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ON HYPERSURFACES OF EVEN DIMENSIONAL CONTACT RIEMANNIAN MANIFOLDS

Dedicated to Professor S. Sasaki on his 60th birthday

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1. Introduction.

Let $\tilde{M}$ be a $(2n+2)$-dimensional even dimensional contact Riemannian manifold with structure tensors η, g and ϕ, where η is a 1-form, g is a Riemannian metric and ϕ is an almost complex structure with the following properties:

(1.1) $$d\eta(\tilde{X}, \tilde{Y})=g(\phi\tilde{X}, \tilde{Y})$$

(1.2) $$g(\phi\tilde{X}, \phi\tilde{Y})=g(\tilde{X}, \tilde{Y}),$$

where $\tilde{X}$ and $\tilde{Y}$ are tangent vectors of $\tilde{M}$ (S. Sasaki [5]). Let $\eta^{\#}$ be the associated vector field of η and let σ be the square of the length of $\eta^{\#}$, which is a non-negative function on $\tilde{M}$:

(1.3) $$\sigma=g(\eta^{\#}, \eta^{\#}).$$

Throughout this note, we assume that $\sigma>0$ holds on $\tilde{M}$. Let $\xi^{\#}$ be a vector field, defined by

(1.4) $$\xi^{\#}=\phi\eta^{\#},$$

and let ξ be the associated 1-form of $\xi^{\#}$. Then we get

(1.5) $$g(\xi^{\#}, \xi^{\#})=\sigma$$

(1.6) $$g(\eta^{\#}, \xi^{\#})=0$$

(1.7) $$\phi\xi^{\#}=-\eta^{\#}.$$

In this note, we study hypersurfaces of even dimensional contact Riemannian manifolds. In § 2, we study a hypersurface M with a normal vector field $\zeta^{\#}$ of the form

(1.8) $$\zeta^{\#}=\lambda\xi^{\#}+\mu\eta^{\#},$$

where λ and μ are scalar functions on M such that $\lambda^2+\mu^2=1$. In § 3 and § 4, we study hypersurfaces M with $\xi^{\#}$ and $\eta^{\#}$ as their affine normal vector fields, respectively.

2. Hypersurfaces with $\zeta^{\#}=\lambda\xi^{\#}+\mu\eta^{\#}$.

Let M be a hypersurface of an even dimensional contact Riemannian manifold $\tilde{M}$ with the structure tensors η, g, and ϕ such that

(2.1) $$\zeta^{\#}=\lambda\xi^{\#}+\mu\eta^{\#}, \qquad \lambda^2+\mu^2=1$$

is a normal vector field. For a tangent vector X of M, put

(2.2) $$\phi X=AX+\alpha(X)\zeta^{\#},$$

where AX and $\alpha(X)\zeta^{\#}$ are respectively tangential part and normal part of ϕX. Operating ϕ to the both sides of (2.2), we get

(2.3) $$A^2X=-X+\alpha(X)\bar{\zeta}^{\#}$$

(2.4) $$\alpha(AX)=0,$$

where we have put

(2.5) $$\bar{\zeta}^{\#}=-\phi\zeta^{\#}.$$

Putting $\bar{\zeta}^{\#}$ in (2.2), we get

(2.6) $$A\bar{\zeta}^{\#}=0$$

(2.7) $$\alpha(\bar{\zeta}^{\#})=1.$$

Consequently, (2.3) and (2.7) imply that $(A, \bar{\zeta}^{\#}, \alpha)$ is an almost contact structure of M (cf. S. Sasaki [2], [4]).

