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FINITE ELEMENT METHOD OF MIXED TYPE AND ITS CONVERGENCE IN LINEAR SHELL PROBLEMS

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Introduction

The purpose of this article is to present a general way for proving the convergence of the finite element method of mixed type in linear elastic problems and as an application of this general theory, to prove the convergence of a mixed method applied to certain shell problems.

One can indicate the method of Rayleigh-Ritz and of Trefftz as the traditional variational methods to the solutions of partial differential equations. Corresponding to these methods the finite element methods are also classified as the displacement and the forced method. Since both methods are regarded, if the trial functions are conforming, as least square approximation to the exact solution, the problems concerning the accuracy are reduced to the approximation theory of functions.

So far as one treats the second order equations, these "classical" finite element methods serve the purpose sufficiently, but for higher order equations the situation becomes different. Hence various new finite element models have been proposed to overcome the difficulties. For the details of these developments one can refer to the survey paper of Pian and Tong [7].

However, it is not necessarily correct to say that these models were proposed always with sufficient theoretical justifications. For convenience' sake let us classify these finite element methods based on the "nonclassical" variational principles as follows.

[a] Methods of classical type using non-conforming trial functions.

[b] Methods of hyblid type.

[c] Methods of mixed type.

The first important theoretical problem is the convergence of these methods. Concerning the first methods some convergence proofs are given by the author [5] and Strang [8]. But their results can not necessarily cover all elements of non-conforming type, and more unified theory is expected. A paper of Tong and Pian [9] is known as a pioneering work which studied the convergence of the hyblid methods. But, as well known, their convergence proof is based on an unproved assertion and not necessarily complete (see the proof in p. 469 of [9]). A work of Kikuchi and Ando [4] is another recent result of this field.

The author's previous paper [6] deals with the convergence of a mixed method applied to plate bending problem, although his motivation comes from the coupled equations approach to bi-harmonic equations. He proves the convergence and gives the order of convergence of several approximating schemes, and proposes some stability criterions of the schemes for dynamic problems.

Although it is to be desired that some comparison theorems concerning the accuracy of each individual method are established, but as the first step of this study it will be also important to know which method is convergent or not and how the order of convergence is estimated.

The present paper is an extension of the author's results in [6] to shell problems. First we generalize the convergence proof of [6] in terms of three dimensional elasticity theory, and then apply the result to the convergence proof in linear shell problems taking cylindricall shell problem as an example. It is not difficult to apply our method to more general shells. Although our results do not cover all methods of mixed type, we expect that our method of proof used in this paper will be applied to the other mixed finite element methods too.

1. A characterization of the mixed method

The mixed method is based on Hellinger-Reissner's principle. The functional Π in this principle is expressed as

$$(1.1)\qquad \begin{aligned}\Pi(\sigma, u)=&\int_V [c(\sigma)-\frac{1}{2}\sigma_{ij}(u_{i,j}+u_{j,i})+\bar{F}_i u_i]dV\\ &+\int_{S_\sigma}\bar{T}_i u_i dS+\int_{S_u} T_i(u_i-\bar{u}_i)dS,\end{aligned}$$

where, the summation convention is employed and

σ_{ij}: stress tensor component $\sigma_{ij}=\sigma_{ji}$,
$c(\sigma)$: complimentally energy density,
u_i: displacement component,
$u_{i,j}$: $\partial u_i/\partial x_j$,
$\bar{F}$: prescribed body force component,
V: volume,

S: surface of V,
S_σ: portion of S over which the surface tractions are prescribed,
S_u: portion of S over which the displacements are prescribed,
T_i: surface traction component, i. e.,
$T_i=\sigma_{ij}\nu_j \qquad \nu_j=\cos(n, x_j)$, n: outward normal to S,
$\bar{T}_i$: prescribed surface traction component,
$\bar{u}_i$: prescribed displacement component.

The strain-displacement and strain-stress relations are given as follows.

(1.2) $$\varepsilon_{ij}=\frac{1}{2}(u_{i,j}+u_{j,i}),$$

(1.3) $$\varepsilon_{ij}=c_{klij}\sigma_{kl}$$

Hence the complimental energy density $c(\sigma)$ is written, by the symmetry of c_{klij}, as

(1.4) $$c(\sigma)=\frac{1}{2}\sigma_{ij}\varepsilon_{ij}=\frac{1}{2}c_{ijkl}\sigma_{ij}\sigma_{kl}$$

One may easily prove that the stationary condition of the functional $\Pi(\sigma,u)$ with respect to σ_{ij} and u_i is

$$\frac{\partial c(\sigma)}{\partial\sigma_{ij}}=\frac{1}{2}(u_{i,j}+u_{j,i})$$
$$\sigma_{ij,j}+\bar{F}_i=0 \qquad \text{in } V$$
$$\begin{cases} T_i=\bar{T}_i & \text{on } S_\sigma \\ u_i=\bar{u}_i \text{ on} & S_u. \end{cases}$$

We use this functional in a little modified form as shown below.

(1.5) $$\Pi'(\sigma,u)=\int_V [c(\sigma)+\sigma_{ij,j}u_i+\bar{F}_iu_i]dV -\int_{S_\sigma}(T_i-\bar{T}_i)u_idS-\int_{S_u}T_i\bar{u}_idS$$

As easily seen, this functional is obtained by the integration by part for the second term of $\Pi(\sigma, u)$. In the present paper we assume, for the convenience of the description, that the boundary condition is homogeneous, that is, $\bar{u}_i=\bar{T}_i=0$.

In the mixed finite element method, the stress and the displacement are assumed separately for each individual element. Suppose that these quantities are written by a suitable system of functions $\{\phi^{(p)}\}$ as

(1.6) $$\hat{\sigma}_{ij}=\sum_{p\in N}\hat{\sigma}_{ij}(p)\phi^{(p)}, \quad \hat{u}_i=\sum_{p\in N'}\hat{u}_i(p)\phi^{(p)}$$

where N is the set of grid points in $\bar{V}$ and N' in $\bar{V}-\bar{S}_u$. Here $\hat{u}_i$ have to vanish on S_u. The unknowns $\{\hat{\sigma}_{ij}(p)\}$ and $\{\hat{u}_i(p)\}$ are determined by the system of equations obtained by putting the first derivatives of $\Pi'(\hat{\sigma},\hat{u})$ with respect to these unknowns equal zero.

We introduce some new bilinear forms to describe these system of equations briefly. Define

(1.7) $$K_l(\hat{u}_k, \phi^{(p)})=\int_V \hat{u}_k\phi,_l{}^{(p)}dV-\int_{S_\sigma}\hat{u}_k\nu_l\phi^{(p)}dS$$

(1.8) $$C_{kl}(\hat{\sigma}, \phi^{(p)})=\int_V c_{ijkl}\hat{\sigma}_{ij}\phi^{(p)}dV.$$

Then the equations by which the unknowns in mixed method are determined can be written as follows.

(1.9) $$\frac{1}{2}[K_l(\hat{u}_k, \phi^{(p)})+K_k(\hat{u}_l, \phi^{(p)})]+C_{kl}(\hat{\sigma}, \phi^{(p)})=0 \ : \ p\in N$$

(1.10) $$K_l(\phi^{(p)}, \hat{\sigma}_{kl})+\int_V \bar{F}_k\phi^{(p)}dV=0 \ : \ p\in N'.$$

Note that these equations are satisfied also by the exact displacements and stresses u_i and σ_{ij}, since

$$K_l(\phi^{(p)}, \sigma_{kl})=\int_V \sigma_{kl},_l\phi^{(p)}dV-\int_{S_\sigma}T_k\phi^{(p)}dS.$$

Hellinger-Reissner's principle is not a minimum principle. But it can be called a conditional minimum principle. Although this is suggested in the previous paper [6] for bi-harmonic problems, we state below in a general form.

THEOREM 1.1. *The mixed method formulated above is equivalent to the following algorithm: Seek the minimizing displacements of the functional*

(1.11) $$F(\hat{u})=\int_V[c(\hat{\sigma})-\bar{F}_i\hat{u}_i]dV$$

when $\hat{\sigma}_{ij}$ are determined by the system of equations (1.9).

