Kumamoto J. Sci. (Math.)
Vol. 11, 1-8 (1974)

SOME REMARKS ON LIE TRIPLE SYSTEMS*

T. S. RAVISANKAR

(Received August 31, 1973)

The notion of Killing form plays a fundamental role in the study of Lie algebras. A similar notion for Lie triple systems was formally introduced by Yamaguti [9] while considering a 'Casimir Operator' associated with representations of certain types of Lie triple systems. However, an independent study of the properties of that Killing form was not undertaken in the paper of Yamaguti. The main object of the present paper is to introduce and study a somewhat similar concept of Killing form for a Lie triple system. We show that this form enjoys some of the nice properties of its classical analogue, thereby extending some of the results on Lie algebras to Lie triple systems. As is usual with the case of a Lie triple system (L. t. s.) the technique mainly consists of passing on to a suitable enveloping Lie algebra; indeed we relate the Killing forms of an L.t.s. and of its associated Lie algebra. This relation enables us to study in detail the structure of an L. t. s. with nondegenerate Killing form. Our final remarks relate to L. t. s.' viewed in the setting of symmetric spaces.

1. A Lie triple system (L. t. s.) T over a field F (assumed to be of characteristic $\neq 2$ throughout this paper) is a (finite dimensional) vector space over F endowed with a trilinear composition $[xyz]$ satisfying (i) $[xxy]=0$ (ii) $[xyz]+[yzx]+[zxy]=0$ and (iii) $[uv[xyz]]=[[uvx]yz]+[x[uvy]z]+[xy[uvz]]$, for x,y,z,u,v in T (see [3], [9]). For subsets A,B,C of T we denote by $[ABC]$ the subspace of T generated by elements of the form $[abc]$ for a,b,c in A,B,C respectively. Then a subspace B of T is called a subsystem (ideal) of T if $[BBB]\subseteq B$ $([BTT]\subseteq B)$; B itself is an L.t.s. relative to [ ] in these cases. For an ideal B of T, define by induction $B^{(1)}=B$, $B^{(k)}=[TB^{(k-1)}B^{(k-1)}]$. Then $B^{(k)}$ are ideals of T and $B\supseteq B^{(2)}\supseteq\ldots\supseteq B^{(k)}\supseteq\ldots$. B is said to be solvable if $B^{(k)}=0$ for some k. T is said to be semisimple if it has no solvable ideals other than 0. T is said to be simple if $[TTT]\neq 0$ and T has no proper ideals. A linear map D of T into itself is a derivation of T if

* An abstract of this paper was presented at a Symposium on Modern Algebra and General Topology, held at the I. I. T., Delhi, during August 1971.

$[xyz]D=[xDyz]+[xyDz]+[xyzD]$ for x,y,z in T. From (iii) above we have that the map $D(x,y)$: $z\to[xyz]$ is a derivation of T. Derivations of T of the form $\sum_{i=1}^{n} D(x_i,y_i)$ are called inner derivations of T. For x,y in T we denote by $R(x,y)$ the linear map: $z\to[zxy]$ of T into itself. Clearly a subspace B of T is an ideal if and only if it is invariant under all the $R(x,y)$'s for x,y in T.

DEFINITION 1.1. *The symmetric bilinear form* α *defined on an L.t.s.* T *by* $\alpha(x,y)=\frac{1}{2}$ trace $(R(x,y)+R(y,x))$ *is called the Killing form of* T.

Following properties of α are clear (see also [9]).
(i) $\alpha([uvx],y)+\alpha(x,[uvy])=0$ or more generally $\alpha(xD,y)+\alpha(x,yD)=0$ for a derivation D of T.
(ii) $\alpha(x\theta,y\theta)=\alpha(x,y)$ for an automorphism θ of T.

REMARK 1. If L be a Lie algebra with multiplication [,] then L becomes an L.t.s. (denoted by T_L) with respect to the composition $[xyz]=[[x,y],z]$. Then the Killing form of L as a Lie algebra and the Killing form of L as an L.t.s. T_L are one and the same.

Now let L be a Lie algebra with an involutory automorphism μ. Let $L=L_-+L_+$ be the decomposition of L into eigenspaces corresponding to the eigenvalues ∓ 1 relative to μ. Then $L_-=T$ is a subsystem of the L.t.s. T_L. Let α,β denote the respective Killing forms of T and L. For x in L_- and y in L_+, $\beta(x,y)=\beta(x\mu,y\mu)=\beta(-x,y)$; $\beta(x,y)=0$. We also have

LEMMA 1.1. *For* x,y *in* $T=L_-$, $\beta(x,y)=2\alpha(x,y)$.

