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A NOTE ON THE ORTHOGONALITY RELATIONS FOR GROUP CHARACTERS

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Let $\mathfrak{G}$ be a group of finite order g and p be a fixed prime number. Let $\mathfrak{K}_1, \mathfrak{K}_2, \cdots, \mathfrak{K}_k$ be the classes of conjugate elements in $\mathfrak{G}$ and $G_1, G_2, \cdots, G_k$ be a complete system of representatives for the classes. $\mathfrak{G}$ has k absolutely irreducible ordinary characters $\chi_1, \chi_2, \cdots, \chi_k$, which are partitioned into p-blocks $B_1, B_2, \cdots, B_t$. The following orthogonality relations are fundamental: If two elements G and H are not conjugate in $\mathfrak{G}$, then

$$\sum_{i=1}^{k} \chi_i(G)\chi_i(H^{-1})=0$$

holds. If two characters χ_i and χ_j are not equal, then

$$\sum_{G\in\mathfrak{G}} \chi_i(G)\chi_j(G^{-1})=0$$

holds. To the former, R. Brauer added an important result: If two elements G and H belong to different p-sections, *i.e.*, the p-part of G and that of H are not conjugate in $\mathfrak{G}$, then

$$\sum_{\chi_i\in B_\tau} \chi_i(G)\chi_i(H^{-1})=0$$

holds for each p-block B_τ ([1]). To the latter, R. Brauer and M. Osima added independently the following: If χ_i and χ_j belong to different p-blocks, then

$$\sum_{G\in\mathfrak{S}} \chi_i(G)\chi_i(G^{-1})=0$$

holds for each p-section $\mathfrak{S}$ ([8]). Moreover, in [7], M. Osima gave the following theorem: If B is a set of characters χ_i such that for each pair of p-regular element R and p-singular element Q,

$$\sum_{\chi_i\in B} \chi_i(R)\chi_i(Q^{-1})=0$$

holds, then B is a union of p-blocks B_τ.

But p-sections are not best refined for p-blocks in the sense of the above theorem. The symmetric group S_4 and the Mathieu groups M_{22}, M_{23}, M_{24} give such examples for $p=2$. The purpose of this note is to give a refinement of p-sections where the orthogonality relations hold for p-blocks (§1). On the problem, confer with [4].

Let K be the field of g-th roots of unity, $\mathfrak{p}$ be a prime ideal divisor of p in K, $\mathfrak{o}$ be the ring of $\mathfrak{p}$-integers in K and $\mathfrak{o}^*=\mathfrak{o}/\mathfrak{p}$ be the residue class field of $\mathfrak{o}$ by $\mathfrak{p}$. Let Z, Z_0, Z^* be the centers of the group algebras $K\mathfrak{G}$, $\mathfrak{o}\mathfrak{G}$, $\mathfrak{o}^*\mathfrak{G}$ of $\mathfrak{G}$ over K, $\mathfrak{o}$, $\mathfrak{o}^*$, respectively. Let $K_1, K_2, \cdots, K_k$ be the class sums in $\mathfrak{o}\mathfrak{G}$ corresponding to $\mathfrak{K}_1, \mathfrak{K}_2, \cdots, \mathfrak{K}_k$ and $\eta_1, \eta_2, \cdots, \eta_t$ be the block idempotents of Z_0 corresponding to $B_1, B_2, \cdots, B_t$:

$$\eta_\tau = \frac{1}{g}\sum_\nu \sum_{\chi_i\in B_\tau} \chi_i(1)\chi_i(G_\nu^{-1})K_\nu, \tag{1}$$

$$K_\mu\eta_\tau = \frac{|\mathfrak{K}_\mu|}{g}\sum_\nu \sum_{\chi_i\in B_\tau} \chi_i(G_\mu)\chi_i(G_\nu^{-1})K_\nu.$$

On account of (1), it seems to us that the study of the block idempotents η_τ is successful for our problem.