PROOF. We show that the derivatives of $\bar{F}(\hat{u})$ coincide with the left side of the equations (1.10). Since $\hat{\sigma}_{ij}$ is a function of $\hat{u}_\alpha(\beta)$, the first derivatives of $\hat{\sigma}_{ij}$ with respect to $\hat{u}_\alpha(\beta)$ satisfy

$$\frac{1}{2}[K_l(\partial\hat{u}_k/\partial\hat{u}_\alpha(\beta), \phi^{(p)})+K_k(\partial\hat{u}_l/\partial\hat{u}_\alpha(\beta), \phi^{(p)})] \\ +C_{kl}(\partial\hat{\sigma}/\partial\hat{u}_\alpha(\beta), \phi^{(p)})=0.$$

Multiplying by $\hat{\sigma}_{kl}(p)$ and summing up first on p and then on k and l we have

$$\frac{1}{2}[K_l(\partial\hat{u}_k/\partial\hat{u}_\alpha(\beta), \hat{\sigma}_{kl})+K_k(\partial\hat{u}_l/\partial\hat{u}_\alpha(\beta), \hat{\sigma}_{kl})] \\ +C_{kl}(\partial\hat{\sigma}/\partial\hat{u}_\alpha(\beta), \hat{\sigma}_{kl})=0.$$

Substituting (1.7) into the first term of this equation we have

$$\begin{aligned}
&-C_{kl}(\partial\hat{\sigma}/\partial\hat{u}_\alpha(\beta), \hat{\sigma}_{kl}) \\
&=\frac{1}{2}\int_V(\partial\hat{u}_k/\partial\hat{u}_\alpha(\beta)\hat{\sigma}_{kl,l}+\partial\hat{u}_l/\partial\hat{u}_\alpha(\beta)\hat{\sigma}_{kl,k})dV \\
&-\frac{1}{2}\int_{S_\sigma}(\partial\hat{u}_k/\partial\hat{u}_\alpha(\beta)\nu_l+\partial\hat{u}_l/\partial\hat{u}_\alpha(\beta)\nu_k)\hat{\sigma}_{kl}dS \\
&=\int_V\hat{\sigma}_{\alpha l,l}\phi^{(\beta)}dV-\int_{S_\sigma}\bar{T}_\alpha\phi^{(\beta)}dS \\
&=K_l(\phi^{(\beta)}, \hat{\sigma}_{\alpha l}).
\end{aligned}$$

Therefore by (1.4) and (1.8) we have

$$\begin{aligned}
\frac{\partial}{\partial\hat{u}_\alpha(\beta)}F(\hat{u})&=\int_V[c_{ijkl}(\partial\hat{\sigma}_{ij}/\partial\hat{u}_\alpha(\beta))\hat{\sigma}_{kl}-\bar{F}_\alpha\phi^{(\beta)}]dV \\
&=C_{kl}(\partial\hat{\sigma}/\partial\hat{u}_\alpha(\beta), \hat{\sigma}_{kl})-\int_V\bar{F}_\alpha\phi^{(\beta)}dV \\
&=-K_l(\phi^{(\beta)}, \hat{\sigma}_{\alpha l})-\int_V\bar{F}_\alpha\phi^{(\beta)}dV.
\end{aligned}$$

This completes the proof. (Q. E. D.)

REMARK (1). Strictly speaking the functional $F(\hat{u})$ has no minimizing point unless the system (1.9), (1.10) has a unique solution. A necessary and sufficient condition in order that this system has a unique solution is given by "*if* $\int_V c(\hat{\sigma})dV=0$ *then* $\hat{u}_i=0$, *when* $\hat{\sigma}_{ij}$ *are determined by the equations* (1.9)". This condition has to make sure in the practical application of the mixed method.

The mixed method can not be regarded as any least square method in the rigorous sense. But approximately we may call it a least square method to the exact solution. This assertion is based on the following fact. Define

$$C(\sigma, \sigma')=\frac{1}{2}C_{kl}(\sigma, \sigma'_{kl}) = \frac{1}{2}\int_V c_{ijkl}\sigma_{ij}\sigma'_{kl}dV \tag{1.12}$$

THEOREM 1.2. *Let u_i and σ_{ij} be the exact displacements and stresses respectively. Then for any $\dot{\sigma}_{ij}$ represented by the basis $\{\phi^{(p)}\}$ holds*

$$F(\hat{u})=C(\sigma-\hat{\sigma}, \sigma-\hat{\sigma})-C(\sigma, \sigma)-2C(\sigma-\hat{\sigma}, \sigma-\dot{\sigma}) -K_j(u_i-\hat{u}_i, \sigma_{ij}-\dot{\sigma}_{ij}), \tag{1.13}$$

where $\hat{u}_i$ and $\hat{\sigma}_{ij}$ are connected by the equations (1.9).

PROOF. Since the exact solution u_i, σ_{ij} satisfies the equations (1.9) and (1.10) we see

$$K_j(u_i, \sigma_{ij})+C_{ij}(\sigma, \sigma_{ij})=0,$$
$$K_j(u_i, \dot{\sigma}_{ij})+C_{ij}(\sigma, \dot{\sigma}_{ij})=0,$$
$$K_j(\hat{u}_i, \dot{\sigma}_{ij})+C_{ij}(\hat{\sigma}, \dot{\sigma}_{ij})=0,$$
$$K_j(\hat{u}_i, \sigma_{ij})+\int_V \bar{F}_i\hat{u}_i dV=0.$$

We have the desired identity by substituting these relations into the right side of (1.13). (Q.E.D.)

The importance of the identity (1.13) will be clear. Since $\dot{\sigma}_{ij}$ is arbitrary we can expect that

$$|2C(\sigma-\hat{\sigma}, \sigma-\dot{\sigma})+K_j(u_i-\hat{u}_i,\sigma_{ij}-\dot{\sigma}_{ij})| \leqq const.\ \varepsilon\|\sigma-\hat{\sigma}\|, \tag{1.14}$$

for some small constant ε depending on the quantity $\|\sigma-\dot{\sigma}\|$.
If this is correct, the dominate variable term in $F(\hat{u})$ is $C(\sigma-\hat{\sigma}, \sigma-\hat{\sigma})$ and thus it will be reasonable to say that the approximate solution is determined, roughly speaking, so as to minimize the quantity $C(\sigma-\hat{\sigma}, \sigma-\hat{\sigma})$ under the constraint (1.9). In such case, if $\{\phi^{(p)}\}$ is complete, the approximate solutions will converge to the exact solution in a suitable norm. Therefore, the convergence depends on whether the second term of the left side of (1.14) becomes small relatively or not, as the element size tends to zero.

For bi-harmonic problems it is already demonstrated that a estimate like (1.14) holds indeed, and by using this fact the order of covergence is also obtained [6]. The differential equations governing the deformation of shell is, unlike the plate case, a system of second and fourth order equations, and thus one can choose various shape functions of different orders for the different unknown functions. But what kind of shape functions is necessary for the convergence? In what follows we study this problem taking the equations of cylindrical shells as examples.

2. Governing equations of cylindrical shells

The deformation of a cylindrical shell is prescribed by a right-handed coordinate system (x_1, x_2, x_3) which is taken on the middle surface of the undeformed shell as shown in Fig. 1, where x_1 is along the generating line of the shell and x_2 is along the paralell ring.

