# ON NORMAL COORDINATES IN PROJECTIVELY CONNECTED SPACES WITH HOMOGENEOUS COORDINATES.

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This paper deals with normal coordinates in projectively connected spaces with homogeneous coordinates. O. Veblen, J. M. Thomas and L. P. Eisenhart extended the normal coordinates in Riemann spaces to affinely connected spaces and projectively connected spaces.

On the other hand, J. A. Schouten and J. Hantjes defined the normal coordinates in projectively connected spaces with homogeneous coordinates.

This paper is concerned with the normal coordinates in their papers.

**1.** We assume that the coefficients of the connexion are given by $\Pi^{\lambda}_{\mu\nu}$ and the coordinate transformations

$$(1.1)\qquad \begin{aligned} &x^{\lambda'} = x^{\lambda'}(x^0, \cdots, x^n), \\ &x^{\lambda'} = \rho x^{\lambda} \ (\rho \neq 0), \quad (\lambda', \mu', \nu' \cdots = 0, 1, \cdots, n) \end{aligned}$$

transform $\Pi^{\lambda}_{\mu\nu}$ into $\Pi^{\lambda'}_{\mu'\nu'}$ as follows:

$$(1.2)\qquad \begin{aligned} &\Pi^{\alpha'}_{\beta'\gamma'} A^{\lambda}_{\alpha'} = \Pi^{\lambda}_{\mu\nu} A^{\mu}_{\beta'} A^{\nu}_{\gamma'} + \partial_{\beta'} A^{\lambda}_{\gamma'} \quad \left(A^{\lambda}_{\mu'} = \frac{\partial x^{\lambda}}{\partial x^{\mu'}}\right), \\ &\Pi^{\lambda'}_{\mu'\nu'} = \rho^{-1} \Pi^{\lambda}_{\mu\nu}, \end{aligned}$$

respectively, where $x^{\lambda'}$ are analytic functions of the first degree in $x^{\lambda}$, and $\det|A^{\lambda}_{\mu'}| \neq 0$.

D. van. Dantig defined the differential equations of paths as follows:

$$(1.3)\qquad \frac{\partial^2 x^{\lambda}}{\partial u^a \partial u^b} + \Pi^{\lambda}_{\mu\nu} \frac{\partial x^{\mu}}{\partial u^a} \frac{\partial x^{\nu}}{\partial u^b} = \Gamma^{c}_{ab} \frac{\partial x^{\lambda}}{\partial u^c}.$$

Solutions $x^{\lambda}(u^0, u^1)$ of this differential equations are regarded as analytic functions of the first degree in $u^0, u^1$.

For convenience we assume that $\Pi^{\lambda}_{\mu\nu}$ are symmetric with respect to $\mu$ and $\nu$. For even if $\Pi^{\lambda}_{\mu\nu} \neq \Pi^{\lambda}_{\mu\nu}$, we can obtain the same results as the symmetric case.

If we put

$$(1.4)\qquad t = u^1/u^0,$$

we have, for arbitrary solutions $x^{\lambda}(u^0, u^1)$, the following relations:

$$(1.5)\quad \begin{cases} \dfrac{\partial x^\lambda}{\partial u^0} = \bar{x}^\lambda - \dfrac{u^1}{u^0}\dfrac{d\bar{x}^\lambda}{dt}, & \dfrac{\partial x^\lambda}{\partial u^1} = \dfrac{d\bar{x}^\lambda}{dt}, \\ \dfrac{\partial^2 x^\lambda}{\partial u^{0^2}} = \dfrac{u^{1^2}}{u^{0^3}}\dfrac{d^2\bar{x}^\lambda}{dt^2}, & \dfrac{\partial^2 x^\lambda}{\partial u^0 \partial u^1} = -\dfrac{u^1}{u^{0^2}}\dfrac{d^2\bar{x}^\lambda}{dt^2}, \\ \dfrac{\partial^2 x^\lambda}{\partial u^{1^2}} = \dfrac{1}{u^0}\dfrac{d^2\bar{x}^\lambda}{dt^2}, & \end{cases}$$

where $\bar{x}^\lambda = x^\lambda(1,t)$. Therefore, by (1.3) and (1.5), we get the following equations:

