).$$

References

(1) S. Sasaki and K. Yano: On the structure of spaces with normal projective connexion whose groups of holonomy fix a hyperquadric. Tôhoku Math. Journ. Vol. 1, (1949).
(2) T. Otsuki: On the projective connected spaces whose groups of holonomy fix a hyperquadric. Journ. of Math. Soc. of Japan, Vol. 1, No. 2, (1950).
(3) J. Hantjes: On the projective geometry of paths. University of Edinburgh, Math. (1937).
(4) K. Yano: Shaeibaikaihensû ni tsuite. Osaka Shijō Danwakai 262 (in Japannese).