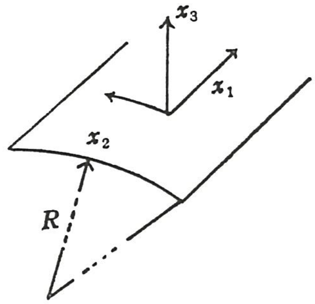

Fig. 1

Let Ω be the domain in (x_1, x_2)-plane occupied by the middle surface of the undeformed shell and u_i be the displacement along the x_i coordinate. The strain-displacement and stress-strain relations of this shell are usually given by the following equations [10].

$$(2.1)\quad \begin{cases} \varepsilon_{11}=u_{1,1}-x_3u_{3,11} \\ \varepsilon_{22}=u_{2,2}+\dfrac{1}{R}u_3-x_3\left(u_{3,22}-\dfrac{1}{R}u_{2,2}\right) \\ 2\varepsilon_{12}=u_{2,1}+u_{1,2}-2x_3\left(u_{3,12}-\dfrac{1}{R}u_{2,1}\right) \end{cases}$$

$$
(2.2)\qquad \begin{cases} \sigma_{11}=\dfrac{E}{1-\nu^2}(\varepsilon_{11}+\nu\varepsilon_{22}), \\ \sigma_{22}=\dfrac{E}{1-\nu^2}(\nu\varepsilon_{11}+\varepsilon_{22}), \\ \sigma_{12}=\dfrac{E}{1+\nu}\varepsilon_{12}, \end{cases}
$$

where

R: radious of the undeformed middle surface,
E: Young's modulus,
ν: Poisson's ratio.

In this article we assume, for the simplicity of the description, that the boundary condition is homogeneous and the geometrical boundary condition is given as

$$
(2.3)\qquad \begin{cases} u_1=u_2=0 & \text{on } \Gamma_1, \\ u_3=\dfrac{du_3}{dn}=0 & \text{on } \Gamma_2, \\ u_3=0 & \text{on } \Gamma_3, \end{cases}
$$

where Γ_i is a non-empty portion of the edge which may possibly be overlapping. Other boundary condition is the natural boundary condition given later.

Let t be the plate thickness. Then the strain energy of the shell is given by

$$
(2.4)\qquad E_s=\frac{1}{2}\int_\Omega\int_{-t/2}^{t/2}\sigma_{ij}\varepsilon_{ij}dx
$$

Let us define E_m and E_b as follows.

$$
E_m(u_1, u_2, u_3;\ \phi, \phi^*, \psi)
$$

$$
=\int_\Omega[u_{1,1}\phi_{,1}+u_{2,2}\phi^*_{,2}+\frac{1-\nu}{2}(u_{2,1}+u_{1,2})(\phi^*_{,1}+\phi_{,2})+\nu(u_{1,1}\phi^*_{,2}+\phi_{,1}u_{2,2})
$$

$$
+\frac{1}{R}\{(\nu u_{1,1}+u_{2,2})\psi+(\nu\phi_{,1}+\phi^*_{,2})u_3+\frac{1}{R}u_3\psi\}]dx
$$

$$
E_b(u_2, u_3;\ \phi^*, \psi)
$$

$$
=\int_\Omega[u_{3,11}\psi_{,11}+(u_{3,22}-\frac{1}{R}u_{2,2})(\psi_{,22}-\frac{1}{R}\phi^*_{,2})+
$$

$$+\nu\{u_{3,11}(\psi_{,22}-\frac{1}{R}\phi^*_{,2})+\psi_{,11}(u_{3,22}-\frac{1}{R}u_{2,2})\}$$
$$+2(1-\nu)(u_{3,12}-\frac{1}{R}u_{2,1})(\psi_{,12}-\frac{1}{R}\phi^*_{,1})]dx$$

As easily seen the quantities

$$E_m(u;u)=E_m(u_1, u_2, u_3;\ u_1, u_2, u_3),$$
$$\frac{t^2}{12}E_b(u;u)=\frac{t^2}{12}E_b(u_2, u_3; u_2, u_3)$$

correspond to the membrane and the bending energy respectively and

$$E_s=\frac{Et}{2(1-\nu^2)}[E_m(u;u)+\frac{t^2}{12}E_b(u;u)].$$

Hereafter we assume, without loss of generality, that

$$\frac{Et}{2(1-\nu^2)}=1. \tag{2.5}$$

Now let us define

$$\begin{cases} W_{11}=u_{3,11}, \quad W_{12}=u_{3,12}-\frac{1}{R}u_{2,1}, \\ W_{22}=u_{3,22}-\frac{1}{R}u_{2,2}, \end{cases} \tag{2.6}$$

and

$$B_1(u_1, u_2, u_3; \phi)=\int_\Omega[(u_{1,1}+\nu u_{2,2}+\frac{\nu}{R}u_3)\phi_{,1}$$
$$+\frac{1-\nu}{2}(u_{2,1}+u_{1,2})\phi_{,2}]dx,$$
$$B_2(u_1, u_2, u_3; \phi^*)=\int_\Omega[\frac{1-\nu}{2}(u_{2,1}+u_{1,2})\phi^*_{,1}$$
$$+(u_{2,2}+\nu u_{1,1}+\frac{1}{R}u_3)\phi^*_{,2}$$
$$-\frac{t^2}{12R}\{(\nu W_{11}+W_{22})\phi^*_{,2}+2(1-\nu)W_{12}\phi^*_{,1}\}]dx.$$
$$B_3(u_1, u_2, u_3; \psi)=\int_\Omega[\frac{1}{R}(\nu u_{1,1}+u_{2,2}+\frac{1}{R}u_3)\psi$$
$$+\frac{t^2}{12}\{(W_{11}+\nu W_{22})\psi_{,11}+(\nu W_{11}+W_{22})\psi_{,22}$$
$$+2(1-\nu)W_{12}\psi_{,12}\}]dx.$$

Further define

$$B_i(u;\ \phi)=B_i(u_1,\ u_2,\ u_3;\ \phi).$$

Then the principle of minimum potential energy yields a system of equations which are written as follows.

$$(2.7)\qquad \begin{cases} B_1(u;\ \phi)=\int_\Omega \bar{f}_1\phi dx \\ B_2(u;\ \phi^*)=\int_\Omega \bar{f}_2\phi^* dx \\ B_3(u;\ \psi)=\int_\Omega \bar{f}_3\psi dx, \end{cases}$$

where $\bar{f}_i$ corresponds to the component of the given body forces and ϕ, ϕ^* and ψ have to satisfy the geometrical boundary conditions imposed on u_1, u_2 and u_3 respectively. The explicit form of the differential equations and the natural boundary conditions of this shell are obtained by the integration by part of these equations. But we treat the equations in the above *weak* form, since the differential equations themselves are not necessary for our discussion.

3. Existence and uniqueness of the exact solution

Let $W_2^k(\Omega)$ be the Sobolev space of functions. The norm in this space is denoted by $\|u\|_k$. For $k=0$ this space is regarded as the space $L_2(\Omega)$ and hence

$$\|u\|_0^2=(u,\ u)=\int_\Omega u^2 dx$$

By $W_2^1(\Omega;\ \Gamma_1)$ we denote the space consisting of all $u\in W_2^1(\Omega)$ vanishing on Γ_1, and by $W_2^2(\Omega;\ \Gamma_{2,3})$ all $u\in W_2^2(\Omega)$ satisfying the boundary conditions imposed in u_3 in (2.3).

We call the triple $(u_1,\ u_2,\ u_3)(u_1,\ u_2\in W_2^1(\Omega;\ \Gamma_1),\ u_3\in W_2^2(\Omega;\ \Gamma_{2,\ 3}))$ a weak solution of the system (2.7) if the equations (2.7) are satisfied by any $\phi,\ \phi^*\in W_2^1(\Omega;\ \Gamma_1)$ and any $\psi\in W_2^2(\Omega;\Gamma_{2,3})$.

The weak solution of the system (2.7) exists and is unique, if R is sufficiently large. This is a immediate consequence of Riesz' representation theorem if the following theorem is proved. Let us define

$$E_s(u_1,\ u_2,\ u_3;\ \phi,\ \phi^*,\ \psi)=B_1(u;\ \phi)+B_2(u;\ \phi^*)+B_3(u;\ \psi),$$
$$E_s(u;\ u)=E_s(u_1,\ u_2,\ u_3;\ u_1,\ u_2,\ u_3).$$

THEOREM 3.1.

(1) $E_s = E_s(u; u)$,

(2) $E_s(u_1, u_2, u_3; \phi, \phi^*, \psi) = E_s(\phi, \phi^*, \psi; u_1, u_2, u_3)$,

(3) *If R is sufficiently large holds*

$$E_s(u; u) \geqq \bar{C}[\|u_1\|_1^2 + \|u_2\|_1^2 + \|u_3\|_2^2], \text{(*)} \tag{3.1}$$

where $\bar{C}$ is a constant independent of u_i.

