Kumamoto J. Sci. (Math.)
Vol. 11, 15-20 (1974)

ON MANIFOLDS WITH MANY GEODESIC LOOPS

Masanori KÔZAKI

(Received March 14, 1974)

1. Introduction.

Let M be a complete Riemannian manifold satisfying the following condition:

(*) There exists a point p in M such that all geodesics starting from p are simple geodesic loops and of same length $2l$.

Bott [3] and Nakagawa [8] determined the cohomology structure of M. Is the assumption of same length in (*) superfluous ? (Berger [2]) In § 2, we shall show that this assumption is not necessarily required (cf. [7]).

On the other hand, Nakagawa [9] investigated the structure of M satisfying (*), $l \leqq \pi$ and with sectional curvature $K \leqq 1$. In § 3, we shall consider the structure of M satisfying (*), $\pi \leqq l < 3\pi/2$ and with $K \leqq 1$.

The author wishes to express his hearty thanks Prof. H. Nakagawa for his valuable advices.

Notations

K : the sectional curvature of M.
$L(c)$: the length of geodesic c.
Ind c : the index of geodesic c.
d : the distance function of M.
$C(p)$: the cut locus of a point p.
$Q(p)$: the first conjugate locus of p.
$\dot{c}(t)$: the tangent vector of geodesic c at $c(t)$.

Let us suppose that all geodesics are parameterized by arc length.

2. A generalization of Bott and Nakagawa's theorem.

In this section, we shall give a slight generalization of Bott and Nakagawa's theorem. The proof is the analogous argument as in Nakagawa [8].

Let M be an $n(\geqq 2)$-dimensional complete Riemannian manifold satisfying the following condition: There exists a point p in M such that all geodesics starting from p are non-trivial simple geodesic loops and the lengths of these loops depend differentiably on their initial tangent vectors at p. Then we have the following

THEOREM A. *If M is simply connected, then M has the same integral cohomology ring as a symmetric space of compact type of rank one. If M is not simply connected, then the universal covering manifold of M is homeomorphic to a sphere.*

In order to prove Theorem A, we state some lemmas.

LEMMA 2.1. *M is simply connected or a fundamental group of M is of order 2.*

LEMMA 2.2. *For any geodesic loop $c(t)$ at p, $0\leqq t\leqq L(c)$, $p=c(0)=c(L(c))$ is a conjugate point of p along c and with multiplicity $n-1$.*

A geodesic segment c in the loop space $\Omega(p,q)$ is said to be of order k, if there exist k real numbers $s_1, \cdots\cdots, s_k$ satisfying $0<s_1<\cdots\cdots<s_k<L(c)$ and $c(s_i)=p$ for $i=1, \cdots\cdots, k$.

LEMMA 2.3. *For any geodesic loop σ at p of index μ, there exists a non-degenerate point q in σ different from p such that, for any integer $k(\geqq 0)$, there exist in $\Omega(p,q)$ two and only two geodesic segments σ_1 and σ_2 of order k, whose indices satisfy*

$$\operatorname{Ind}\sigma_1\geqq k(n-1), \qquad \operatorname{Ind}\sigma_2\geqq k(n-1)+\mu.$$

PROOF. It suffices to show the following: For any point q' sufficiently close to p, there exists no geodesic loop at p passing through q', whose subarcs joining p to q' meet $C(p)$. Suppose that such geodesic loops at p exist. Set $\delta:=d(p,C(p))$. For any number $\varepsilon_1<\delta$, let $U(p,\varepsilon_1)$ be an open ball of center p and radius ε_1. Then there exist such a geodesic loop c_1 at p and a point q_1 in $U(p,\varepsilon_1)$. Similarly for any number $\varepsilon_2<d(p,q_1)$, there exist such a geodesic loop c_2 at p and a point q_2 in $U(p,\varepsilon_2)$. By continuing this process, we obtain two sequences $\{\varepsilon_j\}$ and $\{q_j\}$. Suppose that $\varepsilon_j\to 0$ as $j\to\infty$. For a sequence $\{\dot{c}_j(0)\}$, (if necessary, take its subsequence) there exists a geodesic loop γ at p such that $c_j\to\gamma$ and $\dot{c}_j(0)\to\dot{\gamma}(0)$. From $q_j\to p$, γ is the product of

two simple geodesic loops of length $\geqq 2\delta$. This contradicts the continuity of loop lengths. Thus $\varepsilon_j \to \varepsilon > 0$, so there exists an open ball U of p such that any point in $U-\{p\}$ has the desired property. Therefore for any point q in $\sigma \cap U-\{p\}$, we obtain that $\Omega(p,q)$ is non-degenerate by using Lemma 2.2.

