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EVEN DIMENSIONAL K-CONTACT RIEMANNIAN MANIFOLDS ISOMETRICALLY IMMERSED IN A SPACE OF CONSTANT CURVATURE

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1. Introduction.

An even dimensional K-contact Riemannian manifold $(M, \eta^{\#}, g, \phi)$ is an almost Hermitian manifold (M, g, ϕ) admitting a Killing vector field $\eta^{\#}$ satisfying

(1.1) $$d\eta(X, Y)=g(\phi X, Y),$$

where η is the associated 1-form of $\eta^{\#}$. In this case, (M, g, ϕ) is automatically an almost Kählerian manifold, and

(1.2) $$\nabla_X \eta^{\#}=\phi X$$

holds good (Y. Ogawa [1] and T. Takahashi [5]).

In the preceding paper [3], the present auther and S. Tanno studied (odd dimensional) K-contact Riemannian manifolds which are hypersurfaces of a space of constant curvature. In this paper, we study even dimensional K-contact Riemannian manifolds which are hypersurfaces of a space of constant curvature.

2. A lemma.

Let $(M, \eta^{\#}, g, \phi)$ be an even dimensional K-contact Riemannian manifold. Let η be the associated 1-form of $\eta^{\#}$ and let σ be a non-negative function on M defined by

(2.1) $$\sigma=g(\eta^{\#}, \eta^{\#}).$$

Throughout this paper, we assume $\sigma>0$ *on* M. Since $\eta^{\#}$ is a Killing vector field satisfying (1.2), we get

$$R(X, \eta^{\#})Y=(\nabla_X\phi)Y, \tag{2.2}$$

where R is the curvature tensor of (M, g). Since (g, ϕ) is an almost Kählerian structure, we get

$$\sum(\nabla_{e_i}\phi)_{e_i}=0, \tag{2.3}$$

where $\{e_1, e_2, \cdots, e_m\}$ is an orthonormal basis of a tangent space. Hence, we get

$$S(X, \eta^{\#})=0, \tag{2.4}$$

where S is the Ricci tensor of (M, g).

Now, suppose (M, g) is a hypersurface of a space $(\tilde{M}, \tilde{g})$ of constant curvature $\tilde{c}$. The equations of Gauss and Codazzi are

$$R(X, Y)Z=\tilde{c}\{g(Y, Z)X-g(X, Z)Y\}+g(AY, Z)AX-g(AX, Z)AY, \tag{2.5}$$

$$(\nabla_X A)Y=(\nabla_Y A)X, \tag{2.6}$$

where A denotes the operator defined by the second fundamental form with respect to some (local) field of unit normals. By (2.5), we see that the Ricci tensor S is given by

$$S(X, Y)=(m-1)\tilde{c}g(X, Y)+\theta g(AX, Y)-g(AAX, Y), \tag{2.7}$$

where θ=trace A and m=dim M. By (2.4) and (2.7), we get

$$g(AAX, \eta^{\#})-\theta g(AX, \eta^{\#})-(m-1)\tilde{c}g(X, \eta^{\#})=0. \tag{2.8}$$

Suppose, moreover, that a sectional curvature of any 2-dimensional section of (M, g) containing $\eta^{\#}$ is zero:

$$g(R(X, \eta^{\#})\eta^{\#}, X)=0. \tag{2.9}$$

This is equivalent to the following (2.10):

(2.10) $$g(R(X, \eta^{\#})\eta^{\#}, Y)=0,$$

that is,

(2.11) $$R(X, \eta^{\#})\eta^{\#}=0.$$

In (2.5), we put $Y=Z=\eta^{\#}$ and use (2.11), we get

(2.12) $$\bar{c}\{\sigma X-g(X, \eta^{\#})\eta^{\#}\}+g(A\eta^{\#}, \eta^{\#})AX-g(AX, \eta^{\#})A\eta^{\#}=0.$$

LEMMA 2.1. *If $\eta^{\#}$ is not an eigen vector of A at a point p of M, then $\bar{c}=0$ and $\eta^{\#}=ae_1+be_2$ at p, where $Ae_1=\lambda e_1$ $(\lambda\neq 0)$ and $Ae_2=0$.*