$$(1.6)\quad \begin{aligned} \frac{u^{1^2}}{u^{0^3}}\left(\frac{d^2\bar{x}^\lambda}{dt^2} + \bar{\Pi}^\lambda_{\mu\nu}\frac{d\bar{x}^\mu}{dt}\frac{d\bar{x}^\nu}{dt}\right) &= \frac{1}{u^0}\bar{\Pi}^\lambda_{\mu\nu}\bar{x}^\mu\bar{x}^\nu + 2\frac{u^1}{u^{0^2}}\bar{\Pi}^\lambda_{\mu\nu}\bar{x}^\mu\frac{d\bar{x}^\nu}{dt} \\ &\quad + \frac{1}{u^0}\bar{\Gamma}^0_{00}\bar{x}^\lambda + \left(-\frac{u^1}{u^{0^2}}\bar{\Gamma}^0_{00} + \frac{1}{u^0}\bar{\Gamma}^1_{00}\right)\frac{d\bar{x}^\lambda}{dt}, \\ \frac{u^1}{u^{0^2}}\left(\frac{d^2\bar{x}}{dt^2} + \bar{\Pi}^\lambda_{\mu\nu}\frac{d\bar{x}^\mu}{dt}\frac{d\bar{x}^\nu}{dt}\right) &= \frac{1}{u^0}\bar{\Pi}^\lambda_{\mu\nu}\bar{x}^\mu\frac{d\bar{x}^\nu}{dt} - \frac{1}{u^0}\bar{\Gamma}^0_{01}\bar{x}^\lambda \\ &\quad - \left(\frac{u^1}{u^{0^2}}\bar{\Gamma}^0_{01} + \frac{1}{u^0}\bar{\Gamma}^0_{01}\right)\frac{d\bar{x}^\lambda}{dx}, \\ \frac{1}{u^0}\left(\frac{d^2\bar{x}^\lambda}{dt^2} + \bar{\Pi}^\lambda_{\mu\nu}\frac{d\bar{x}^\mu}{dt}\frac{d\bar{x}^\nu}{dt}\right) &= \frac{1}{u^0}\bar{\Gamma}^0_{11}\bar{x}^\lambda \\ &\quad + \left(-\frac{u^1}{u^{0^2}}\bar{\Gamma}^0_{01} + \frac{1}{u^0}\bar{\Gamma}^1_{11}\right)\frac{d\bar{x}^\lambda}{dt}, \end{aligned}$$

where $\bar{\Pi}^\lambda_{\mu\nu} = \Pi^\lambda_{\mu\nu}(1, t)$. and $\bar{\Gamma}^c_{ab} = \Gamma^c_{ab}(1, t)$.

The above three equations contain the common factor $\dfrac{d^2\bar{x}^\lambda}{dt^2} + \bar{\Pi}^\lambda_{\mu\nu}\dfrac{d\bar{x}^\mu}{dt}\dfrac{d\bar{x}^\nu}{dt}$ in the left hand sides and these relations must be compatible. Therefore we find from the last equations that $(1.6)_1$, $(1.6)_2$ ought to be written by the same form as $(1.6)_3$. Hence we obtain the following necessary condition for the compatibility of (1.6):

$$\bar{\Pi}^\lambda_{\mu\nu}\bar{x}^\mu\frac{d\bar{x}^\nu}{dt} = \sigma\frac{d\bar{x}^\lambda}{dt} + \tau\bar{x}^\lambda.$$

As $\Pi^\lambda_{\mu\nu}x^\mu$ is a tensor, we get from the last equation

$$\Pi^\lambda_{\mu\nu}x^\mu = p_\nu x^\lambda + \sigma\delta^\lambda_\nu,$$

where $p_\mu$ is a covariant vector and $\sigma$ is a scalar. Hence we find that $\Pi^\lambda_{\mu\nu}x^\mu x^\nu = Px^\lambda$ $(P = p_\nu x^\nu + \sigma)$, where $P$ is a scalar. Now the above relation is written as

$$(1.7)\qquad \Pi^\lambda_{\mu\nu}x^\mu = p_\nu x^\lambda + (P - p_\rho x^\rho)\delta^\lambda_\nu.$$