PROOF. Similar to one given by Loos [4] for the case of Malcev algebras.

Given any L.t.s. T, we denote by $\mathscr{D}_0(T)$ the Lie algebra of all inner derivations of T. Let $L^*=T\oplus\mathscr{D}_0(T)$ (vector space direct sum). L^* is a Lie algebra with respect to the multiplication defined by $[x+D_1,\ y+D_2]=-xD_2+yD_1+D(x,y)+[D_1,D_2]$ (x,y in T and D_1,D_2 in $\mathscr{D}_0(T)$). The linear map μ: $L^*\to L^*$ defined by $(x+D_1)\mu=-x+D_1$ is an involutory automorphism of L^* and L^*_- is precisely T. Clearly $[T,T]=\mathscr{D}_0(T)$ and with this identification $L^*=T\oplus[T,T]$. L^* is said to be the Lie algebra obtained from T by standard imbedding (see [4]). This construction can be used to prove the following

LEMMA 1.2. *If* α *be the Killing form of an L.t.s.* T, *then* $\alpha(xR(z,w),y)=\alpha(x,yR(w,z))$ *for* x,y,w,z *in* T.

PPOOF. Let [,] denote the multiplication in L^* and β be the Killing form of L^*. Then we have: $\alpha(xR(z,w),y)=\alpha([xzw],y)=\frac{1}{2}\beta([xzw],y)$ (by Lemma 1.1) $=\frac{1}{2}\beta([[x,z],w],y)=\frac{1}{2}\beta([x,z],[w,y])$ (by the associativity of β [1, p. 71]) $=\frac{1}{2}\beta(x,[z,[w,y]])=\frac{1}{2}\beta(x,[[y,w],z])=\frac{1}{2}\beta(x,[ywz])=\alpha(x,yR(w,z))$ (again by Lemma 1.1).

An immediate consequence of Lemma 1.2 is the fact that the α-orthogonal complement $B^{\perp}=\{x\in T\mid\alpha(x,y)=0$ for all y in $B\}$ of an ideal B of T is again an ideal of T. One can, in fact, define a symmetric bilinear form $\lambda(x,y)$ on an L.t.s. T to be an invariant or associative form if $\lambda(xR(z,w),y)=\lambda(x,yR(w,z))$ and note that an associative form on a Lie algebra L is also associative as a form on the L.t.s. T_L. The λ-orthocomplement of an ideal of T will again be an ideal of T. In this setting we can easily prove the following analogue of the classical Dieudonne's Lemma [1, p. 71], along the same lines as the cited result.

LEMMA 1.3. *Let* T *be a finite dimensional L.t.s. over a field* F *such that (i)* T *has a nondegenerate associative form* λ *and (ii)* T *has no nonzero ideals* B *with* $[TBB]=0$. *Then* T *is a direct sum of ideals* T_i *which are simple as L.t.s.'.*

Now let B be an ideal of an L.t.s. T such that $[TBB]=0$. Then, clearly $[BTB]=0$ and hence also $[BBT]=0$. For x in B and y in T, $(R(x,y)+R(y,x))^2=0$ on T so that trace $(R(x,y)+R(y,x))=0$ or $\alpha(x,y)=0$. In particular, if the Killing form α of T is nondegenerate, such a nonzero ideal B can't exist. We have in fact proved (in view of Lemmas 1.2 and 1.3)

THEOREM 1.4. *If the Killing form of an L.t.s.* T *over a field* F *of characteristic* $\neq 2$ *is nondegenerate then* T *is a direct sum of simple ideals* T_i.

REMARK. It is clear that the ideals T_i of Theorem 1.4 would themselves have nondegenerate Killing forms.

2. The structure of a Lie triple system with nondegenerate Killing form is completely determined by that of the simple ones of the same type (by Theorem 1.4). The study of the latter class can be related to that of their associated Lie algebras (standard imbedding). An important step in this direction is the

THEOREM 2.1. *Let T be an L.t.s. and $L^*=T\oplus\mathscr{D}_0(T)=T\oplus[T,T]$ be the Lie algebra obtained from T by standard imbedding. Then the Killing form α of T is nondegenerate if and only if the Killing form β of L^* is nondegenerate.*

PROOF. (A proof similar to that of [6, Theorem 3.1] can be given here also, we give the details again for completeness.)