PROOF. Let $\{e_1, e_2, \cdots, e_m\}$ be an orthonormal basis of a tangent space at p such that each e_i is an eigen vector of A: $Ae_i=\lambda_i e_i$. Put $\eta^{\#}=\sum\alpha^i e_i$, where α^i are constant. Since $\eta^{\#}$ is not an eigen vector of A by the assumption, at least two of α^i are non-zero. Assume $\alpha^1, \alpha^2, \cdots, \alpha^s$ are non-zero. If we put $X=e_j$ in (2.8), we get

(2.13) $$\lambda_j^2-\theta\lambda_j-(m-1)\bar{c}=0, \qquad j=1, 2, \cdots, s.$$

Hence, $\lambda_1, \lambda_2, \cdots, \lambda_s$ take at most two values. Since $\eta^{\#}$ is not an eigen vector of A, at least two of λ_j $(j=1, 2, \cdots, s)$ are different. Thus there are exactly two values, say λ and μ, in $\lambda_1, \lambda_2, \cdots, \lambda_s$. We may assume $\lambda_1=\lambda_2=\cdots=\lambda_r=\lambda$, $\lambda_{r+1}=\lambda_{r+2}=\cdots=\lambda_s=\mu$, $\lambda\neq\mu$. According to (2.13), λ and μ satisfy

(2.14) $$\lambda\mu=-(m-1)\bar{c}.$$

By a change of eigen vectors, we may assume

(2.15) $$\eta^{\#}=ae_1+be_2, \qquad Ae_1=\lambda e_1,\ Ae_2=\mu e_2,$$

where $a^2+b^2=\sigma$, $a\neq 0$ and $b\neq 0$. In (2.12), we put $X=e_1$ and consider the inner product with e_1. Then we get

(2.16) $$\bar{c}(\sigma-a^2)+b^2\lambda\mu=0.$$

Since $a^2+b^2=\sigma$, (2.14) and (2.16) imply

(2.17) $$(m-2)\bar{c}b^2=0.$$

Hence we get $\tilde{c}=0$ and hence $\lambda\mu=0$.

REMARK 1. Since we have

$$R(X, \eta^{\#})\eta^{\#}=\nabla_X\xi^{\#}+X \tag{2.18}$$

by (2.2), where $\xi^{\#}=\phi\eta^{\#}$, (2.11) is equivalent to

$$\nabla_X\xi^{\#}=-X. \tag{2.19}$$

On the other hand, if the almost Kählerian structure (g, ϕ) in consideration is a Kählerian structure, we get (2.19), and hence (2.11) holds good (cf. T. Takahashi [4]).

REMARK 2. The statement and proof of Lemma 2.1 are analogous to those of Lemma 2.1 in [3].

3. The case $\tilde{c}=0$.

THEOREM 3.1. *Let $(M^m, \eta^{\#}, g, \phi)$ be an even dimensional K-contact Riemannian manifold, $m\leqq 4$. If a sectional curvature of any 2-dimensional section of (M^m, g) containing $\eta^{\#}$ is zero and if (M^m, g) is isometrically immersed in a space $(\tilde{M}^{m+1}, \tilde{g})$ of constant curvature $\tilde{c}=0$, then (M^m, g, ϕ) is a Kählerian manifold.*

PROOF. Let $e_1, e_2, \cdots, e_m$ be orthonormal eigen vectors of A at a point p of M^m such that $Ae_i=\lambda_i e_i$. Suppose $\eta^{\#}$ is an eigen vector of A at p. We may assume $\eta^{\#}=\sqrt{\sigma}\,e_1$. Then (2.12) implies that $\lambda_1\lambda_j=0$ holds for $j=2,3, \cdots, m$. If $\lambda_1\neq 0$, then $\lambda_2=\lambda_3=\cdots=\lambda_m=0$ and hence M^m is flat at p. Thus (2.2) implies that $(\nabla_X\phi)Y=0$ holds at p. If $\lambda_1=0$. Then (2.5) implies $R(X, \eta^{\#})\,Y=0$ and hence (2.2) implies that $(\nabla_X\phi)\,Y=0$ holds at p.