By applying Morse's fundamental theorem to $\Omega(p,q)$, we have the following two lemmas (cf. Nakagawa [8]).

LEMMA 2.4. *In the case $n \geqq 3$, if there exists a geodesic loop at p of index 0, then M is not simply connected.*

LEMMA 2.5. *If M is simply connected, then the index of each geodesic loop c at p satisfies* Ind $c \leqq n-1$. *In particular, if $n \geqq 3$,* $0 <$ Ind $c \leqq n-1$.

REMARK. In particular, if M satisfies (*), then the statements of Lemma 2.4 and Lemma 2.5 are true for $n=2$.

PROOF OF THEOREM A. The first assertion is proved by using Lemma 2.5, the theorem 2.2 of Nakagawa [8] and cohomology theory, and the index of each geodesic loop at p is same and $1, 3, 7 (n=16)$ or $n-1$. If M is not simply connected and $n \geqq 3$, then each geodesic loop at p is of index 0 by Lemma 2.5, and therefore the universal covering manifold of M is homeomorphic to a sphere (cf. [7]).

3. Restricted loop length.

Firstly we shall show the following

LEMMA 3.1. *Let M be an $n(\geqq 2)$-dimensional complete Riemannian manifold satisfying (*). If M is simply connected, then $C(p)$ and $Q(p)$ intersect, and also the converse is true.*

PROOF. There exists a point q in $C(p)$ satisfying $d(p, C(p)) = d(p,q)$. Suppose that q is not contained in $Q(p)$, then there exist two minimal geodesic segments c_1 and c_2 joining p to q, and $c_1 \circ c_2$ is a geodesic loop at p of length $2d(p,q) = 2l$. Thus p is the first conjugate point of p along $c_1 \circ c_2$ by Lemma 2.2. Therefore M is not simply connected by Lemma 2.4 and its remark. This is a contradiction.

Restricting loop length of M satisfying (*), we have the following

THEOREM B. *Let M be an $n(\geqq 2)$-dimensional complete simply connected Riemannian manifold satisfying* (*) *and with $K\leqq 1$. If $\pi\leqq l<3\pi/2$, then M satisfies $(\Sigma, n-1)$ or $(\Pi, \lambda;\ 1, 4l/3\pi, 2l/\pi, 4l/3\pi+1)$ in the sense of* [5].

PROOF. Let λ be the index of each geodesic loop at p, then λ must be equal to $1, 3, 7(n=16)$ or $n-1$ by proof of Theorem A. By Morse and Schoenberg's theorem (cf. [4, p. 176]) and Lemma 2.2, along each loop at p, there are no conjugate points of p in $[0,\pi)$ and there are λ conjugate points in $[\pi, 2l-\pi]$. If $\lambda=n-1$, then M satisfies $(\Sigma, n-1)$. If $\lambda=1, 3$ or $7(n=16)$, then M satisfies $(\Pi, \lambda; 1, 4l/3\pi,\ 2l/\pi, 4l/3\pi+1)$. Moreover we have $d(p, C(p))\geqq\pi$ from the proof of Lemma 3.1.

REMARK 1. Under the assumption of Theorem B, it seems to us that M is diffeomorphic to a symmetric space of compact type of rank one (cf. [6]). In particular, if $n\neq 4m, m\geqq 2$, then M is a homotopical sphere or M has the same homotopy type as a complex projective space by Theorem B and Klingenberg's theorem [5].

Let M be a Kählerian manifold. Let σ be a plane section of M and X, Y an orthonormal pair of tangent vectors of σ. Set $\cos\theta := |\langle X, JY\rangle|$ and moreover $\bar{K}(\sigma) := (1+3\cos^2\theta)/4$, where J is the almost complex structure of M. Then we have the following

THEOREM C. *Let M be a complete Kählerian manifold of complex dimension $n(\geqq 2)$ satisfying* (*) *and with $K\leqq 1$. If $\pi\leqq l<3\pi/2$, then M has the same homotopy type as a complex projective space. In particular, if $l=\pi$ and K satisfies $K(\sigma)\leqq\bar{K}(\sigma)$ for any plane section σ of M, then M is isometric to a complex projective space with constant holomorphic sectional curvature* 1.