Let β be nondegenerate and let $\alpha(x,z)=0$ for some x and all z in T. By the remark just preceding Lemma 1.1, we have $\beta(x,[T,T])=0$; by Lemma 1.1 itself one has $\beta(x,z)=0$ for all z in T. Thus $\beta(x,T+[T,T])=0$; $\beta(x,L^*)=0$. Nondegeneracy of β implies that $x=0$ or that α is nondegenerate.

Conversely, let α be nondegenerate. Let $\beta(z+D,L^*)=0$, for an $z+D$ in L^* with z in T and D in $[T,T]$. In particular $\beta(z+D,T)=0$, $\beta(z,T)=0$; by Lemma 1.1 $\alpha(z,T)=0$ and hence $z=0$. Thus $\beta(D,L^*)=0$; $\beta(D,[x,y])=0$ for all x,y in T. By the associativity of $\beta,\beta([D,x],y)=0$; by the multiplication rule in $L^*,\beta(xD,y)=0$. xD belongs to T and so by Lemma 1.1 again $\alpha(xD,y)=0$ for all y in T. $xD=0$ by the nondegeneracy of α. x being arbitrary in $T,D=0$. Q.E.D.

REMARK. For the first part of the above proof, it suffices that $T=L_-$ for a Lie algebra L with an involutory automorphism μ and β, the Killing form of L. For the second part of the proof it suffices that L_+ acts faithfully on L_- by the adjoint action. However this latter condition essentially amounts to our own hypothesis; indeed L_+ will be isomorphic to $[T,T]=\mathscr{D}_0(T)$, in this case: when L_+ acts faithfully on $L_-=T$, elements of L_+ can be identified with ad $t|_{L_-}$ (ad denoting the adjoint in the Lie algebra L); ad $t|_{L_-}$ can be considered as a derivation D_t of the L.t.s. $T=L_-$; thus L_+ can be identified with a subalgebra of the Lie algebra $\mathscr{D}(T)$ of all derivations of T (through the isomorphism $t\to D_t,t$ in L_+). For x,y in $T,[x,y]$ belongs to L_+ and the associated $D_{[x,y]}$ is nothing but the inner derivation $D(x,y)$ of T. Thus L_+ is a subalgebra of $\mathscr{D}(T)$ containing $\mathscr{D}_0(T)$. It is also clear that the multiplication in L after this identification is the same as the one defined for L^*, the standard Lie algebra of T. When the Killing form of $T(=L_-)$ is nondegenerate, every derivation of T is inner (as we shall see presently); L_+ indeed becomes $\mathscr{D}_0(T)$, i.e. L is effectively the same as L^* associated with $T=L_-$.

REMARK 2. The above situation essentially corresponds to the situation obtaining in the case of involutive Lie algebras (considered by Koh [2], in connection with his study of affine symmetric spaces).

COROLLARY 2.2. *The Killing form of a semisimple L.t.s. T over a field of characteristic zero is nondegenerate.*

PROOF. Let T be semisimple. Then L^* is known to be semisimple [3, Theorem 2.7]. Consequently the Killing form of L^* is nondegenerate. The corollary now follows from Theorem 2.1.

From Corollary 2.2 and Theorem 1.4 one immediately has

COROLLARY 2.3. ([3, Theorem 2.9]). *A semisimple L.t.s. over a field of characteristic zero is a direct sum of simple ideals.*

REMARK 3. If α be the Killing form of an L.t.s. T over a field of characteristic zero, then the radical of T (maximal solvable ideal of T)$=\{x\in T \mid \alpha(x,[TTT])=0\}$ (see [6, Section 3, Remark 3]). Also T will be solvable if and only if $\alpha(x,x)=0$ for all x in $[TTT]$ (cf. [1, p.69]). An immediate consequence of the first assertion is that the radical of an L.t.s. over a field of characteristic zero is a characteristic ideal (see [4, Lemma 5]).

The following result extends to Lie triple systems a well-known result of Zassenhaus for Lie algebras ([1, p.74]).

THEOREM 2.4. *If the Killing form of an L.t.s. T over a field of characteristic$\neq 2$ is nondegenerate, then every derivation of T is inner.*

PROOF. The Killing form of L^* is nondegenerate (by Theorem 2.1). Then every derivation of L^* is inner, by the cited theorem of Zassenhaus. The result now follows in the same way as in the proof of [6, Theorem 4.1].