Taking account of (1.2), (2.2) and (2.5), we get

(2.8) $$g(AX, AY)=g(X, Y)-\alpha(X)\bar{\zeta}(Y)-\bar{\zeta}(X)\alpha(Y)+\sigma\alpha(X)\alpha(Y),$$

where $\bar{\zeta}$ is a 1-form of M, defined by

(2.9) $$\bar{\zeta}(X)=g(X, \bar{\zeta}^{\#}).$$

Since we have $g(X, \bar{\zeta}^{\#})=g(X,-\phi\zeta^{\#})=g(\phi X, \zeta^{\#})=g(AX+\alpha(X)\zeta^{\#}, \zeta^{\#})=\sigma\alpha(X)$, we get

(2.10) $$\zeta=\sigma\alpha.$$

Hence (2.8) becomes

(2.11) $$g(AX, AY)=g(X, Y)-\sigma\alpha(X)\alpha(Y).$$

Thus, if we put

(2.12) $$\gamma=\frac{1}{\sigma}g,$$

γ is a Riemannian metric of M and satisfies the followings:

(2.13) $$\gamma(AX, AY)=\gamma(X,Y)-\alpha(X)\alpha(Y)$$

(2.14) $$\gamma(\zeta^{\#}, \zeta^{\#})=1$$

(2.15) $$\gamma(\zeta^{\#}, X)=\alpha(X).$$

Hence $(A, \zeta^{\#}, \alpha, \gamma)$ is an almost contact Riemannian structure of M (cf. S. Sasaki [2], [3]).

THEOREM 2.1. *Let $\tilde{M}$ be an even dimensional contact Riemannian manifold with structure tensors η, g and ϕ. We suppose that $\eta\neq 0$ holds everywhere. Let $\eta^{\#}$ be the associated vector field of η and let $\xi^{\#}=\phi\eta^{\#}$. Let M be a hypersurface of $\tilde{M}$ with a normal vector field $\zeta^{\#}$ of the form*

$$\zeta^{\#}=\lambda\xi^{\#}+\mu\eta^{\#}, \qquad \lambda^2+\mu^2=1.$$

Then $(A, \zeta^{\#}, \alpha, \gamma)$, defined by (2.2), (2.5) and (2.12), is an almost contact Riemannian structure of M.

For tangent vector fields X and Y of M, we have the equations of Gauss and Weingarten:

(2.16) $$\tilde{\nabla}_X Y=\nabla_X Y+h(X, Y)\zeta^{\#}$$

(2.17) $$\tilde{\nabla}_X\zeta^{\#}=-HX+\omega(X)\zeta^{\#}.$$

Now, suppose $\eta^{\#}$ is a Killing vector field. Then (1.1) implies

(2.18) $$\tilde{\nabla}_{\tilde{X}}\eta^{\#}=\phi\tilde{X}$$

for a tangent vector $\tilde{X}$ of $\tilde{M}$. Suppose, moreover, $\tilde{M}$ is a Kählerian manifold. Then (2.18) implies

(2.19) $$\tilde{\nabla}_{\tilde{X}}\xi^{\#}=-\tilde{X}.$$

In this case, for a tangent vector X of M, we get

(2.20) $$\begin{aligned}\tilde{\nabla}_X\zeta^{\#}&=\tilde{\nabla}_X(\lambda\xi^{\#}+\mu\eta^{\#})\\&=(X\lambda)\xi^{\#}+(X\mu)\eta^{\#}-\lambda X+\mu AX+\mu\alpha(X)\zeta^{\#}.\end{aligned}$$

LEMMA 2.2. *$(X\lambda)\xi^{\#}+(X\mu)\eta^{\#}$ is tangent to M and satisfies*

(2.21) $$A\{(X\lambda)\xi^{\#}+(X\mu)\eta^{\#}\}=0.$$

PROOF. Since $\lambda^2+\mu^2=1$, we get

(2.22) $$\lambda(X\lambda)+\mu(X\mu)=0.$$

Hence $g((X\lambda)\xi^{\#}+(X\mu)\eta^{\#},\ \zeta^{\#})=\{\lambda(X\lambda)+\mu(X\mu)\}\sigma=0$ holds good, that is, $(X\lambda)\xi^{\#}+(X\mu)\eta^{\#}$ is tangent to M.