PROOF. From the proof of Theorem A, M is simply connected and the index of each geodesic loop at p is equal to 1. Therefore the first assertion is proved by remark 1.

We prove the second assertion. Along any geodesic loop at p, the first conjugate point of p is the midpoint of the loop from the proof of Lemma 3.1. For any unit vector X to M at p, take a geodesic $c(t)$ satisfying $c(0)=$

$p, \dot{c}(0)=X$ and then there exists a Jacobi field $Y(t)$ $(\neq 0, 0<t<\pi)$ along c satisfying $Y(0)=Y(\pi)=0$. By the same argument as in Nakagawa [9], we obtain that the sectional curvature for the plane section $\sigma(t)$ spanned by $\dot{c}(t)$ and $Y(t)$ is equal to 1. By the assumption $K \leqq \bar{K}$, $\sigma(t)$ must be the holomorphic plane section and then we have $Y(t)=A(\sin t)J\dot{c}(t)$ (A: constant). The holomorphic sectional curvature for X is equal to 1 by the continuity of curva ture. Thus by using the proof of [4, p.132], we obtain that all geodesics starting from p in a direction contained in the holomorphic plane section spanned by X and JX form a totally geodesic 2-dimensional sphere with constant curvature 1. This holds for each holomorphic plane section in the tangent space to M at p, and therefore the second assertion is proved.

REMARK 2. Nakagawa [9] conjectured that an even dimensional complete simply connected manifold M satisfying (*), $l=\pi$ and with $K \leqq 1$ is isometric to a symmetric space of compact type of rank one. Theorem C states that the Kählerian analogue of this conjecture is true under the curvature condition $K \leqq \bar{K}$.

Finally for manifolds with positive curvature we have the following

THEOREM D. *Let M be an $n(\geqq 2)$-dimensional complete Riemannian manifold satisfying (*) and with $K \geqq k > 0$, where k is a constant. If $\pi/2\sqrt{k} < l \leqq \pi/\sqrt{k}$, then M is a homotopical sphere. In particular, if $l=\pi/\sqrt{k}$, then M is isometric to a sphere with constant curvature k.*

PROOF. If $\pi/2\sqrt{k} < l \leqq \pi/\sqrt{k}$, by using Morse and Schoenberg's theorem (cf. [4, p.176]), we have Ind $c \geqq n-1$ for any geodesic loop c at p. Thus combining Lemma 2.5 we have Ind $c=n-1$, from which it follows that M is simply connected. Suppose that M is not a homotopical sphere. By the result of Berger [1], there exists a non-trivial geodesic loop at p whose length is not greater than $\pi/\sqrt{k}$. This contradicts the assumption of loop length. Therefore M is a homotopical sphere.

If $l=\pi/\sqrt{k}$, then the midpoint of any geodesic loop at p is the first conjugate point. By the proof of Lemma 3.1, Myers' theorem (cf. [4, p.212]) and Toponogov's theorem (cf. [4, p.213]), M is isometric to a sphere with constant curvature k.

References

[1] M. Berger, On the diameter of some Riemannian manifolds, Technical report, Univ. of California, 1962.

[2] M. Berger, Lectures on geodesics in Riemannian geometry, Tata Institute, Bombay, 1965, p.134.

[3] R. Bott, On manifolds all of whose geodesics are closed, Ann. of Math., 60 (1954), 375-382.

[4] D. Gromoll, W. Klingenberg and W. Meyer, Riemannsche Geometrie im Grossen, Springer-Verlag. Lecture note 55, 1968.

[5] W. Klingenberg, Manifolds with restricted conjugate locus, Ann. of Math., 78 (1963), 527-547.

[6] S. Kobayashi and J. Eells, Problems in differential geometry, Proc. U.S.- Japan Sem. Diff. Geom., Kyoto (1965), p. 169.

[7] M. Kôzaki, On Riemannian manifolds with certain geodesic loops, to appear in Tamkang J. Math.

[8] H. Nakagawa, A note on theorems of Bott and Samelson, J. Math. Kyoto Univ., 7 (1967), 205-220.

[9] H. Nakagawa, Riemannian manifolds with many geodesic loops, J. Math. Soc. Japan, 20 (1968), 648-654.

Saga University