REMARK 4. Let L be a Lie algebra with a nondegenerate Killing form. Then the associated L.t.s. T_L also has the same property (by Remark 1 of

Section 1). If D be a derivation of L then D is also a derivation of T_L so that $D=\Sigma D(x_i,y_i)$ for some x_i,y_i in L, by the above Theorem. By the definition of Lie triple composition in T_L, we have $D=\Sigma\ \mathrm{ad}[x_i,y_i]=\mathrm{ad}(\Sigma[x_i,y_i])$, $[\ ,\ ]$ being the bracket operation in L. D is thus an inner derivation of L. In other words, we have qualified our statement that Theorem 2.4 is an extension of the corresponding result for Lie algebras.

REMARK 5. The above remark could be construed as an instance of terming the results for L.t.s. as extensions of the results for Lie algebras or more generally those of Malcev algebras. Here we recall that a Malcev algebra (of characteristic $\neq 2,3$) is an algebra satisfying (i) $xy=-yx$ (ii) $(xy)(xz)=((xy)z)x+((yz)x)x+((zx)x)y$ and that A forms an L.t.s. T_A relative to the composition $[xyz]=2(xy)z-(yz)x-(zx)y$. If R_x denotes the right multiplication in A, the bilinear form $\theta(x,y)=\mathrm{trace}\ R_xR_y$ is called the Killing form of A and if α denotes the Killing form of T_A then $3\theta(x,y)=\alpha(x,y)$. This fact along with the fact that R_x is a derivation of T_A implies, in view of the observations following Definition 1.1, that θ is an associative form on A. Many results on the structure of Malcev algebras can be obtained from the results on L.t.s.. We skip the details and just refer to [4] and [6].

REMARK 6. In view of Theorem 1.4 and the remark following it, the structure of L.t.s.' with nondegenerate Killing forms is completely determined by that of the simple ones of the same type. If T is a simple L.t.s. over a field of characteristic zero, nondegeneracy is a superfluous assumption (by Corollary 2.2) and the structure of such an L.t.s. is known [3, Section IV]. If T is simple over a field of characteristic $\neq 2$ with nondegenerate Killing form, the associated Lie algebra L^* has nondegenerate Killing form. One then notes that arguments similar to those of Lister [3] can be given in this situation also so that T will be one of the following two types: (i) the L.t.s. T_L associated with a simple Lie algebra L with a nondegenerate Killing form (ii) the simple L.t.s. of skew elements relative to an involutory automorphism of a simple Lie algebra with a nondegenerate Killing form. The simple Lie algebras with nondegenerate Killing forms over an algebraically closed field of characteristic $\neq 2,3,5$ have already been classified (see [7, Chapter II, p.47 especially]); they are analogues of the classical simple Lie algebras over the complex field (with a few excepted). The automorphisms of these algebras have been completely determined by Seligman [8] and they are againfound to be the exact analogues of the characteristic zero case (see [1, Chapter IX, Section 5]). As such an analysis similar to one carried out by Lister leads to a similar characterization of the simple L.t.s.' with nondegenerate Killing forms.

REMARK 7. Let L be an involutive Lie algebra, i.e. L is a Lie algebra with an involutive automorphism σ such that L_+ acts faithfully on $L_-=T$. Then T is an L.t.s. associated with the iLa L (see [2]). Conversely, if T be any L.t.s. and L' be an iLa whose associated L.t.s. is T, then L'_+ can be identified with a subalgebra of the derivation algebra $\mathscr{D}(T)$ containing the inner derivation algebra $\mathscr{D}_0(T)$ of T (in view of Remark 1). Thus if L' is an iLa with T as the associated L.t.s., $\mathscr{D}_0(T)\subseteq L'_+\subseteq\mathscr{D}(T)$, after suitable identification. In particular, the Lie algebra L^* (standard imbedding) associated with an L.t.s. T is determined uniquely (up to isomorphism) as *the* iLa L with $L_-=T$ and $[L_-,L_-]=L_+$. Also if the Killing form of T is nondegenerate, T is the associated L.t.s. of an unique iLa, namely L^*.