By the definition of A, applying (2.10), we get

$$\begin{aligned}&A\{(X\lambda)\xi^{\#}+(X\mu)\eta^{\#}\}\\&=\phi\{(X\lambda)\xi^{\#}+(X\mu)\eta^{\#}\}-\alpha((X\lambda)\xi^{\#}+(X\mu)\eta^{\#})\zeta^{\#}\\&=\{(1-\lambda^2)(X\mu)+\lambda\mu(X\lambda)\}\xi^{\#}-\{(1-\mu^2)(X\lambda)+\lambda\mu(X\mu)\}\eta^{\#}\\&=\mu\{\mu(X\mu)+\lambda(X\lambda)\}\xi^{\#}-\lambda\{\lambda(X\lambda)+\mu(X\mu)\}\eta^{\#}\\&=0.\end{aligned}$$
Q. E. D.

According to Lemma 2.2, (2.17) and (2.20) imply

(2.23) $$HX=\lambda X-\mu AX-(X\lambda)\xi^{\#}-(X\mu)\eta^{\#}$$

(2.24) $$\omega(X)=\mu\alpha(X).$$

LEMMA 2.3. *If X is orthogonal to $\zeta^{\#}$, then $(X\lambda)\xi^{\#}+(X\mu)\eta^{\#}=0$ holds good.*

PROOF. Lemma 2.2 and (2.23) imply that $\zeta^{\#}$ is an eigenvector of H. Hence, if X is orthogonal to $\zeta^{\#}$, then HX is also orthogonal to $\zeta^{\#}$. Thus (2.23) implies $HX-\lambda X+\mu AX=-\{(X\lambda)\xi^{\#}+(X\mu)\eta^{\#}\}=0$. *Q. E. D.*

From (2.2), (2.16) and (2.17), we get

(2.25) $$\begin{aligned}\tilde{\nabla}_X(\phi Y)&=(\nabla_XA)Y+A\nabla_XY+h(X,\ AY)\zeta^{\#}\\&+\{(\nabla_X\alpha)(Y)+\alpha(\nabla_XY)\}\zeta^{\#}+\alpha(Y)\{-HX+\omega(X)\zeta^{\#}\}.\end{aligned}$$

On the other hand, we get

(2.26) $$\tilde{\nabla}_X(\phi Y)=\phi\tilde{\nabla}_X Y$$
$$=A\nabla_X Y+\alpha(\nabla_X Y)\zeta^{\#}+h(X, Y)\phi\zeta^{\#}.$$

Comparing (2.25) and (2.26), we get

(2.27) $$(\nabla_X A)Y-\alpha(Y)HX=-h(X, Y)\zeta^{\#}$$

(2.28) $$h(X,AY)+(\nabla_X\alpha)(Y)+\mu\alpha(X)\alpha(Y)=0,$$

where we have used (2.24).

Now, we calculate the torsion tensor N of the almost contact structure $(A, \zeta^{\#}, \alpha)$. Using (2.27), we get

(2.29) $$N(X,Y)=[AX,AY]-A[AX,Y]-A[X,AY]+A^2[X,Y]+2d\alpha(X,Y)\zeta^{\#}$$
$$=(\nabla_{AX}A)Y-(\nabla_{AY}A)X+A(\nabla_Y A)X-A(\nabla_X A)Y+2d\alpha(X,Y)\zeta^{\#}$$
$$=\alpha(Y)(HAX-AHX)-\alpha(X)(HAY-AHY)$$
$$-\{h(AX,Y)-h(X,AY)-2d\alpha(X,Y)\}\zeta^{\#}.$$

According to (2.28), we get $h(AX, Y)-h(X, AY)=2d\alpha(X, Y)$. Hence (2.29) becomes