The condition (1.7) is necessary in order that the equations are compatible. Conversely, suppose that the condition (1.7) is satisfied by $\Pi^{\lambda}_{\mu\nu}$, then (1.6) can be written as follows:

$$\frac{d^2\bar{x}^\lambda}{dt^2} + \bar{\Pi}^{\lambda}_{\mu\nu}\frac{d\bar{x}^\mu}{dt}\frac{d\bar{x}^\nu}{dt} = \frac{u^{0^2}}{u^{1^2}}\left(\bar{P} + 2\frac{u^1}{u^0}\bar{p}_\nu\frac{d\bar{x}^\nu}{dt} + \bar{\Gamma}^0_{00}\right)\bar{x}^\lambda$$
$$+ \frac{u^{0^2}}{u^{1^2}}\left\{2\frac{u^1}{u^0}\left(\bar{P} - p_\rho\,\bar{x}_\rho\right) + \left(-\frac{u^1}{u^0}\bar{\Gamma}^0_{00} + \bar{\Gamma}^1_{00}\right)\right\}\frac{d\bar{x}^\lambda}{dt},$$

(1.6)
$$\frac{d^2\bar{x}^\lambda}{dt^2} + \bar{\Pi}^{\lambda}_{\mu\nu}\frac{d\bar{x}^\mu}{dt}\frac{d\bar{x}^\nu}{dt} = \frac{u^0}{u^1}\left(\bar{p}_\nu\frac{d\bar{x}^\nu}{dt} - \bar{\Gamma}^0_{01}\right)\bar{x}_\lambda$$
$$+ \frac{u^0}{u^1}\left\{\left(\bar{P} - \bar{p}_\rho\,\bar{x}_\rho\right) - \left(\frac{u^1}{u^0}\bar{\Gamma}^0_{01} + \bar{\Gamma}^1_{01}\right)\right\}\frac{d\bar{x}^\lambda}{dt},$$

$$\frac{d^2\bar{x}^\lambda}{dt^2} + \bar{\Pi}^{\lambda}_{\mu\nu}\frac{d\bar{x}^\mu}{dt}\frac{d\bar{x}^\nu}{dt} = \bar{\Gamma}^0_{11}\,\bar{x}_\lambda + \left(-\frac{u^1}{u^0}\bar{\Gamma}^0_{11} + \bar{\Gamma}^1_{11}\right)\frac{d\bar{x}^\lambda}{dt},$$

where $\Gamma^c_{ab}$ are symmetric with respect to $a$, $b$ and consist of six elements $\Gamma^0_{00}$, $\Gamma^0_{01}$ $(=\Gamma^0_{10})$, $\Gamma^1_{00}$, $\Gamma^1_{01}$ $\Gamma^1_{01}$ $(=\Gamma^1_{10})$, $\Gamma^0_{11}$, and $\Gamma^1_{11}$. The three equations of (1.6)′ should be the same. Hence we get:

$$\frac{u^{0^2}}{u^{2^1}}\left(\bar{P} + 2\frac{u^1}{u^0}\bar{p}_\nu\frac{d\bar{x}^\nu}{dt} + \bar{\Gamma}^0_{00}\right) = \frac{u^0}{u^1}\left(\bar{p}_\nu\frac{d\bar{x}^\nu}{dt} - \bar{\Gamma}^0_{01}\right) = \bar{\Gamma}^0_{11},$$
$$\frac{u^{0^2}}{u^{1^2}}\left\{2\frac{u^1}{u^0}\left(\bar{P} - \bar{p}_\rho\,\bar{x}_\rho\right) + \left(-\frac{u^1}{u^0}\bar{\Gamma}^0_{00} + \bar{\Gamma}^1_{00}\right)\right\} = \frac{u^1}{u^0}\left\{\left(\bar{P} - \bar{p}_\rho\,\bar{x}_\rho\right)\right.$$
$$\left. - \left(\frac{u^1}{u^0}\bar{\Gamma}^0_{01} + \bar{\Gamma}^1_{01}\right)\right\} = \left(-\frac{u^1}{u^0}\bar{\Gamma}^0_{11} + \bar{\Gamma}^1_{11}\right).$$

Therefore arbitrary four elements of $\Gamma^c_{ab}$ are represented by linear combination of the other two. Hence we obtain the following

**Theorem.** *In a projectively connected space with homogeneous coordinates, the path equations* (1.3) *can de represented by*

(1.8)
$$\frac{d^2x^\lambda}{dt^2} + \Pi^{\lambda}_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt} = \alpha\frac{dx^\lambda}{dt} + \beta\,x^\lambda,$$

*if and only if the coefficients of the connexion satisfies* (1.7), *where* $\alpha$, $\beta$ *are arbirary functions of* $t$.