In §1 and §2, some results on η_τ will be given. In §3, for a certain normal subgroup $\mathfrak{H}$ of $\mathfrak{G}$, we shall relate Proposition [1.2] to the blocks with regard to $\mathfrak{H}$, referred as "$\mathfrak{H}$-blocks" (Cf. [6], [2]).

§ 1

Let $\mathfrak{D}$ be the maximal normal p-subgroup of $\mathfrak{G}$ and $C(\mathfrak{D})$ be the centralizer of $\mathfrak{D}$ in $\mathfrak{G}$.

[1.1] *We set* $\eta_\tau=\sum_\nu a_\nu^\tau K_\nu$. *If* a_ν^τ *does not vanish, then* $\mathfrak{K}_\nu$ *is contained in* $C(\mathfrak{D})$.

PROOF. Let A be the subalgebra of Z_0 spanned by those K_α with $\mathfrak{K}_\alpha\subset C(\mathfrak{D})$ and B be the linear subspace spanned by the remaining class sums K_β. If we set

$$A^*=\{a+\mathfrak{p}Z_0\in Z^* \mid a\in A\},\ B^*=\{b+\mathfrak{p}Z_0\in Z^* \mid b\in B\},$$

we have $Z^*=A^*\oplus B^*$. Since B^* is a nilpotent ideal in Z^*, the block idempotents of Z^* are all in A^*. As is well known, there is a one to one correspondence between the block idempotents of Z_0 and those of Z^*. The same holds for A and A^*. Hence, the block idempotents of Z_0 coincide with those of A.

If $\mathfrak{K}_\alpha \subset C(\mathfrak{D})$ then, by [1.1], we have

$$K_\alpha \eta_\tau = \frac{|\mathfrak{K}_\alpha|}{g} \sum_{\mathfrak{K}_\gamma \subset C(\mathfrak{D})} \sum_{\chi_i \in B_\tau} \chi_i(G_\alpha)\chi_i(G_\gamma^{-1})K_\gamma .$$

Hence, if further $\mathfrak{K}_\beta \not\subset C(\mathfrak{D})$, then

$$\sum_{\chi_i \in B_\tau} \chi_i(G_\alpha)\chi_i(G_\beta^{-1}) = 0$$

holds. Therefore, we obtain the following proposition:

[1.2] *Let $\mathfrak{N}$ be a normal subgroup of $\mathfrak{G}$ containing $C(\mathfrak{D})$. If $\mathfrak{K}_\alpha \subset \mathfrak{N}$* and *$\mathfrak{K}_\beta \not\subset \mathfrak{N}$, then we have*

$$\sum_{\chi_i \in B_\tau} \chi_i(G_\alpha)\chi_i(G_\beta^{-1}) = 0$$

for each p-block B_τ.

In the symmetric group S_4, [1.2] gives the best refinement of 2-sections for 2-blocks. But in the Mathieu groups M_{22}, M_{23}, M_{24}, the same does not remain valid. Applying Proposition [1.2] to the centralizer $C(P)$ of a p-element P in $\mathfrak{G}$, by Theorem (6A) in [1] or Theorem 2 in [3], we may refine the p-section of P in $\mathfrak{G}$ again.

§ 2

For an element x of Z_0, we denote by x^* the element $x + \mathfrak{p}Z_0$ of Z^*.

[2.1] *Let $\mathfrak{K}_\nu$ be a p-regular class such that K_ν^* is not nilpotent, then there exists at least one p-block B_τ such that $a_\nu^\tau \neq 0$, a_ν^τ being in* [1.1].