PROOF. It is easy to prove (1) and (2), hence we prove (3). Let us define

$$E_i(u_1, u_2) = \int_\Omega [u_{1,1}^2 + u_{2,2}^2 + 2\nu u_{1,1} u_{2,2} + \frac{1-\nu}{2}(u_{2,1} + u_{1,2})^2] dx$$

$$E_o(u_3) = \int_\Omega [u_{3,11}^2 + u_{3,22}^2 + 2\nu u_{3,11} u_{3,22} + 2(1-\nu) u_{3,12}^2] dx$$

for $u_1, u_2 \in W_2^1(\Omega; \Gamma_1)$ and $u_3 \in W_2^2(\Omega; \Gamma_{2,3})$ respectively and put

$$\bar{C}_i = \inf \frac{E_i(u_1, u_2)}{\|u_1\|_1^2 + \|u_2\|_1^2}, \quad \bar{C}_o = \inf \frac{E_o(u_3)}{\|u_3\|_0^2}. \tag{3.2}$$

The existence of the constant $\bar{C}_i (>0)$ comes form the Korn's inequality [2], as well known. Now for any positive constants α and $\beta(<1)$ we have

$$\begin{aligned}
E_s = & E_i(u_1, u_2) + \frac{2}{R}\int_\Omega (u_{2,2} + \nu u_{1,1}) u_3 dx + \frac{1}{R^2}\int_\Omega u_3^2 dx \\
& + \frac{t^2}{12} E_o(u_3) - \frac{t^2}{6R}\int_\Omega [(u_{3,22} + \nu u_{3,11}) u_{2,2} + 2(1-\nu) u_{3,12} u_{2,1}] dx \\
& + \frac{t}{12R^2}\int_\Omega [u_{2,2}^2 + 2(1-\nu) u_{2,1}^2] dx \\
\geqq & E_i(u_1, u_2) - \alpha \int_\Omega (u_{2,2} + \nu u_{1,1})^2 dx \\
& + \frac{t^2}{12R^2}(1 - \frac{1}{\beta}) \int_\Omega [u_{2,2}^2 + 2(1-\nu) u_{2,1}^2] dx + \frac{t^2}{12} E_o(u_3) \\
& - \frac{t^2}{12}\beta \int_\Omega [(u_{3,22} + \nu u_{3,11})^2 + 2(1-\nu) u_{3,12}^2] dx \\
& + \frac{1}{R^2}(1 - \frac{1}{\alpha}) \int_\Omega u_3^2 dx \\
\geqq & \{(1-\alpha)\bar{C}_i + \frac{t^2}{6R^2}(1 - \frac{1}{\beta})\}\{\|u_1\|_1^2 + \|u_2\|_1^2\} \\
& + \{\frac{t^2}{12}(1-\beta) + \frac{1}{\bar{C}_o R^2}(1 - \frac{1}{\alpha})\} E_o(u_3).
\end{aligned}$$

(*) We use $\bar{C}$ as generic constant depending only on Ω and Γ_i, which is not necessarily the same each time used in this paper.

Therefore if R is sufficiently large we can choose α and β so that the inequality (3.1) holds. For example, choose $\alpha=\beta=1/2$. In this case the condition required for R and t are

$$3\bar{C}_i > \frac{t^2}{R^2}, \quad \frac{t^2}{24} > \frac{1}{\bar{C}_o R^2} \tag{3.3}$$

(Q.E.D.)

We remark that the conditions (3.3) are not so peculiar in practical application, since C_o and $\bar{C}_i$ depend only on Ω and Γ_i and not on t or R and further t/R is assumed to be small in linear shell theory. Through the present article we assume that R is sufficiently large so that the inequalities (3.3) hold. Thus an exact (weak) solution of our problem exists and is unique.

4. Hellinger-Reissner's functional

In this section we derive the explicit form of the system of equations associated with the Hellinger-Reissner's functional in cylindrical shell problem. This will make clear the difference between the system of equations (2.7) which is the Euler's equation of the total potential energy functional and the system of equations derived from Hellinger-Reissner's functional which is employed in our mixed method.

Let $\Pi(W,u)$ be the functional $\Pi(\sigma,u)$ represented by W_{ij} and u_i. Substituting (2.1), (2.2) and (2.6) into (1.1) we have

$$\begin{aligned}\Pi(W,u) &= E_m(u;\, u)+\frac{t^2}{12}E_b(W;\, W)\\ &\quad -2\Big\{E_m(u;\, u)+\frac{t^2}{12}\int_\Omega [(W_{11}+\nu W_{22})u_{3,11}+(\nu W_{11}+W_{22})u_{3,22}\\ &\quad +2(1-\nu)W_{12}u_{3,12}-\frac{1}{R}(\nu W_{11}+W_{22})u_{2,2}-\frac{2}{R}(1-\nu)W_{12}u_{2,1}]dx\Big\}\\ &\quad +2\int_\Omega \bar{f}_i u_i dx,\end{aligned}$$

where $t^2/12\cdot E_b(W;\, W)$ is the bending energy represented by W_{ij}. Note that we assumed $Et/2(1-\nu^2)=1$. The functional Π' necessary for our mixed method takes the following form.

$$
\begin{aligned}
\Pi'(W,u) =& -E_m(u;\, u)+\frac{t^2}{12}E_b(W;W)\\
&+\frac{t^2}{6}\int_\Omega [(W_{11}+\nu W_{22})_{,1}u_{3,1}+(\nu W_{11}+W_{22})_{,2}u_{3,2}\\
&+(1-\nu)(W_{12,1}u_{3,2}+W_{12,2}u_{3,1})\\
&+\frac{1}{R}(\nu W_{11}+W_{22})u_{2,2}+\frac{2}{R}(1-\nu)W_{12}u_{2,1}]dx\\
&-\frac{t^2}{6}\int_{\Gamma_2^c}[(W_{11}+\nu W_{22})\nu_1 u_{3,1}+(\nu W_{11}+W_{22})\nu_2 u_{3,2}\\
&+(1-\nu)(u_{3,2}\nu_1+u_{3,1}\nu_2)W_{12}]ds\\
&+2\int_\Omega \bar{f}_i u_i dx,
\end{aligned}
$$

where Γ_2^c denotes the complement of Γ_2 in the boundary of Ω. In order to express this functional briefly let us introduce bilinear form K_{kl} as follows.

$$
(4.1)\quad
\begin{cases}
K_{11}(u_2,\, u_3;\, \phi)=\int_\Omega u_{3,1}\phi_{,1}dx-\int_{\Gamma_2^c}u_{3,1}\nu_1\phi ds,\\
K_{22}(u_2,\, u_3;\, \phi)=\int_\Omega u_{3,2}\phi_{,2}dx-\int_{\Gamma_2^c}u_{3,2}\nu_2\phi ds+\frac{1}{R}\int_\Omega u_{2,2}\phi dx,\\
K_{12}(u_2,\, u_3;\, \phi)=\frac{1}{2}\{\int_\Omega(u_{3,1}\phi_{,2}+u_{3,2}\phi_{,1})dx\\
\quad -\int_{\Gamma_2^c}(u_{3,1}\nu_2+u_{3,2}\nu_1)\phi ds+\frac{2}{R}\int_\Omega u_{2,1}\phi dx\}.
\end{cases}
$$

Further define

$$
(4.2)\quad
\begin{cases}
K_{11}^{(0)}(u_3;\, \phi)=K_{11}(u_2,\, u_3;\, \phi),\\
K_{22}^{(0)}(u_3;\, \phi)=K_{22}(u_2,\, u_3;\, \phi)-\frac{1}{R}\int_\Omega u_{2,2}\phi dx,\\
K_{12}^{(0)}(u_3;\, \phi)=K_{12}(u_2,\, u_3;\, \phi)-\frac{1}{R}\int_\Omega u_{2,1}\phi dx,
\end{cases}
$$

and

$$
(4.3)\quad
\begin{aligned}
K(u_2,\, u_3;\, W)=&K_{11}(u_2,\, u_3;\, W_{11}+\nu W_{22})\\
&+K_{22}(u_2,\, u_3;\nu W_{11}+W_{22})+2(1-\nu)K_{12}(u_2,\, u_3;\, W_{12}),
\end{aligned}
$$

$$
(4.4)\quad
\begin{aligned}
K^{(0)}(u_3;\, W)=&K_{11}^{(0)}(u_3;W_{11}+\nu W_{22})\\
&+K_{22}^{(0)}(u_3;\, \nu W_{11}+W_{22})+2(1-\nu)K_{12}^{(0)}(u_3;W_{12}).
\end{aligned}
$$

THEOREM 4.1.