In the above argument we assumed the symmetry of $\Pi^{\lambda}_{\mu\nu}$. If we remove this assumption, then (1.6) and (1.6)′ will be replaced by four equations.

If $\Gamma^c_{ab}$ are not symmetric with respect to $a, b$, then these consist of eight elements $\Gamma^0_{00}$, $\Gamma^0_{10}$, $\Gamma^0_{01}$, $\Gamma^1_{01}$, $\Gamma^1_{10}$, $\Gamma^0_{11}$, $\Gamma^1_{00}$, and $\Gamma^1_{11}$. Therefore arbitrary six elements of $\Gamma^c_{ab}$ are represented by linear conbination of the other two. Hence the above theorem is applicable to the case: $\Pi^{\lambda}_{\nu\mu} \mp \Pi^{\lambda}_{\nu\mu}$. But (1.7) must be replaced by the following relations:

$$\Pi^{\lambda}_{\mu\nu}\, x^{\mu} = p_{\nu}\, x^{\lambda} + (P - p_{\rho}\, x_{\rho})\, \delta^{\lambda}_{\nu}, \tag{1.7}'$$

$$\Pi^{\lambda}_{\mu\nu}\, x^{\nu} = q_{\mu}\, x^{\lambda} + (P - q_{\rho}\, x^{\rho})\, \delta^{\lambda}_{\mu}.$$

**2.** The path equations (1.8) are transformed by $(1.1)_2$ into

$$\frac{d^2x^{\lambda}}{dt^2} + \Pi^{\lambda}_{\mu\nu}\frac{dx^{\mu}}{dt}\frac{dx^{\nu}}{dt} = \bar{\alpha}\frac{dx^{\lambda}}{dt} + \bar{\beta}\, x^{\lambda},$$

where $\bar{\alpha} = \alpha\,\rho - 2\,P\,\dfrac{d\rho}{dt} + p_{\chi}\dfrac{dx^{\chi}}{dt}\dfrac{d\rho}{dt} + q_{\chi}\dfrac{dx^{\chi}}{dt} - 2\,\dfrac{d\rho}{dt}$,

$$\bar{\beta} = \alpha\,\frac{d\rho}{dt} + \beta\,\rho - \rho\,P - \frac{d^2\rho}{dt^2} - p_{\chi}\frac{dx^{\chi}}{dt} - q_{\chi}\frac{dx^{\chi}}{dt}.$$

Let $\rho$ be a solution of $\bar{\beta} = 0$, then (1.8) reduces to

$$\frac{d^2x^{\lambda}}{dt^2} + \Pi^{\lambda}_{\mu\nu}\frac{dx^{\mu}}{dt}\frac{dx^{\nu}}{dt} = \bar{\alpha}\frac{d\bar{x}^{\lambda}}{dt}.$$

Hence we get as the simplest form of the path equations:

$$\frac{d^2x^{\lambda}}{ds^2} + \Pi^{\lambda}_{\mu\nu}\frac{dx^{\mu}}{ds}\frac{dx^{\nu}}{ds} = 0, \tag{2.1}$$

taking the parameter $s$ defind by $\dfrac{ds}{dt} = c.\exp\left(\int\bar{\alpha}dt\right)$ ($c$=const). J. Hantjes and K. Yano proved that under the conditions $\Pi^{\lambda}_{\nu\mu} = \Pi^{\lambda}_{\mu\nu}$ and $\Pi^{\lambda}_{\mu\nu}\, x^{\mu} = 0$, $s$ is a projective parameter. In general, $s$ is an affine parameter.