PROOF. We may set $K_\nu = \sum_\rho b_{\nu\rho}\eta_\rho + r$, where r^* is an element of the radical of Z^*. Let s be the sum of all p-elements of $\mathfrak{G}$ in $\mathfrak{o}\mathfrak{G}$. Since, by Lemma 3 in [5], the radical of Z^* is the annihilator ideal of s^* in Z^*, we have

$$K_\nu s \equiv \sum_\rho b_{\nu\rho}\eta_\rho s \not\equiv 0 \pmod{\mathfrak{p}Z_0}.$$

Hence, there is a p-block B_τ such that $b_{\nu\tau}\not\equiv 0 \pmod{\mathfrak{p}}$. Let $\mathfrak{K}_\mu$ be a class such that $a^\tau_\mu \not\equiv 0 \pmod{\mathfrak{p}}$ and that the defect of $\mathfrak{K}_\mu$ is equal to the defect of B_τ. We compare the coefficients of K_μ in both sides of the congruence

$$K_\nu s\eta_\tau \equiv b_{\nu\tau}\eta_\tau s \pmod{\mathfrak{p}Z_0}. \tag{2}$$

The coefficient of K_μ in the left side of (2) is

$$\sum_{\chi_i\in B_\tau} \frac{\chi_i(s)|\mathfrak{K}_\nu|\chi_i(G_\nu)\chi_i(G_\mu^{-1})}{g\chi_i(1)} = \sum_{\chi_i\in B_\tau} \frac{\chi_i(s)|\mathfrak{K}_\nu|\chi_i(G_\nu)|\mathfrak{K}_\mu|\chi_i(G_\mu^{-1})}{g|\mathfrak{K}_\mu|\chi_i(1)}$$
$$\equiv \omega_{i_0}(K_{\mu'}) \sum_{\chi_i\in B_\tau} \frac{\chi_i(s)\chi_i(G_\nu)|\mathfrak{K}_\nu|}{g|\mathfrak{K}_\mu|} \pmod{\mathfrak{p}},$$

where $\mathfrak{K}_{\mu'}$ is the class containing G_μ^{-1} and ω_{i_0} is a linear character of Z which corresponds to a character χ_{i_0} belonging to B_τ. Since $\eta_\tau s$ is a sum of η_τ and a linear combination of those K_λ for which $\mathfrak{K}_\lambda$ are p-singular classes, the coefficient of K_μ in the right side of (2) is $b_{\nu\tau}a^\tau_\mu$, which is not congruent to zero modulo $\mathfrak{p}$. Hence, we get

$$\sum_{\chi_i\in B_\tau} \chi_i(s)\chi_i(G_\nu) \neq 0.$$

Since G_ν is a p-regular element and $s-1$ is a sum of p-singular elements, we have

$$\sum_{\chi_i\in B_\tau} \chi_i(s)\chi_i(G_\nu) = \sum_{\chi_i\in B_\tau} \chi_i(1)\chi_i(G_\nu) = ga^\tau_\nu,$$

hence $a^\tau_\nu \neq 0$.

The following fact is seen from [2.1] also.

[2.2] *If $\mathfrak{G}$ has a normal Sylow p-subgroup or has a normal p-complement, then the union of all p-regular classes $\mathfrak{K}_\nu$ with $\mathfrak{K}_\nu \subset C(\mathfrak{D})$ is not refined for p-blocks.*

§ 3

Let $\mathfrak{H}$ be a normal subgroup of $\mathfrak{G}$ and $\chi_i|\mathfrak{H}$ be the restriction of χ_i to $\mathfrak{H}$. By Clifford's theorem, we may set

$$\chi_i \,|\, \mathfrak{H} = e_i \Psi_\sigma,$$

where Ψ_σ is the sum of all irreducible characters $\psi_{\sigma 1}, \psi_{\sigma 2}, \cdots$ of $\mathfrak{H}$ belonging to a class of associated irreducible characters of $\mathfrak{H}$ in $\mathfrak{G}$. After Osima [8], we say that two characters χ_i and χ_j belong to the same $\mathfrak{H}$-block if and only if they correspond to the same Ψ_σ. χ_i and χ_j belong to the same $\mathfrak{H}$-block if and only if

$$\chi_i(H)/\chi_i(1)=\chi_j(H)/\chi_j(1)$$

holds for every $H \in \mathfrak{H}$ ([6], Theorem 7).