(2.30) $$N(X, Y)=\alpha(Y)(HAX-AHX)-\alpha(X)(HAY-AHY).$$

On the other hand, (2.23), Lemma 2.2 and Lemma 2.3 imply that

(2.31) $$AHX=\lambda AX-\mu A^2X=HAX$$

holds good. Thus we get $N=0$; i.e., the almost contact structure $(A, \zeta^{\#}, \alpha)$ is normal (cf. S. Sasaki and Y. Hatakeyama [3]). Furthermore, (2.23) and Lemma 2.3 imply that, for tangent vectors X and Y which are orthogonal to $\zeta^{\#}$, $\lambda g(X, Y)-\mu g(AX, Y)=g(HX, Y)=g(X, HY)=\lambda g(X, Y)-\mu g(X, AY)$ holds good. Hence, since A is skew symmetric with respect to g, we get $\mu=0$ and hence $\lambda=\pm1$. Thus we get $\zeta^{\#}=\varepsilon\xi^{\#}$, where $\varepsilon=\pm1$, and hence $X\sigma=2g(\tilde{\nabla}_X\eta^{\#}, \eta^{\#})=2g(\phi X, \eta^{\#})=-2g(X, \xi^{\#})=0$ holds for any tangent vector X to M, i.e., σ is constant along M. Thus the Levi-Civita connections for $\gamma=(1/\sigma)g|M$ and $g|M$ are coincide. On the other hand, (2.23) implies $H=\varepsilon I$, where I is the identity transformation of tangent spaces. Thus (2.27) and $h(X, Y)=(1/\sigma)g(HX,Y)$ imply

(2.32) $$(\bar{\nabla}_X A)Y=\varepsilon\{\alpha(Y)X-\gamma(X,Y)\zeta^{\#}\},$$

where $\bar{\nabla}$ is the Levi-Civita connection for γ. In general, it is known that an almost contact Riemannian structure $(A, \zeta^{\#}, \alpha, \gamma)$ is a Sasakian structure, i.e., a normal contact Riemannian structure, it is necessary and sufficient that

$$(\bar{\nabla}_X A)Y=\alpha(Y)X-\gamma(X, Y)\zeta^{\#} \tag{2.33}$$

holds good (cf. S. Sasaki [4]). Hence if $\varepsilon=1$ (resp. $\varepsilon=-1$) holds, then $(A, \zeta^{\#}, \alpha, \gamma)$ (resp. $(-A, \zeta^{\#}, \alpha, \gamma)$) is a Sasakian structure. To sum up, we get the following:

THEOREM 2.4. *Under the same notations and assumptions of Theorem 2.1, if, furthermore, $\eta^{\#}$ is a Killing vector field and if $\tilde{M}$ is a Kählerian manifold, then the almost contact Riemannian structure $(A, \zeta^{\#}, \alpha, \gamma)$ is normal and $\zeta^{\#}=\varepsilon\xi^{\#}$ holds good, where $\varepsilon=\pm 1$. Furthermore, if $\varepsilon=1$ (resp. $\varepsilon=-1$), then $(A, \zeta^{\#}, \alpha, \gamma)$ (resp. $(-A, \zeta^{\#}, \alpha, \gamma)$) is a Sasakian structure, and vice versa.*

REMARK. In the case when $\zeta^{\#}=\xi^{\#}$, H. Taketa [7] has shown that $(A, \zeta^{\#}, \alpha, \gamma)$ is a Sasakian structure.

3. Hypersurfaces with an affine normal vector field $\xi^{\#}$.

Let M be a hypersurface of an even dimensional contact Riemannian manifold $\tilde{M}$ with the structure tensors η, g and ϕ such that $\xi^{\#}$ is never tangent to M and $\eta^{\#}$ is tangent to M. The restriction of $\eta^{\#}$ to M is also denoted by $\eta^{\#}$. We put

$$\phi X=BX+\beta(X)\xi^{\#} \tag{3.1}$$

for a tangent vector X of M, where BX and $\beta(X)\xi^{\#}$ are respectively tangential and (affine) normal components of ϕX with respect to $\xi^{\#}$. Applying ϕ to the both sides of (3.1), we get

$$B^2X=-X+\beta(X)\eta^{\#} \tag{3.2}$$

$$\beta(BX)=0. \tag{3.3}$$

Putting $\eta^{\#}$ in (3.1), and noticing $\phi\eta^{\#}=\xi^{\#}$, we get

$$B\eta^{\#}=0 \tag{3.4}$$

$$\beta(\eta^{\#})=1. \tag{3.5}$$

Hence $(B, \eta^{\#}, \beta)$ is an almost contact structure of M.