**3.** The normal coordinates at a point $x^{\lambda}_0$ are defined as a coordinate system which satisfies the conditions:

$$\left(\Pi^{\lambda}_{(\mu\nu)}\right)_0 = \left(\partial_{(\omega}\Pi^{\lambda}_{\mu\nu)}\right)_0 = \left(\partial_{(\chi}\partial_{\omega}\Pi^{\lambda}_{\mu\nu)}\right)_0 = \cdots = 0, \tag{3.1}$$

$$\left(A^{\lambda'}_{\mu}\right)_0 = \delta^{\lambda}_{\mu},$$

where $(\ \ )_0$ denote the value at $x^{\lambda}_0$.

To get such a coordinate system we consider a coordinate transformation

$$x^{\lambda'} = x^{\lambda'}\left(x^{\mu}\right). \tag{3.2}$$

Let $x^{\lambda'}$ be an arbitrary coordinate whose path equations are given by (2.1) and $x^{\lambda}$ be a normal coordinate, then $(3.1)_1$ must be valid. Hence under these conditions we can determine, from $(1.2)_1$, the values of $\dfrac{\partial^2 x^{\lambda'}}{\partial x^{\alpha}\partial x^{\beta}}$, $\dfrac{\partial^3 x^{\lambda'}}{\partial x^{\alpha}\partial x^{\beta}\partial x^{\gamma}}\cdots$, at $x^{\lambda}_0$: namely, by $\left(\Pi^{\lambda}_{(\beta\gamma)}\right)_0 = 0$, we have $\left(\dfrac{\partial^2 x^{\lambda'}}{\partial x^{\alpha}\partial x^{\beta}}\right)_0 = -\left(\Pi^{\lambda'}_{(\alpha'\beta')}\right)_0.$

Similarly, from $(\partial_{(\gamma}\Pi^{\lambda}_{\alpha\beta)})_0 = 0, \cdots$ , etc., we have

$$\left(\frac{\partial^3 x^{\lambda'}}{\partial x^\alpha \partial x^\beta \partial x^\gamma}\right)_0 = -\left(\partial_{(\gamma'}\Pi^{\lambda'}_{\alpha'\beta')} - \Pi^{\lambda'}_{\mu'(\beta'}\Pi^{\mu'}_{\alpha'\gamma')} - \Pi^{\lambda'}_{(\beta'|\mu'|}\Pi^{\mu'}_{\alpha'\gamma')}\right)_0,$$

$$\left(\frac{\partial^4 x^{\lambda}}{\partial x^\alpha \partial x^\beta \partial x^\gamma \partial x^\delta}\right)_0 = -\left(\partial_{(\delta'}\partial_{\gamma'}\Pi^{\lambda'}_{\alpha'\beta')} - \partial_{\rho'}\Pi^{\lambda'}_{(\alpha'\beta'}\Pi^{\rho'}_{\delta'\gamma')} - 2\,\partial_{(\gamma'}\Pi^{\lambda'}_{|\mu'|\beta'}\Pi^{\mu'}_{\alpha'\beta')}\right.$$
$$- 2\,\partial_{(\gamma'}\Pi^{\lambda'}_{\alpha'|\mu'|}\Pi^{\mu'}_{\beta'\delta')} + 2\,\Pi^{\lambda'}_{\mu'\nu'}\Pi^{\mu'}_{(\alpha'\delta'}\Pi^{\nu'}_{\beta'\gamma')}$$
$$\left. - \Pi^{\lambda'}_{\alpha'|\mu'|}\Pi^{\mu'}_{\beta'\gamma'\delta'} - \Pi^{\lambda'}_{\mu'(\alpha'}\Pi^{\mu'}_{\beta'\gamma'\delta')}\right)_0,$$

where $\left(\frac{\partial^3 x^{\lambda'}}{\partial x^\beta \partial x^\gamma \partial x^\delta}\right)_0 = -\Pi^{\lambda'}_{\beta'\gamma'\delta'}$ .