(1) *The functional* $\Pi'(W,u)$ *is represented also as follows.*

$$(4.5) \qquad \Pi'(W, u) = -E_m(u; u) + \frac{t^2}{12} E_b(W; W) + \frac{t^2}{6} K(u_2, u_3; W) + 2(\bar{f}_i, u_i).$$

(2) *The stationary condition of* $\Pi'(W, u)$ *with respect to* $\{W_{ij}\}$ *and* $\{u_i\}$ *is given by the following system of equations.*

$$(4.6) \qquad K_{ij}(u_2, u_3; w) + (W_{ij}, w) = 0,$$

$$(4.7) \qquad \begin{cases} B_1(u; \phi) = (\bar{f}_1, \phi), \\ B_2(u; \phi^*) = (\bar{f}_2, \phi^*), \\ \dfrac{1}{R}(\nu u_{1,1} + u_{2,2} + \dfrac{1}{R} u_3, \psi) - \dfrac{t^2}{12} K^{(0)}(\psi; W) = (\bar{f}_3, \psi), \end{cases}$$

where w *is arbitrary smooth function.*

Proof of this theorem is easy. Note that the third equation in (4.7) is obtained by the integration by part for $B_3(u; \psi)$ defined in the previous section. This is the only difference between the Euler's equations of total potential energy functional and the one of Hellinger-Reissner's functional.

5. Approximating schemes

Hereafter we assume, for simplicity, that Ω is a closed polygon. Let Ω_h be a triangulation of Ω. We assume that Ω_h is a regular triangulation of Ω in the sense defined in the previous paper [6]. Roughly speaking, this means that no triangle in Ω_h is crushed as the triangulation becomes fine.

In this article we employ two kinds of basis $\{\phi^{(p)}\}$ (p runs all vertexes of Ω_h) and $\{\psi^{(p)}\}$ (p runs all vertexes and midpoints of the sides of Ω_h). $\{\phi^{(p)}\}$ is a system of functions such that (i) linear in each triangle and continuous on Ω_h (ii) $\phi^{(p)}$ vanishes at any vertexes except p and assumes unity at p. Similarly, $\{\psi^{(p)}\}$ is (i) perfectly quadratic in each triangle and continuous on Ω_h (ii) $\psi^{(p)}$ vanishes at any nodes except p and assumes unity at p. We call $\{\phi^{(p)}\}$ and $\{\psi^{(p)}\}$ the first and the second order basis respectively.

Although there are several combinations of these bases we consider the following two algorithms.

Algorithm C11: The approximate solution is sought in the following form.

(5.1) $$\hat{W}_{ij}(x)=\sum_{p\in\Omega_h} W_{ij}^{(p)}\phi^{(p)}(x),$$

(5.2) $$\begin{cases}\hat{u}_i(x)=\sum_{p\in\Omega_h-\bar{\Gamma}_1}\hat{u}_i^{(p)}\phi^{(p)}(x) \quad (i=1,2)\\ \hat{u}_3(x)=\sum_{p\in\Omega_h-\bar{\Gamma}_{2,3}}\hat{u}_3^{(p)}\phi^{(p)}(x)\end{cases}$$

The unknowns are determined by the following equations.

(5.3) $$K_{ij}(\hat{u}_2, \hat{u}_3; \phi^{(p)})+(\hat{W}_{ij}, \phi^{(p)})=0 \quad p\in\Omega_h,$$

(5.4) $$\begin{cases}B_1(\hat{u}_1, \hat{u}_2, \hat{u}_3; \phi^{(p)})=(\bar{f}_1, \phi^{(p)}) \quad p\in\Omega_h-\bar{\Gamma}_1,\\ B_2(\hat{u}_1, \hat{u}_2, \hat{u}_3; \phi^{(p)})=(\bar{f}_2, \phi^{(p)}) \quad p\in\Omega_h-\bar{\Gamma}_1,\\ \frac{1}{R}(\nu\hat{u}_{1,1}+\hat{u}_{2,2}+\frac{1}{R}\hat{u}_3, \phi^{(p)})-\frac{t^2}{12}K^{(0)}(\phi^{(p)}; \hat{W})\\ \qquad =(\bar{f}_3, \phi^{(p)}) \quad p\in\Omega_h-\bar{\Gamma}_{2,3}.\end{cases}$$

Algorithm C12: $\hat{u}_3$ and $\hat{W}_{ij}$ are sought by the same basis as in the Algorithm C_{11} but $\tilde{u}_1$ and $\tilde{u}_2$ by the second order basis $\{\psi^{(p)}\}$. The unknowns are determined by the following equations.

(5.5) $$K_{ij}(\hat{u}_2, \hat{u}_3; \phi^{(p)})+(\hat{W}_{ij}, \phi^{(p)})=0 \quad p\in\Omega_h,$$

(5.6) $$\begin{cases}B_1(\tilde{u}_1, \tilde{u}_2, \hat{u}_3; \psi^{(p)})=(\bar{f}_1, \psi^{(p)}) \quad p\in\Omega_h-\bar{\Gamma}_1\\ B_2(\tilde{u}_1, \tilde{u}_2, \hat{u}_3; \psi^{(p)})=(\bar{f}_2, \psi^{(p)}) \quad p\in\Omega_h-\bar{\Gamma}_1,\\ \frac{1}{R}(\nu\tilde{u}_{1,1}+\tilde{u}_{2,2}+\frac{1}{R}\hat{u}_3, \phi^{(p)})-\frac{t^2}{12}K^{(0)}(\phi^{(p)}; \hat{W})\\ \qquad =(\bar{f}_3, \phi^{(p)}) \quad p\in\Omega_h-\bar{\Gamma}_{2,3}.\end{cases}$$

We note that these system of equations can be derived by putting the first derivatives of $\Pi'(\hat{W}, \hat{u})$ equal zero after substituting (5.1) and (5.2), & C..

The unique solvability of these systems are not necessarily evident. To ensure this we put

ASSUMPTION: *Let $\hat{W}_{ij}$ be determined by the equations* (5.3) (*in Algorithm C_{11}*) *or* (5.5) (*in Algorithm C_{12}*). *Then holds*

(5.7) $$\|\hat{u}_3\|_0^2\leqq\bar{C}(E_b(\hat{W}; \hat{W})+\frac{1}{R^2}\|\hat{u}_2\|_1^2) \text{ (in Algorithm } C_{11}),$$

or

(5.8) $$\|\hat{u}_3\|_0^2\leqq\bar{C}(E_b(\hat{W}; \hat{W})+\frac{1}{R^2}\|\tilde{u}_2\|_1^2) \text{ (in Algorithm } C_{12}).$$