By the definition of B, we get

(3.6) $g(BX, BY)=g(X, Y)-\beta(X)\eta(Y)-\beta(Y)\eta(X)+\sigma\beta(X)\beta(Y)$

(3.7) $\xi(BX)=\eta(X)-\sigma\beta(X)$

(3.8) $\eta(BX)=-\xi(X).$

Thus, if we put

(3.9) $$\gamma=\frac{1}{\sigma}(g-\beta\otimes\eta-\eta\otimes\beta+2\sigma\beta\otimes\beta),$$

γ is a Riemannian metric of M and

(3.10) $\gamma(BX, BY)=\gamma(X, Y)-\beta(X)\beta(Y)$

(3.11) $\gamma(\eta^{\#}, X)=\beta(X)$

(3.12) $\gamma(\eta^{\#}, \eta^{\#})=1$

hold good. Hence $(B, \eta^{\#}, \beta, \gamma)$ is an almost contact Riemannian structure of M.

THEOREM 3.1. *Let $\tilde{M}$ be an even dimensional contact Riemannian manifold with structure tensors η, g and ϕ. We suppose $\eta\neq 0$ everywhere. Let $\eta^{\#}$ be the associated vector field of η and let $\xi^{\#}=\phi\eta^{\#}$. Let M be a hypersurface of $\tilde{M}$ with an affine normal vector field $\xi^{\#}$. Then, if $\eta^{\#}$ is tangent to M, the structure $(B, \eta^{\#}, \beta, \gamma)$, defined by* (3.1) *and* (3.9), *is an almost contact Riemannian structure of M.*

Now, suppose $\eta^{\#}$ is a Killing vector field. Then (2.18) and (3.1) imply

(3.13) $\tilde{\nabla}_X\eta^{\#}=BX+\beta(X)\xi^{\#},$

where $\tilde{\nabla}$ is the Levi-Civita connection for g of $\tilde{M}$. On the other hand, we have the equation of Gauss:

(3.14) $\tilde{\nabla}_X\eta^{\#}=\nabla_X\eta^{\#}+h(X, \eta^{\#})\xi^{\#},$

where $\nabla_X\eta^{\#}$ and $h(X, \eta^{\#})\xi^{\#}$ are tangential and normal parts of $\tilde{\nabla}_X\eta^{\#}$ with respect to the affine normal $\xi^{\#}$, respectively. Comparing (3.13) and (3.14), we get

(3.15) $\nabla_X\eta^{\#}=BX,$

(3.16) $h(X, \eta^{\#})=\beta(X).$

Suppose, furthermore, $\tilde{M}$ is a Kählerian manifold. Then (3.1) and (2.19) imply

$$(3.17)\qquad \tilde{\nabla}_X(\phi Y)=(\nabla_X B)Y+B\nabla_X Y+h(X, BY)\xi^{\#} + \{(\nabla_X \beta)(Y)+\beta(\nabla_X Y)\}\xi^{\#}-\beta(Y)X.$$

On the other hand, we get

$$(3.18)\qquad \tilde{\nabla}_X(\phi Y)=\phi\tilde{\nabla}_X Y = B\nabla_X Y+\beta(\nabla_X Y)\xi^{\#}-h(X, Y)\eta^{\#}.$$

