A PROOF OF THE SPECTRAL THEOREM

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In this note we shall give a proof of the well known spectral formula $f(H) = \int f(\chi)\, dE(\chi)$ for unbounded operators in the Hilbert space.

1. Let **A** denote the uniformly-closed, self-adjoint and commutative B*-algebra, with the unit 1, of continuous operators in a Hilbert space $\mathfrak{H}$. The spectrum of **A** is a compact Hausdorff space Ω whose element is a continuous character χ of **A**, that is, a continuous homomorphism of **A** in the field of complex numbers: $A \rightarrow \chi(A)$ $(A \varepsilon \mathbf{A})$.

The following theorem is the result of Stone [1] and Gelfand-Neumark [2].

Theorem 1. For every x, y, there is an uniquely determined Radon measure $\mu_{x,y}(\chi)$ on Ω such that $(Ax,\ y) = \int_{\Omega} \chi(A)\, d\mu_{x,y}(\chi)$ for all $A \varepsilon$ **A**. This measure $\mu_{x,y}$ has the properties: μ_{xy}, depends linearly on x; $\bar{\mu}_{x,y} \doteq \mu_{y,x}$; $\mu_{x,x} \geqq 0$; $\| \mu_{x,y} \| \leqq \| x \|\ \| y \|$; $A(\chi) \equiv \chi(A)$ is a continuous function on Ω, and $d\mu_{Ax,y}(\chi) = A(\chi)\, d\mu_{x,y}$ for all $A \varepsilon \mathbf{A}$; a bounded operator T commutes with A if and only if $\mu_{Tx,y} = \mu_{x,T^*y}$ for all $x,\ y \varepsilon \mathfrak{H}$.

A complex-valued function $f(\chi)$ on Ω or a subset ω of Ω will be called measurable if it is measurable for every measure $\mu_{x,y}$. From Theorem 1, the following facts [3] are deduced: For every bounded and measurable function $f(\chi)$ there corresponds a bounded operator T_f on $\mathfrak{H}$ with $(T_f x, y) = \int f(\chi)\, d\mu_{x,y}(\chi)$ for all $x,\ y \varepsilon \mathfrak{H}$. This correspondence has the properties; $f \rightarrow T_f$ is an homomorphism; $T_{\bar{f}} = T_f^*$; $\| T_f \| \leqq \operatorname{Sup}_{\chi} | f(\chi) |$; a bounded operator which commutes with A commutes with every T_f; if a sequence f_n is uniformly bounded and converges to f for every x, then T_{f_n} converges strongly to T_f.

Let $\varphi_\omega(\chi)$ be the characteristic function of a measurable set ω and put $E(\omega) = T\varphi(\omega)$. $E(\omega)$ is a projection. Now, more generally, we shall prove next lemmas.

Lemma 1. For every function $f(\chi)$ of $\chi \varepsilon \Omega$, there is an additive and homogeneous operator T_f such that $(T_f x,\ y) = \int f(\chi)\, d\mu_{x,y}(\chi)$ for every $x \varepsilon \mathfrak{D}_f$ and every $y \varepsilon \mathfrak{H}$, where $\mathfrak{D}_f$ is the set of all x such that $\int |f|^2\, d\mu_{x,x} < +\infty$. T_f has the following properties:

(1) $T_{\alpha f} = \alpha T_f$ in $\mathfrak{D}_f$;

(2) $T_{f+g} = T_f + T_g$ in $\mathfrak{D}_f \cap \mathfrak{D}_g$; (3) $T_f^* = T_{\bar{f}}$ in $\mathfrak{D}_f = \mathfrak{D}_{\bar{f}}$;

(4) $E(\omega)\mathfrak{D}_f \subset \mathfrak{D}_f$ and $T_f E(\omega) = E(\omega)\, T_f$ in $\mathfrak{D}_f$;

(5) $(T_f x,\ T_g y) = \int f(x)\, \overline{g(x)}\, d\mu_{x,y}(x)$ for every $x \varepsilon \mathfrak{D}_f$ and every $y \varepsilon \mathfrak{D}_g$;

(6) if $x \varepsilon \mathfrak{D}_f$, we have $T_f x \varepsilon \mathfrak{D}_g$ if and only if $x \varepsilon \mathfrak{D}_{f.g}$, and, when this condition is

satisfied, $T_g T_f x = T_{f.g} x$ where $fg(x)$ equals to $f(x)g(x)$ whenever both factors are defined and equals to 0 elsewhere.