REMARK 8. The study of iLa's is closely related to that of the affine symmetric spaces (see [2]): Let G be a connected Lie group with an involutive automorphism σ. The totality of fixed points H_σ of σ is a closed subgroup of G. If H be a closed subgroup of G lying between H_σ and its identity component, then the homogeneous space G/H is called a symmetric space ([5]). Let also G act effectively on G/H as a transformation group. We denote again by σ the automorphism induced by σ on the associated Lie algebra $\mathfrak{g}$ of G. Let $\mathfrak{m}=\mathfrak{g}_-$ and $\mathfrak{h}$ be the subalgebra of $\mathfrak{g}$ corresponding to the identity component H_0 of H. Then $\mathfrak{g}=\mathfrak{m}+\mathfrak{h};\mathfrak{m}=\mathfrak{g}_-$ and $\mathfrak{h}=\mathfrak{g}_+$. The effective action of G on G/H just means that $\mathfrak{g}$ is an iLa relative to σ. G/H is said to be irreducible if $\mathrm{ad}(\mathfrak{h})$ is irreducible on $\mathfrak{m}$. In this set up, if $\mathfrak{g}$ is semisimple, then by the preceding Remark $[\mathfrak{m},\mathfrak{m}]=\mathfrak{h}$ (see (16.2) of [5]). If G/H is irreducible then the radical of the L.t.s. $\mathfrak{m}$ being $\mathrm{ad}(\mathfrak{h})$-invariant (by Remark 3) it is either 0 or the whole of $\mathfrak{m}$. In the former case $\mathfrak{m}$ is semisimple; hence $\mathfrak{g}$ is semisimple (by the above Remark and Theorem 2.1); in fact $\mathfrak{m}$ has to be simple and $\mathfrak{g}$ will be itself simple or a direct sum of two simple ideals (see [3, Theorem 2.13]). In the latter case $\mathfrak{m}$ is a solvable L.t.s.; $[\mathfrak{m}\,\mathfrak{m}\,\mathfrak{m}]$ being again an $\mathrm{ad}(\mathfrak{h})$-invariant subspace of $\mathfrak{m}$ properly contained in $\mathfrak{m}$, $[\mathfrak{m}\,\mathfrak{m}\,\mathfrak{m}]=0$; faithful action of $\mathfrak{h}$ on $\mathfrak{m}$ implies that $[\mathfrak{m}\,\mathfrak{m}]=0$ and this is precisely the assertion of [5, (16.2)]. In other words, the algebraic study of L.t.s.

may be useful in the geometric study of symmetric spaces.

ADDED 29TH MAY 1974: Professor Yamaguti has kindly pointed out to the author that there is a certain overlap of material between the present paper and a paper of Professor J. A. Wolf entitled "On the geometry and classification of absolute parallellisms I, II", appearing in the Journal of differential geometry (Vol.6: 317–334; Vol.7: 19–44). The author himself since finds that similar ideas are also developed partly in the lectures on algebras and triple systems delivered by K. Meyberg at the University of Virginia in 1972.

The author takes this opportunity to record an acknowledgement. Theorem 2.4 of this paper was proved earlier by the author (see [6, Theorem 4.1]) for Malcev algebras with the assumption of characteristic of the base field F being$\neq 2, 3$. Dr. Renate Carlsson of Hamburg kindly points out to me that this result is valid even when the characteristic of F is $\neq 2$, and that this follows easily from a result of Meyberg. This is also recorded as Satz 24 in the Habilitationschrift of Dr. Renate Carlsson submitted to the Hamburg University in 1966.

References

[1] N. JACOBSON, Lie Algebras, Wiley Interscience, New York, 1962.
[2] S.S. KOH, On affine symmetric spaces, Trans. Amer. Math. Soc., 119(1965) 291-309.
[3] W. G. LISTER, A structure theory of Lie triple systems, Trans. Amer. Math. Soc., 72(1952) 217-242.
[4] O. LOOS, Über eine Beziehung zwischen Malcev-Algebren und Lie-Tripel Systemen, Pacific J. Math., 18(1966) 553-562.
[5] K. NOMIZU, Invariant affine connections on homogeneous spaces, Amer. J. Math., 76 (1954) 33-65.
[6] T. S. RAVISANKAR, On Malcev Algebras, Pacific J. Math., 42(1972) 227-234.
[7] G. B. SELIGMAN, Modular Lie algebras, Springer, Berlin, 1967.
[8] ——————, On automorphisms of Lie algebras of classical type, II, III, Trans. Amer. Math. Soc., 94(1960) 452-482; 97(1960) 286-316.
[9] K. YAMAGUTI, On the cohomology space of Lie triple system, Kumamoto J. Sci. Ser. A, 5(1960) 44-52.

Department of Mathematics
Birla Institute of Technology and Science
Pilani, INDIA.