If we put

$$\xi^i = \frac{x^i}{x_0} \qquad (i.j.k. \cdots = 1,\ 2,\ \cdots,\ n),$$

then we can determine from the following formulae the values of $\frac{\partial}{\partial \xi^k}A^{\mu'}_{\alpha}, \frac{\partial^2}{\partial \xi^i \partial \xi^k}A^{\mu'}_{\alpha}, \cdots$ at $x^\lambda_0$.

$$\frac{\partial}{\partial \xi^k}A^{\mu'}_{\alpha} = \left(\partial_k A^{\mu'}_{\alpha}\right) x^0,$$

$$\frac{\partial^2}{\partial \xi^j \partial \xi^k}A^{\mu'}_{\alpha} = \left(\partial_j \partial_k A^{\mu'}_{\alpha}\right) x^{0^2},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\frac{\partial^r}{\partial \xi^{j_1} \cdots \partial \xi^{j_r}}A^{\mu'}_{\alpha} = \left(\partial_{j_1}\cdots\partial_{j_r}\cdots A^{\mu'}_{\alpha}\right) x^{0r} .$$

Hence we can define $A^{\mu'}_{\alpha}$ as follows:

$$A^{\mu'}_{\alpha} = \delta^{\mu}_{\alpha} + \frac{1}{1!}\left(\partial_{\xi^i}A^{\mu'}_{\alpha}\right)\left(\xi^i - \xi^i_0\right) + \cdots , \tag{3.4}$$

where $\xi^i_0$ denote the values of $\xi^i$ at $x^\lambda_0$. Now we can conclude that when $A^{\mu'}_{\alpha}$ are defined for suitable ranges of $|\xi^i - \xi^\lambda_0|$, the coordinate transformation should be defined by

$$x^{\lambda'} = A^{\lambda'}_{\alpha} x^\alpha . \tag{3.5}$$

It remains to show that $A^{\mu'}_{\alpha}$ have the properties $A^{\mu'}_{\alpha} = \frac{\partial x^{\mu'}}{\partial x^\alpha}$. But this is easily seen on account of $(\partial_\beta A^{\lambda'}_{\alpha})_0 = (\partial_\alpha A^{\lambda'}_{\beta})_0$, $(\partial_\gamma \partial_\beta A^{\mu'}_{\alpha})_0 = (\partial_\gamma \partial_\alpha A^{\mu'}_{\beta})_0, \cdots$ .

Let $x^\lambda(s)$ be a solution of (2.1) amd a normal coordinate system, then $x^\lambda(s)$ are expanded, for a suitable range of $s$, in power series, namely,

$$x^\lambda(s) = x^\lambda_0 + \frac{s}{1!}\left(\frac{dx^\lambda}{ds}\right)_0 + \cdots ,$$

where $s=0$ corresponds to the origin $x_0^\lambda$. Then we find from the property of normal coordinates $(3.1)_1$ that

$$\left(\frac{d^2x^\lambda}{ds^2}\right)_0 = \left(\frac{d^3x^\lambda}{ds^3}\right)_0 = \cdots = 0.$$

Hence the above power series are expressible by

$$x^\lambda(s) = x_0^\lambda + \left(\frac{dx^\lambda}{ds}\right)_0 s\,. \tag{3.7}$$

If we put $s=u^1/u^0$ in (3.7) then we get from (1.5)

$$x^\lambda(u^0,\ u^1) = \left(\frac{\partial x^\lambda}{\partial u^0}\right)_0 u^0 + \left(\frac{\partial x^\lambda}{\partial u^1}\right)_0 u^1,$$

because of $x^\lambda(u^0, u^1) = u^0 x^\lambda(1, s)$. Thus we have the analogous conclusion to affinely (or projectively) connected spaces, namely we get the following theorem.

**Theorem.** *In a projectively connected space with homogeneous coordinates, when the power series* (3.4) *converge, we can locally define a normal coordinate system at a given point in terms of which any geodesics through the point is expressible by*

$$x^\lambda(u^0,\ u^1) = C_0^\lambda u^0 + C_1^\lambda u^1,$$

*where* $C_0^\lambda$, $C_1^\lambda$ *are arbitrary constants.*

Furthermore we can prove the following theorem by the quite similar method in Riemann spaces.

**Theorem.** *In a projectively connected space with homogenous coordinates, under an arbitrary coordinate transformation, normal coordinates at a given point are related by linear transformation with constant coefficients to the other normal coordinates at the same point.*

Specially if the coefficients of the connection $\Pi_{\mu\nu}^\lambda$ are symmetric with respect to $\mu, \nu$, then the normal coordinates are also applicable to construct normal tensors and extensions of tensors with the quite similar methods in an affine (or a projective) space with symmetric connection.

## References

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