Proof. By Theorem 1, $\mathfrak{D}_f$ is a linear subspace of $\mathfrak{H}$, whence T_f exists and is additive and homogeneous. (1), (2) and (3) follow from Theorem 1. Since $E(\omega)$ commutes with A, we have $E(\omega)\mathfrak{D}_f \subset \mathfrak{D}_f$. If $x \in \mathfrak{D}_f$, then $(T_f E(\omega)x, y) = \int f(x)\, d\mu_{E(\omega)x,y} = (T_f x, E(\omega)y) = (E(\omega) T_f x, y)$, which proves (4).

Since we have $d\mu_{Tfx,y}(x) = f(x)\, d\mu_{x,y}(x)$ by (4), (5) is valid.

Furthermore, we have (6) from $\int |g(x)|^2\, d\mu_{Tfx,Tfx} = \int |f(x)g(x)|^2\, d\mu_{x,x}$.

Lemma 2. If $f(x)$ is a measurable function defined almost everywhere in Ω, then T_f is a closed operator with the domain $\mathfrak{D}_f$ everywhere dense in $\mathfrak{H}$. The adjoint operator T_f^* exists and is identical with $T_{\bar{f}}$.

Proof. It follows from Lemma 1 that $T_{(|f|+1)} T_{(|f|+1)^{-1}} x = x$ for every $x \in \mathfrak{H}$ and $T_{(|f|+1)^{-1}} T_{(|f|+1)} x = x$ for every $x \in \mathfrak{D}_f$. Therefore, $T^{-1}_{(|f|+1)} = T_{(|f|+1)^{-1}}$. Since $T_{(|f|+1)^{-1}}$ is bounded and measurable, the domain of $T_{(|f|+1)^{-1}}$ is $\mathfrak{H}$ and $T_{(|f|+1)^{-1}}$ is self-adjoint. It follows that $T_{(|f|+1)}$ is also self-adjoint and $\mathfrak{D}_{(|f|+1)}$ is everywhere dense in $\mathfrak{H}$. Since $\mathfrak{D}_f = \mathfrak{D}_{(|f|+1)}$, $\mathfrak{D}_f$ is everywhere dense. We get $g(x) = f(x)(|f(x)|+1)^{-1}$. By Lemma 1, it follows that $T_f = T_{(|f|+1)} \cdot T_g = T_g T_{(|f|+1)}$ in $\mathfrak{D}_f$ and $T_g^* = T_{\bar{g}}$. Consequently, $(x, T_f^* y) = (x, T_{(|f|+1)} T_{\bar{g}} y) = (x, T_{\bar{f}} y)$, which proves that $T_f^* = T_{\bar{f}}$ and T_f is closed.

Let λ be a complex number. If a measurable function $f(x)$ is defined almost everywhere in Ω, then the operator $(T_f - \lambda 1)$ exists and is bounded if and only if there exists a positive real number C such that $|f(x) - \lambda| \geqq C$ almost everywhere. λ is a characteristic value of T_f if and only if there exists $x \in \mathfrak{H}$ such that $\mu_{x,x}(\omega) > 0$, where ω is the set of x such that $f(x) = \lambda$. If λ is not a characteristic value of T_f, $T_{(f-\lambda)^{-1}}$ is defined and $T_{(f-\lambda)^{-1}} = (T_f - \lambda 1)^{-1}$.

2. We shall denote by **A′** the set of bounded operators which commute with **A**, and by $\mathbf{C}(A)$ the set of the closed operators which commute with **A′**.

Lemma 3. If $U \in \mathbf{C}(A)$, then

(1) for any x contained in the domain $\mathfrak{D}(U)$ of U, there exists a measurable function $f_x = f_x(x)$ such that $Ux = T_{f_x} x$.

(2) for any x and y contained in $\mathfrak{D}(U)$, there exists a measurable function $f_{x,y} = f_{x,y}(x)$ such that $Ux = T_{f_{x,y}} x$ and $Uy = T_{f_{x,y}} y$.