In many cases this assumption is fulfilled. See [6]. If this assumption is filfilled then the system which is proposed above is uniquely solvable for sufficiently large R. To prove this consider, for instance, the system (5.3), (5.4). Assume that $(\bar{f}_i, \phi^{(p)})=0$. Then, by (5.7) and (3.2) we have

$$
\begin{aligned}
0 &= B_1(\hat{u}_1, \hat{u}_2, \hat{u}_3; \hat{u}_1)+B_2(\hat{u}_1, \hat{u}_2, \hat{u}_3; \hat{u}_2) \\
&\quad +\frac{1}{R}(\nu\hat{u}_{1,1}+\hat{u}_{2,2}+\frac{1}{R}\hat{u}_3, \hat{u}_3)-\frac{t^2}{12}K^{(0)}(\hat{u}_3; \hat{W}) \\
&= E_m(\hat{u}; \hat{u})+\frac{t^2}{12}E_b(\hat{W}; \hat{W}) \\
&\geqq E_i(\hat{u}_1, \hat{u}_2)+\frac{2}{R}(\hat{u}_{2,2}+\nu\hat{u}_{1,1}, \hat{u}_3)+\frac{1}{R^2}\|\hat{u}_3\|_0^2 \\
&\quad +\frac{t^2}{12}[\frac{1}{\bar{C}}\|\hat{u}_3\|_0^2-\frac{1}{R^2}\|\hat{u}_2\|_1^2] \\
&\geqq [(1-\alpha)\bar{C}_i-\frac{t^2}{12R^2}][\|\hat{u}_1\|_1^2+\|\hat{u}_2\|_1^2] \\
&\quad +[\frac{t^2}{12\bar{C}}+\frac{1}{R^2}(1-\frac{1}{\alpha})]\|\hat{u}_3\|_0^2 \quad (0<\alpha<1),
\end{aligned}
$$

where $\bar{C}$ is the constant appearing in (5.7). Therefore, if R is sufficiently large we can take α so that

$$
(5.9) \qquad \begin{cases} (1-\alpha)\bar{C}_i-\dfrac{t^2}{12R^2}>0, \\ \dfrac{t^2}{12\bar{C}}+\dfrac{1}{R^2}(1-\dfrac{1}{\alpha})>0. \end{cases}
$$

Hence $\hat{u}_i=\hat{W}_{ij}=0$, which proves the unique solvability of (5.3), (5.4). Inequalities (5.9) are realized, for example, by putting $\alpha=1/2$ and assuming that R satisfies the following inequalities.

$$
(5.10) \qquad 6\bar{C}_i>\frac{t^2}{R^2}, \quad \frac{t^2}{12}<\frac{\bar{C}}{R^2}.
$$

Note that this condition is similar to that of (3.3) which is a sufficient condition for the unique existence of the exact solution. We assume that R is sufficiently large so that the inequalities of (5.10) hold together with those of (3.3)

Now corresponding to the Theorem 1.1 in the general theory we have

THEOREM 5.1. *Algorithm C_{11} and Algorithm C_{12} are equivalent to the following algorithms respectively.*

(C_{11}): *Seek the minimizing functions of the functional*

(5.11) $$F_1(\hat{u})=E_m(\hat{u};\hat{u})+\frac{t^2}{12}E_b(\hat{W};\hat{W})-2(\bar{f}_i,\hat{u}_i),$$

where $\hat{W}_{ij}$ *are determined by the equations* (5.3).

(C_{12}): *Seek the minimizing functions of the functional*

(5.12) $$F_2(\hat{u})=E_m(\tilde{u}_1,\tilde{u}_2,\hat{u}_3;\tilde{u}_1,\tilde{u}_2,\hat{u}_3)+\frac{t^2}{12}E_b(\hat{W};\hat{W})$$
$$-2(\bar{f}_1,\tilde{u}_1)-2(\bar{f}_2,\tilde{u}_2)-2(\bar{f}_3,\hat{u}_3),$$

where $\hat{W}_{ij}$ *are determined by the equations* (5.5).

Similarly, corresponding to Theorem 1.2 we have

THEOREM 5.2. $F_1(\hat{u})$ and $F_2(\hat{u})$ *in Theorem* 5.1 *are represented also as follows.*

(5.13) $$F_1(\hat{u})=E_s(u-\hat{u};u-\hat{u})-E_s(u;u)$$
$$-\frac{t^2}{6}[E_b(W-\hat{W};W-\dot{W})+K(u_2-\hat{u}_2,u_3-\hat{u}_3;W-\dot{W})]$$

(5.14) $$F_2(\hat{u})=E_s(u_1-\tilde{u}_1,u_2-\tilde{u}_2,u_3-\hat{u}_3;u_1-\tilde{u}_1,u_2-\tilde{u}_2,u_3-\hat{u}_3)$$
$$-E_s(u;u)$$
$$-\frac{t^2}{6}[E_b(W-\hat{W};W-\dot{W})+K(u_2-\tilde{u}_2,u_3-\hat{u}_3;W-\dot{W})],$$

where $\{W_{ij},u_i\}$ is the exact solution and $\dot{W}_{ij}$ are arbitrary functions represented by the basis $\{\phi^{(p)}\}$.

PROOF. We first prove (5.13). The second one is then obvious. By (4.3), (4.2) and (4.7) we see

$$\frac{t^2}{12}K(\hat{u}_2,\hat{u}_3;W)=\frac{t^2}{12}K^{(0)}(\hat{u}_3;W)+\frac{t^2}{12R}[(\nu W_{11}+W_{22},\hat{u}_{2,2})$$
$$+2(1-\nu)(W_{12},\hat{u}_{2,1})]$$
$$=\frac{1}{R}(\nu u_{1,1}+u_{2,2}+\frac{1}{R}u_3,\hat{u}_3)-(\bar{f}_3,\hat{u}_3)$$
$$+\frac{t^2}{12R}[(\nu W_{11}+W_{22},\hat{u}_{2,2})+2(1-\nu)(W_{12},\hat{u}_{2,1})].$$

Therefore it holds that

$$\frac{t^2}{12}K(u_2-\hat{u}_2,u_3-\hat{u}_3;W-\dot{W})$$
$$=-\frac{t^2}{12}[E_b(W;W)-E_b(W;\dot{W})+E_b(\hat{W};\dot{W})]$$

$$-\frac{1}{R}(\nu u_{1,1}+u_{2,2}+\frac{1}{R}u_3,\ \hat{u}_3)+(\bar{f}_3,\ \hat{u}_3)$$
$$-\frac{t^2}{12R}[(\nu W_{11}+W_{22},\ \hat{u}_{2,2})+2(1-\nu)(W_{12},\ \hat{u}_{2,1})],$$

and this implies the following identity.

$$\frac{t^2}{12}E_b(\hat{W};\ W)$$
$$(5.15)\qquad =-\frac{t^2}{12}E_b(W-\hat{W};\ W-\dot{W})-\frac{t^2}{12}K(u_2-\hat{u}_2,\ u_3-\hat{u}_3;\ W-\dot{W})$$
$$-\frac{1}{R}(\nu u_{1,1}+u_{2,2}+\frac{1}{R}u_3,\ \hat{u}_3)+(\bar{f}_3,\ \hat{u}_3)$$
$$-\frac{t^2}{12R}[(\nu W_{11}+W_{22},\ \hat{u}_{2,2})+2(1-\nu)(W_{12}.\ \hat{u}_{2,1})].$$

On the other hand, by (2.7) we have

$$(\bar{f}_1,\ \hat{u}_1)+(\bar{f}_2,\ \hat{u}_2)$$
$$=B_1(u;\ \hat{u}_1)+B_2(u;\ \hat{u}_2)$$
$$=E_m(u;\hat{u})-\frac{1}{R}(\nu u_{1,1}+u_{2,2}+\frac{1}{R}u_3,\ \hat{u}_3)$$
$$-\frac{t^2}{12R}[(\nu W_{11}+W_{22},\ \hat{u}_{2,2})+2(1-\nu)(W_{12},\ \hat{u}_{2,1})],$$

and hence by (5.15)

$$E_s(u-\hat{u};\ u-\hat{u})=E_s(u;\ u)-2E_s(u;\ \hat{u})+E_s(\hat{u};\ \hat{u})$$
$$=E_s(u;\ u)+E_s(\hat{u};\ \hat{u})$$
$$-2[(f_1,\ \hat{u}_1)+(f_2,\hat{u}_2)+\frac{1}{R}(\nu u_{1,1}+u_{2,2}+\frac{1}{R}u_3,\ \hat{u}_3)$$
$$+\frac{t^2}{12R}\{(\nu W_{11}+W_{22},\ \hat{u}_{2,2})+2(1-\nu)(W_{12},\hat{u}_{2,1})\}$$
$$+\frac{t^2}{12}E_b(W;\hat{W})]$$
$$=E_s(u;\ u)+E_s(\hat{u};\ \hat{u})-2(\bar{f}_i,\ \hat{u}_i)$$
$$+\frac{t^2}{6}[E_b(W-\hat{W};\ W-\dot{W})+K(u_2-\hat{u}_2,\ u_3-\hat{u}_3;\ W-\dot{W})],$$

which proves the equality (5.13). (Q.E.D.)