For each $\mathfrak{H}$-block $\mathfrak{B}_\sigma$, denote by E_σ the sum of all primitive idempotents of Z corresponding to those χ_i which belong to $\mathfrak{B}_\sigma$.

[3.1] *We set* $E_\sigma=\sum_\nu \alpha_\nu^\sigma K_\nu$. *If* $\alpha_\nu^\sigma \neq 0$, *then* $\mathfrak{K}_\nu$ *is contained in* $\mathfrak{H}$. *Hence, if* $\mathfrak{K}_\alpha$ *is contained in* $\mathfrak{H}$ *and* $\mathfrak{K}_\beta$ *is not contained in* $\mathfrak{H}$, *then*

$$\sum_{\chi_i \in \mathfrak{B}_\sigma} \chi_i(G_\alpha)\chi_i(G_\beta^{-1})=0$$

holds for each $\mathfrak{H}$-block $\mathfrak{B}_\sigma$. ([2])

If $\mathfrak{K}_\nu$ is a class contained in $\mathfrak{H}$, then α_ν^σ does not vanish for some $\mathfrak{B}_\sigma$, in fact, for the $\mathfrak{H}$-block $\mathfrak{B}_\sigma$ which contains the principal character of $\mathfrak{G}$. Hence, we see that $\mathfrak{H}$ is not refined for $\mathfrak{H}$-blocks. In the analogous way as in the proof of Theorem 3 in [7], we obtain

[3.2] *If* $\mathfrak{B}$ *is a set of characters* χ_i *such that for each element* G *outside of a normal subgroup* $\mathfrak{H}$ *of* $\mathfrak{G}$

$$\sum_{\chi_i \in \mathfrak{B}} \chi_i(1)\chi_i(G^{-1})=0$$

holds, then $\mathfrak{B}$ *is a union of* $\mathfrak{H}$*-blocks* $\mathfrak{B}_\sigma$.

On account of [1.1] and [3.2], we have

[3.3] *If* $\mathfrak{H}$ *is a normal subgroup of* $\mathfrak{G}$ *which contains the centralizer of the maximal normal* p*-subgroup of* $\mathfrak{G}$, *then every* p*-block* B_τ *is a union of* $\mathfrak{H}$*-blocks* $\mathfrak{B}_\sigma$.

Proposition [1.2] follows from [3.3] and [3.1] also.

REMARK. Proposition [1.2] and [3.1] correspond to Brauer's orthogonality relations. We have, of course, propositions corresponding to Brauer-Osima's orthogonality relations.

References

[1] R. BRAUER, Zur Darstellungstheorie der Gruppen endlicher Ordnung II, Math. Zeitschr. **72** (1959), 25-46.

[2] K. IIZUKA, Some studies on the orthogonality relations for group characters, Kumamoto J. Sci. Ser. A **5** (1961), 111-118.

[3] ———, On Brauer's theorem on sections in the theory of blocks of group characters, Math, Zeitschr. **75** (1961), 299-304.

[4] K. IIZUKA and T. NAKAYAMA, A remark on orthogonality relations in finite groups, Nagoya Math. J. **20** (1962), 185-194.

[5] K. IIZUKA and A. WATANABE, On the number of blocks of irreducible characters of a finite group with a given defect group, Kumamoto J. Sci. (Math.) **9** (1973), 55-61.

[6] M. OSIMA, On the representations of groups of finite order, Math. J. Okayama Univ. **1** (1952), 33-61.

[7] ———, Notes on blocks of group characters, Math. J. Okayama Univ. **4** (1955), 175-188.

[8] ———, On some properties of group characters, Proc. Japan Acad. **36** (1960), 18-21.

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