Proof. (1) Let $\mathfrak{M}_x$ be the set of all $T_f x$ such that $f \in L^2(\mu_{x,x})$: i, e., $\int |f|^2\, d\mu_{x,x}(x) < +\infty$. Since $\mathfrak{M}_x$ is isometrically isomorphic to $L^2(\mu_{x,x})$, it is a closed linear subspace. It is clear that $E(\omega)\mathfrak{M}_x \subset \mathfrak{M}_x$, therefore, the projection $P = P(\mathfrak{M}_x)$

commues with $E(\omega)$, so that $P \varepsilon A'$.

We have $Ux = UPx = PUx \varepsilon \mathfrak{M}_x$ as we wish to show.

(2) Let $\mathfrak{H} \oplus \mathfrak{H}$ be the direct sum of $\mathfrak{H}$ and $\mathfrak{H}$, whose elements are the pairs [u, v], where $u \varepsilon \mathfrak{H}$ and $v \varepsilon \mathfrak{H}$. We denote by $\mathfrak{M}_{x,y}$ the subset of all $[T_f x, T_f y] \varepsilon \mathfrak{H} \oplus \mathfrak{H}$, such that $f \varepsilon L^2 (\mu_{x,x} + \mu_{y,y})$. It follows that $\mathfrak{M}_{x,y}$ is a closed linear subspace of $\mathfrak{H} \oplus \mathfrak{H}$ and the projection $\hat{P}$ onto $\mathfrak{M}_{x,y}$ commutes with $\hat{E}(\omega)$, where $\hat{E}(\omega)$ is a projection defined by $\hat{E}(\omega)[u, v] = [E(\omega) u, E(\omega) v]$. We define P_{11}, P_{12}, P_{21} and P_{22} by $\hat{P}[u, 0] = [P_{11} u, P_{12} u]$ and $\hat{P}[0, v] = [P_{21} v, P_{22} v]$, then bounded operators P_{11}, P_{12}, P_{21} and P_{22} commute with $E(\omega)$. It follows that $[Ux, Uy] = \hat{U}[P_{11}x + P_{21}y, P_{12}x + P_{22}y]$. $\hat{P}[Ux, Uy] \varepsilon \mathfrak{M}_{x,y}$.

Theorem 2. Let $\mathfrak{H}$ be a separable Hilbert space. For every operator $U \varepsilon C(A)$ with everywhere dense domain, there exists a measurable function $f(\chi)$ such that $U = T_f$: $(Ux, y) = \int f(\chi)\, d\mu_{x,y}(\chi)$ for all $x \varepsilon \mathfrak{D}(U)$ and all $y \varepsilon \mathfrak{H}$.

Proof. By Lemma 3, for any x and y, there exists f_x, f_y and $f_{x,y}$ such that $Ux = T_{f_x} x$, $Uy = T_{f_y} y$, $Ux = T_{f_{x,y}} x$ and $Uy = T_{f_{x,y}} y$. Since T_{f_x}, T_{f_y} and $T_{f_{x,y}}$ commute with A, we have $f_x(\chi) = f_{x,y}(\chi)$ almost everywhere with respect to $\mu_{x,x}$ and $f_y(\chi) = f_{x,y}(\chi)$ almost everywhere with respect to μ_{yy}. By the separability of $\mathfrak{H}$, there exists an $x \varepsilon \mathfrak{D}(U)$ such that $\mu_{y,y}$ is absolutely continuous with respect to $\mu_{x,x}$; therefore, for this x, we have $f_x(\chi) = f_y(\chi)$ almost evervwhere. Since U is closed, $Uy = T_{f_x} y$ for every $y \varepsilon \mathfrak{D}(U)$. Theorem 2 is thereby proved.

Theorem 3. If H is a self-adjoint operator, we have $H \varepsilon C(A)$, where A is a maximal commutative B*-algebra which commutes with H, and $(Hx, y) = \int f(\chi)\, d(E(\chi) x, y)$ for all $x \varepsilon \mathfrak{D}(H)$ and $y \varepsilon \mathfrak{H}$.

References

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