Combining (3.17) and (3.18), we get

$$(3.19)\qquad (\nabla_X B)Y-\beta(Y)X=-h(X, Y)\eta^{\#}$$

$$(3.20)\qquad h(X, BY)+(\nabla_X\beta)(Y)=0.$$

Applying (3.19) and (3.20), we see that the torsion N of the almost contact structure $(B, \eta^{\#}, \beta)$ vanishes:

$$N(X, Y)=(\nabla_{BX}B)Y-(\nabla_{BY}B)X+B(\nabla_Y B)X-B(\nabla_X B)Y+2d\beta(X, Y)\eta^{\#} =0.$$

Hence we get

THEOREM 3.2. *Under the same notations and assumptions of Theorem 3.1, if, furthermore, $\eta^{\#}$ is a Killing vector field and if $\tilde{M}$ is a Kählerian manifold, the almost contact Riemannian structure $(B, \eta^{\#}, \beta, \gamma)$ is normal.*

After a rather long calculation, we see that

$$(3.21)\qquad \gamma(\bar{\nabla}_X\eta^{\#}, Y)=d\alpha(X, Y)$$

holds good, where $\bar{\nabla}$ is the Levi-Civita connection for γ. Thus $\eta^{\#}|M$ is a Killing vector field with respect to γ. But it is not known whether $(B, \eta^{\#}, \beta, \gamma)$ is a K-contact structure or not. If it is a K-contact structure, it is automatically a Sasakian structure.

4. Hypersurfaces with an affine normal vector field $\eta^{\#}$.

Let $\tilde{M}$ be an even dimensional contact Riemannian manifold with the structure tensors η, g and ϕ, and let M be a hypersurface of $\tilde{M}$ with an affine normal vector field $\eta^{\#}$. We assume that $\xi^{\#}$ is tangent to M and that $\eta\neq 0$ everywhere. If we put

$$\phi X=BX+\beta(X)\eta^{\#}, \tag{4.1}$$

$$\gamma=g-\frac{1}{\sigma}(\xi\otimes\xi+\eta\otimes\eta)+\beta\otimes\beta, \tag{4.2}$$

where BX and $\beta(X)\eta^{\#}$ are respectively tangential and normal components of ϕX with respect to the (affine) normal $\eta^{\#}$, we can see that the structure $(B, -\xi^{\#}, \beta, \gamma)$ is an almost contact Riemannian structure of M. Now, suppose $\eta^{\#}$ is a Killing vector field of $\tilde{M}$ and suppose $\tilde{M}$ is a Kählerian manifold. Then we can see that the almost contact Riemannian structure in consideration is normal. But, since we have

$$(\mathscr{L}_{\xi^{\#}}\gamma)(X, Y)=2\beta(X)\beta(Y)-2\gamma(X, Y), \tag{4.3}$$

$-\xi^{\#}$ is not a Killing vector field of M with respect to γ. Thus the normal almost contact Riemannian structure $(B, -\xi^{\#}, \beta, \gamma)$ is not a K-contact structure.

Bibliography

[1] S. I. Goldberg and K. Yano, Noninvariant hypersurfaces of almost contact manifolds, J. Math. Soc. Japan, 22 (1970), 25～34.

[2] S. Sasaki, On differentiable manifolds with certain structures which are closely related to almost contact structure, Tôhoku Math. J., 12 (1960), 459～476.

[3] S. Sasaki and Y. Hatakeyama, On differentiable manifolds with certain structures which are closely related to almost contact structure II, Tôhoku Math. J., 13 (1961), 281～294.

[4] S. Sasaki, Almost Contact Manifolds, Part I, Lecture note at Tôhoku University, 1965.

[5] S. Sasaki, On even dimensional contact Riemannian manifolds, Differential Geometry, in honor of K. Yano, Kinokuniya, 1972.

[6] T. Takahashi, A note on certain hypersurfaces of Sasakian manifolds, Kōdai Math. Sem. Rep., 21 (1969), 510～516.

[7] H. Taketa, An example of Sasakian manifold, to appear.

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