Suppose now, $\eta^{\#}$ is not an eigen vector of A at p. Then, by Lemma 2.1, we can write $\eta^{\#}=ae_1+be_2$, $a^2+b^2=\sigma$, $ab\neq 0$, $\lambda_1\neq 0$ and $\lambda_2=0$, Applying this to (2.12), we get $AX=(1/a)g(AX, \eta^{\#})\,e_1$. Hence the rank of A is 1 and hence (M^m, g) is flat at p. Thus (2.2) implies that $(\nabla_X\phi)Y=0$ holds at p.

Consequently, $(\nabla_X\phi)Y=0$ holds on M^m, and hence (g, ϕ) is a Kählerian structure of M^m.

4. The case $\tilde{c}\neq 0$.

THEOREM 4.1. *Let $(M^m, \eta^\#, g, \phi)$ be an even dimensional K-contact Riemannian manifold, $m\geqq 4$. Suppose a sectional curvature of any 2-dimensional section of (M^m, g) containing $\eta^\#$ is zero and suppose (M^m, g) is a hypersurface of a space of constant curvature $\tilde{c}\neq 0$. Then $\tilde{c}<0$ and (M^m, g, ϕ) is a flat Kählerian manifold.*

PROOF. By Lemma 2.1, $\eta^\#$ is an eigen vector of A. Let $\{e_1, e_2, \cdots, e_m\}$ be an orthonormal basis of a tangent space such that $Ae_i=\lambda_i e_i$ and $\eta^\#=\sqrt{\sigma}\, e_1$. By(2.12), we get $(\tilde{c}+\lambda_1\lambda_i)(\sigma e_i-g(e_i,\eta^\#)\eta^\#)=0$. Hence we get $\lambda_i=-\tilde{c}/\lambda_1$, $i=2,3, \cdots, m$. Thus, if e is any (local) vector field which is orthogonal to $\eta^\#$, then we get

$$Ae=\nu e, \quad \nu=-\tilde{c}/\lambda_1. \tag{4.1}$$

In (2.6), we put $X=e$ and $Y=\eta^\#$. Then, by $A\eta^\#=\lambda_1\eta^\#$ and (1.2), we get

$$(\nabla_e\lambda_1)\eta^\#+\lambda_1\phi e-A\phi e=(\nabla_{\eta^\#}\nu)e+\nu\nabla_{\eta^\#}e-A\nabla_{\eta^\#}e. \tag{4.2}$$

On the other hand, we have $\nabla_{\eta^\#}e=[\eta^\#, e]+\nabla_e\eta^\#=[\eta^\#,e]+\phi e$ and $g([\eta^\#, e],\eta^\#) =g(\nabla_{\eta^\#}e-\nabla_e\eta^\#, \eta^\#)=g(\nabla_{\eta^\#}e, \eta^\#)-g(\nabla_e\eta^\#, \eta^\#)=-g(e, \nabla_{\eta^\#}\eta^\#)-g(\phi e, \eta^\#)=- g(e, \phi\eta^\#)-g(\phi e, \eta^\#)=0$. Thus, (4.2) becomes

$$(\nabla_e\lambda_1)\eta^\#+\lambda_1\phi e=(\nabla_{\eta^\#}\nu)e+\nu\phi e. \tag{4.3}$$

If we take e to be orthogonal to $\eta^\#$ and $\xi^\#=\phi\eta^\#$, then $\eta^\#$, e and ϕe are linearly independent and hence (4.3) implies $\lambda_1=\nu$. Hence, by (4.1), we get $\tilde{c}<0$ and $\lambda_1=\lambda_2=\cdots=\lambda_m=\sqrt{-\tilde{c}}$. Thus (M^m, g) is totally umbilic and flat, and hence (2.2) implies that (g,ϕ) is a Kählerian structure.

5. The case when $\eta^\#$ may vanish at some points.

In the preceding sections, we have assumed that $\eta^\#\neq 0$ holds everywhere. In this section, we treat the general case.