6. Convergence of the approximate solutions

In this section we first prove the convergence of Algorithm C_{11}. The proof for Algorithm C_{12} is almost the same.

DEFINITION 6.1. Let h be the length of the largest side of the triangles in Ω_h. We say a sequence of triangulations $\{\Omega_h\}$ (or briefly, the triangulation Ω_h) is nearly consistent triangulation of Ω, if the triangulation Ω_h is regular and there exists a closed subdomain Ω_h' of Ω_h such that
(i) (the number of vertexes in $\Omega_h-\Omega_h'$)$\leqq \bar{C}h^{-1}$, (ii) for any sufficiently smooth function w satisfying the same boundary condition as is required for $\hat{u}_3$ holds

$$(6.1)\quad K_{ij}^{(0)}(\dot{w},\phi^{(p)})=\begin{cases}-w,_{ij}|_p(1,\phi^{(p)})+Ch^3 & p\in\Omega_h'\\ Ch^2 & p\in\Omega_h-\Omega_h',\end{cases}$$

where $\dot{w}$ is the interpolating function of w represented by the basis $\{\phi^{(p)}\}$, and C is a constant depending only on w.

We remark that if Ω_h is nearly consistent then the difference operator $K_{ij}^{(0)}(\dot{W},\phi^{(p)})(p\in\Omega_h')$ is consistent in the ordinary sense used in finite difference method. Hence there are several triangulations of the above property (see, for instance, [3]). We assume in this section that the triangulation Ω_h is nearly consistent.

Now let us determine $\mathring{W}_{ij}=\Sigma\mathring{W}_{ij}(p)\phi_p(p\in\Omega_h)$ by the following system of equations.

$$(6.2)\qquad K_{ij}(\dot{u}_2,\dot{u}_3;\phi_p)+(\mathring{W}_{ij},\phi_p)=0\quad p\in\Omega_h,$$

where $\dot{u}_i$ is the interpolating function of the exact solution u_i by the basis $\{\phi^{(p)}\}$.

LEMMA 6.1. *Let $\hat{u}_i$ and $\hat{W}_{ij}$ be the solution of the system* (5.3), (5.4). *Suppose that the exact solution is sufficiently smooth in Ω_h. Then holds*

$$(6.3)\qquad \|u_3-\hat{u}_3\|_0^2\leqq Ch+\bar{C}[E_b(W-\hat{W};W-\hat{W})+\frac{1}{R^2}\|u_2-\hat{u}_2\|_1^2]^{(*)}$$

* In what follows, C is a generic constant depending on the exact solution, which is not necessarily the same each time used.

PROOF. Since

$$K_{ij}(\dot{u}_2-\hat{u}_2,\ \dot{u}_3-\hat{u}_3;\ \phi^{(p)})+(\mathring{W}_{ij}-\hat{W}_{ij},\ \phi^{(p)})=0,$$

we have by (5.7)

$$\text{(6.4)}\qquad \|\dot{u}_3-\hat{u}_3\|_0^2\leqq\bar{C}(E_b(\mathring{W}-\hat{W};\ \mathring{W}-\hat{W})+\frac{1}{R^2}\|\dot{u}_2-\hat{u}_2\|_1^2).$$

On the other hand, by (6.1) we see, for instance if $i=j=2$,

$$\begin{aligned}&K_{ij}(u_2-\dot{u}_2,\ u_3-\dot{u}_3;\ \phi^{(p)})\\ &=K_{ij}^{(0)}(u_3-\dot{u}_3;\ \phi^{(p)})-\frac{1}{R}(u_{2,2}-\dot{u}_{2,2},\ \phi^{(p)})\\ &=\begin{cases} Ch^3-\dfrac{1}{R}(u_{2,2}-\dot{u}_{2,2},\ \phi^{(p)}) & p\in\Omega_h' \\ Ch^2-\dfrac{1}{R}(u_{2,2}-\dot{u}_{2,2},\ \phi^{(p)}) & \text{otherwise.}\end{cases}\end{aligned}$$

Taking account of the fact that

$$\text{(6.5)}\qquad \sum_{p\in\Omega_h}[\hat{W}_{ij}(p)-\mathring{W}_{ij}(p)]^2h^2\leqq\bar{C}\|\hat{W}_{ij}-\mathring{W}_{ij}\|_0^2$$

(see [6], p./97) and the well known interpolation theorem we have

$$\begin{aligned}&|(W_{ij}-\mathring{W}_{ij},\ \dot{W}_{ij}-\mathring{W}_{ij})|\\ &\leqq C[h^3\sum_{p\in\Omega'_h}|\dot{W}_{ij}(p)-\mathring{W}_{ij}(p)|+h^2\sum_{p\in\Omega_h-\Omega'_h}|\dot{W}_{ij}(p)-\mathring{W}_{ij}(p)|]\\ &\quad+\frac{1}{R}|(u_{2,2}-\dot{u}_{2,2},\ \dot{W}_{ij}-\mathring{W}_{ij})|\\ &\leqq Ch^{1/2}\|\dot{W}_{ij}-\mathring{W}_{ij}\|_0,\end{aligned}$$

and thus

$$\text{(6.6)}\qquad \|W_{ij}-\mathring{W}_{ij}\|_0\leqq Ch^{1/2}.$$

Inequality (6.3) follows from (6.4) and (6.6). (Q.E.D.)

LEMMA 6.2. *Let $v_i\in W_2^1(\Omega)$. Then holds*

$$\text{(6.7)}\qquad \|v_1\|_1^2+\|v_2\|_1^2\leqq\frac{1}{\bar{C}_i}[2E_m(v;\ v)+\frac{1}{R^2}\|v_3\|_0^2].$$

PROOF. For $\alpha>0$ we have

$$E_m(v;\ v)=E_i(v_1,\ v_2)+\frac{2}{R}(v_{2,2}+\nu v_{1,1},\ v_3)+\frac{1}{R^2}\|v_3\|_0^2$$

$$\geqq E_i(v_1, v_2)-\alpha\|v_{2,2}+\nu v_{1,1}\|_0^2+\frac{1}{R^2}(1-\frac{1}{\alpha})\|v_3\|_0^2$$
$$\geqq (1-\alpha)E_i(v_1, v_2)+\frac{1}{R^2}(1-\frac{1}{\alpha})\|v_3\|_0^2.$$

Therefore, taking $\alpha=1/2$ we obtain the inequality (6.7). (Q.E.D.)

LEMMA 6.3. *If R is sufficiently large hold the following inequalities.*

$$\|u_1-\hat{u}_1\|_1^2+\|u_2-\hat{u}_2\|_1^2\leqq C\frac{h}{R^2}+\bar{C}[E_m(u-\hat{u};\, u-\hat{u}) + \frac{1}{R^2}E_b(W-\hat{W};\, W-\hat{W})], \tag{6.8}$$

$$\|u_3-\hat{u}_3\|_0^2\leqq Ch+\bar{C}[\frac{1}{R^2}E_m(u-\hat{u};\, u-\hat{u})+E_b(W-\hat{W};\, W-\hat{W})]. \tag{6.9}$$

These inequalities are the direct consequence of the previous two lemmas. The condition required for R is

$$\frac{1}{R^4}<\frac{\bar{C}_i}{\bar{C}}, \tag{6.10}$$

where $\bar{C}_i$ and $\bar{C}$ appearing in this inequality are the constants defined by (3.2) and (5.7) respectively.