Let $(M^m, \eta^\#, g, \phi)$ be an even dimensional K-contact Riemannian manifold.

LEMMA 5.1.

(5.1) $$d\sigma(X)=-2g(X, \xi^{\#})$$

holds good, where $\xi^{\#}$ *is a vector field defined by* $\xi^{\#}=\phi\eta^{\#}$.

PROOF. Since $\eta^{\#}$ is a Killing vector field, we have (1.2) and hence, by a direct calculation, we get (5.1).

LEMMA 5.2. *If a sectional curvature of any 2-plane containing* $\eta^{\#}$ *is zero, then the set of vanishing points of* $\eta^{\#}$ *is discrete.*

PROOF. Let O be a vanishing point of $\eta^{\#}$. Take a normal coordinate neighborhood U around O. Consider a unit speed geodesic $\gamma(s)$ through the point O such that $\gamma(0)=O$. Put $f(s)=g(\xi^{\#}, \dot{\gamma}(s))$. By Remark 1 of section 2, we have $\nabla_X\xi^{\#}=-X$. Hence we get

$$\frac{df(s)}{ds}=g(\nabla_{\dot{\gamma}}\xi^{\#}, \dot{\gamma})=-1.$$

Thus, since $f(0)=0$, we get $f(s)=-s$. Hence there is no vanishing point of $\xi^{\#}$, and hence of $\eta^{\#}$, in U except O.

THEOREM 5.3. *Theorems* 3.1 *and* 4.1 *hold good without the assumption that* $\eta^{\#}\neq 0$ *holds everywhere.*

PROOF. If we denote the set of the vanishing points by V and put $M'=M^m-V$ and denote the restrictions of the structure tensors of M^m to M' by the same letters, we see, by Theorems 3.1 and 4.1, that (M', g, ϕ) is a Kählerian manifold and hence, since V is discrete, (g, ϕ) is a Kählerian structure of M^m.

According to Y. Ogawa [1], if an even dimensional K-contact *complete* Riemannian manifold is Kählerian, then it is flat. Thus, if the Riemannian manifold in Theorem 5.3 is complete, then it is flat. We state this as follows:

THEOREM 5.4. *Let* $(M^m, \eta^{\#}, g, \phi)$ *be an even dimensional K-contact Riemannian manifold,* $m\geqq 4$. ($\eta^{\#}$ *may vanish at some point.*) *If a sectional curvature of any 2-dimensional section containing* $\eta^{\#}$ *is zero and if* (M^m, g) *is complete and isometrically immersed in a space* $(\tilde{M}^{m+1}, \tilde{g})$ *of constant curvature* $\tilde{c}$, *then* $\tilde{c}\leqq 0$ *and* (M^m, g, ϕ) *is a flat Kählerian manifold.*

REMARK. Lemma 4.1 in [1] seems to be incomplete. The following Lemma 5.6 gives a proof for that Lemma under the additional condition that a sectional curvature of any 2-plane containing $\eta^{\#}$ is zero.

LEMMA 5.5. *If (M^m, g) is complete, then $\eta^{\#}$ has at least one vanishing point.*

PROOF. Suppose $\sigma \neq 0$ holds everywhere. Then we can consider a maximal orbit $c(t), -\infty < t < +\infty$, of unit vector field $(1/\sqrt{\sigma})\xi^{\#}$. Put $\sigma(t) = \sigma(c(t))$. Then, by Lemma 5.2, we get

$$\frac{d\sigma(t)}{dt} = -2g(\dot{c}(t), \xi^{\#}) = -2\sqrt{\sigma}.$$

Hence we get $\sqrt{\sigma(t)} = -t + a$ for some constant a, and hence we get $\sigma(a) = 0$, which is a contradiction.

LEMMA 5.6. *If (M^m, g) is complete and if a sectional curvature of any 2-plane containing $\eta^{\#}$ is zero, then $\eta^{\#}$ has only one vanishing point.*

PROOF. According to Lemma 5.5, there exist at least one vanishing point, say O, of $\eta^{\#}$. Modifying the proof of Lemma 5.2, we see that there is no vanishing point of $\eta^{\#}$ except O.

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