Now let us estimate the quantity $E_s(u-\hat{u};u-\hat{u})$. Theorem 5.2 implies

$$E_s(u-\hat{u};\, u-\hat{u})-\frac{t^2}{6}[E_b(W-\hat{W};\, W-\mathring{W})+K(u_2-\hat{u}_2, u_3-\hat{u}_3;\, W-\mathring{W})]$$
$$\leqq E_m(u-\dot{u};\, u-\dot{u})+\frac{t^2}{12}E_b(W-\mathring{W};\, W-\mathring{W})-\frac{t^2}{6}[E_b(W-\mathring{W};\, W-\mathring{W}) + K(u_2-\dot{u}_2,\, u_3-\dot{u}_3;\, W-\mathring{W})], \tag{6.11}$$

where $\{\mathring{W}_{ij}\}$ are the functions defined by the equations (6.2) and $\{\dot{u}_i\}$ and $\{\dot{W}_{ij}\}$ are arbitrary functions, where $\dot{u}_i$ satisfies the boundary conditions required for $\hat{u}_i$. We take the interpolating function of the exact solution as these dotted functions. Now rewrite (6.11) as follows.

$$E_s(u-\hat{u};\, u-\hat{u})\leqq E_m(u-\dot{u};\, u-\dot{u})+\frac{t^2}{12}E_b(W-\mathring{W};\, W-\mathring{W})$$
$$-\frac{t^2}{6}[-E_b(W-\hat{W};\, W-\mathring{W})+E_b(W-\dot{W};\, W-\mathring{W}) + K(\dot{u}_2-\hat{u}_2,\, \dot{u}_3-\hat{u}_3;\, W-\mathring{W})]. \tag{6.12}$$

Estimation of $K(\dot{u}_2-\hat{u}_2, \dot{u}_3-\hat{u}_3; W-\mathring{W})$: This term can be written as follows.

$$=K^{(0)}(\dot{u}_3-\hat{u}_3;W-\dot{W})+\frac{2(1-\nu)}{R}(\dot{u}_{2,1}-\hat{u}_{2,1},W_{12}-\dot{W}_{12}) \tag{6.13}$$
$$+\frac{1}{R}(\dot{u}_{2,2}-\hat{u}_{2,2},\ \nu W_{11}+W_{22}-[\nu\dot{W}_{11}+\dot{W}_{22}]).$$

Since Ω_h is nearly consistent, there is $\Omega'_h(\subset\Omega_h)$ such that

$$K^{(0)}(\dot{u}_3-\hat{u}_3;W-\dot{W})$$
$$\leqq C\sum_{i,j}\{\sum_{p\in\Omega'_h}h^3|\dot{u}_3(p)-\hat{u}_3(p)|+\sum_{p\in\Omega_h-\Omega'_h}h^2|\dot{u}_3(p)-\hat{u}_3(p)|\}$$
$$\leqq C\sqrt{\sum_{p\in\Omega'_h}h^4}\cdot\sqrt{\sum_{p\in\Omega'_h}h^2|\dot{u}_3(p)-\hat{u}_3(p)|^2}$$
$$+C\sqrt{\sum_{p\in\Omega_h-\Omega'_h}h^2}\cdot\sqrt{\sum_{p\in\Omega_h-\Omega'_h}h^2|\dot{u}_3(p)-\hat{u}_3(p)|^2}$$
$$\leqq Ch^{1/2}\|\dot{u}_3-\hat{u}_3\|_0\leqq Ch^{1/2}(h^2+\|u_3-\hat{u}_3\|_0)$$
$$\leqq C[h+h^{1/2}E_b(W-\hat{W};\ W-\hat{W})^{1/2}+\frac{h^{1/2}}{R}E_m(u-\hat{u};\ u-\hat{u})^{1/2}].$$

Since the second and third terms of (6.13) are estimated, with the aid of (6.8), as

$$\leqq C\frac{h^2}{R}[h+\frac{h^{1/2}}{R}+E_m(u-\hat{u};\ u-\hat{u})^{1/2}+\frac{1}{R}E_b(W-\hat{W};\ W-\hat{W})^{1/2}],$$

we have

$$|K(\dot{u}_2-\hat{u}_2,\dot{u}_3-\hat{u}_3;W-\dot{W})|$$
$$\leqq C[h+h^{1/2}E_b(W-\hat{W};W-\hat{W})^{1/2}+\frac{h^{1/2}}{R}E_m(u-\hat{u};u-\hat{u})^{1/2}].$$

The remaining terms of the right side of (6.12) are estimated by (6.6) as

$$\leqq C_mh^2+Ct^2[h+h^2E_b(W-\hat{W};\ W-\hat{W})^{1/2}],$$

where we estimated $E_m(u-\dot{u};\ u-\dot{u})\leqq C_mh^2$. Hence we obtain

$$E_s(u-\hat{u};\ u-\hat{u})$$
$$\leqq C_mh^2+Ct^2h^{1/2}[h^{1/2}+E_b(W-\hat{W};\ W-\hat{W})^{1/2} \tag{6.14}$$
$$+\frac{1}{R}E_m(u-\hat{u};u-\hat{u})^{1/2}],$$

which proves the following theorem.

THEOREM 6.1. *Assume that the triangulation is nearly consistent and R is sufficiently large so that the conditions* (3.3), (5.10) *and* (6.10) *are satisfied. If the exact solution $\{W_{ij},u_i\}$ is sufficiently smooth, then the approximate solution*

$\{\hat{W}_{ij}, \hat{u}_i\}$ *obtained by Algorithm* C_{11} *converges to the exact one in the following sense.*

(6.15)
$$\begin{cases} E_s(u-\hat{u};\, u-\hat{u}) \leqq C'_m h^2 + Ct^2 h, \\ \sum_{i=1}^{2} \|u_i - \hat{u}_i\|_1^2 \leqq C'_m\left(\frac{1}{t^2R^2}+1\right)h^2 + C\left(\frac{1}{R^2}+t^2\right)h, \\ \quad \|u_3 - \hat{u}_3\|_0^2 \leqq C'_m \frac{1}{t^2} h^2 + Ch, \\ E_b(W-\hat{W};\, W-\hat{W}) \leqq C'_m \frac{1}{t^2} h^2 + Ch. \end{cases}$$

According to this result the accuracy of $\hat{W}_{ij}$ and $\hat{u}_i$ appears to be considerably lower than those of $\hat{u}_1$ and $\hat{u}_2$, since in practical applications the ratio h/t is not so small. This unbalance of accuracy between the in-plane and normal approximations will occur indeed unless the rate of in-plane deformation is considerably lower than that of normal deformation, since C'_m and C in (6.15) depend, roughly speaking, upon the higher derivatives of u_1, u_2 and of u_3 respectively.

For Algolithm C_{12} we have

THEOREM 6.2. *Under the same assumptions as in Theorem* 6.1, *the approximate solution obtained by Algorithm* C_{12} *converges to the exact solution in the following sense.*

(6.16)
$$\begin{cases} E_s(u_1-\tilde{u}_1,\, u_2-\tilde{u}_2,\, u_3-\hat{u}_3;\, u_1-\tilde{u}_1,\, u_2-\tilde{u}_2,\, u_3-\hat{u}_3) \\ \qquad\qquad \leqq C''_m h^4 + Ct^2 h, \\ \sum_{i=1}^{2} \|u_i - \hat{u}_i\|_1^2 \leqq C''_m\left(\frac{1}{t^2R^2}+1\right)h^4 + C\left(\frac{1}{R^2}+t^2\right)h, \\ \|u_3 - \hat{u}_3\|_0^2 \leqq C''_m \frac{1}{t^2} h^4 + Ch, \\ E_b(W-\hat{W};\, W-\hat{W}) \leqq C''_m \frac{1}{t^2} h^4 + Ch. \end{cases}$$

Proof is almost the same for the previous theorem and hence we omit this.

These two theorems indicate that one must employ higher order basis first for "*in-plane*" if he wants to increase the accuracy of Algorithm C_{11} and that second order in-plane approximation will give satisfactory results. We are thus very interested in the results of numerical test by Connor and Will [1], in which they report the merits of their Model 2 (a mixed method using quadratic basis for u_1, u_2 and linear basis for u_3 and *moments*) for shallow shell problems.

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