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ON THE EXISTENCE OF LIPSCHITZ CONTINUOUS WEAK SOLUTIONS OF THE DIRICHLET PROBLEM FOR SOME QUASI-LINEAR DEGENERATE ELLIPTIC EQUATIONS

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§1. Introduction

Let Ω be a bounded convex domain in two dimensional Euclidian space R^2 and its boundary, say $\partial\Omega$, be of a class $C_{3,\alpha}$ (see p. 7 [7]).

We consider the Dirichlet problem for the quasi-linear degenerate elliptic equation

$$Lu = f(x, y, u, Du)\, u_{yy} + u_{xx} + a(x, y, u, Du) = 0 \qquad (1)$$

with boundary condition

$$u|_{\partial\Omega} = 0, \qquad (2)$$

where $f \geqq 0$, $Du = \left(\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}\right)$ and the equation (1) has been obtained by the equation

$$\frac{\partial}{\partial y}\, a_0(x, y, u, Du) + \frac{\partial}{\partial x}\, a_1(x, y, u, Du) + b(x, y, u, Du) = 0 \qquad (1)'.$$

For the equation

$$y^m u_{yy} + u_{xx} - F(x, y, u, Du) = 0 \qquad (\bar{\Omega} \subseteqq \{y \geqq 0\})$$

the interesting results have been given by M. V. Keldysh, M. I. Aliev and many authors. Our concern here is with the case two dimensional Lebesgue measure of $\{(x, y):\ f(x, y, u(x, y), Du(x, y)) = 0\}$ is non-negative. In fact, in our theorem, the above set which we denote by A will have no restriction neither $A \subset \partial\Omega$ nor mes $A = 0$ (cf. Dubinski [4], [5]).

Our aim is to show the existence of Lipschitz continuous solutions of the Dirichlet problem (1), (2) in Sobolev space $\dot{W}_2^1(\Omega)$ (see p.4 [7]). A central position in the discussion is a construction of barrier function to obtain the bound of the solutions and their derivatives for elliptic regularized equations of (1). These procedures are greatly indebted to J. Serrin, O. A. Ladyzhenskaya, N. N. Ural'tseva, A. M. Il'in, O. A. Oleinik and so many authors.

§ 2. Assumptions and theorem

Throughout this paper we assume that the functions $f(x,y,u,p_1,p_2)$, $a(x,y,u,p_1,p_2)$ are continuously differentiable with their arguments, their derivatives satisfy Hölder condition with exponent α with respect to (x,y) in $\bar{\Omega}\times R^3$, and that $f\geqq 0$.

We set the following conditions for f and a.

(1) $f_x{}^2, f_y{}^2, f_u{}^2, f_{p_i}{}^2 \leqq C_1 f(x,y,u,p_1,p_2)$

for some positive constant C_1 and $i=1,2$.

(2) $f_{p_1}^2 = o(f(x,y,u,p_1,p_2))$ $(|p|=\sqrt{p_1^2+p_2^2}\to\infty)$

(3) $a_u(x,y,u,p_1,p_2)\leqq -C_2<0$ for some positive constant C_2.

(4) $a_{p_1}, a_{p_2}=o(|p|)$ $(|p|\to\infty)$ and $|a_y|\leqq\theta_1 C_2|p|$, $|a_x|\leqq\theta_2 C_2|p|$ for some positive constants C_3 and $0<\theta_i<1$.

(5) $a=o(|p|^2)$ $(|p|\to\infty)$

Furthermore, there exists some neighborhood of any closed boundary portion consisting from the points satisfying $(n,(0,1))\neq\pm 1$, n being a unit inner normal of $\partial\Omega$, such that

(6) $f(x,y,u,p_1,p_2)=o(|p|)$ $(|p|\to\infty)$.

On the other hand, near the boundary points $(n,(0,1))=\pm 1$ one of the two conditions, (7), (7)′ is valid.

(7) $a(x,y,u,p_1,p_2)=o(f(x,y,u,p_1,p_2)|p|^2)$

$f(x,y,u,p_1,p_2)|p|^2>C_4>0$

for sufficiently large $|p|$ and for some positive constant C_4.

(7)′ $-(n,(0,1))\ a_{p_2}(x,y,u,p_1,p_2)>C_5$

$a_{p_1}(x,y,u,p_1,p_2)<-(n,(0,1))\ C_6 a_{p_2}(x,y,u,p_1,p_2)$

for positive constants C_5 and C_6 and for sufficiently large $|p|$.

REMARK The conditions (1), (2), (4), (5), (6) and (7) should be understood as they hold for any compact set $\bar{\Omega}\times[-K,K]$, K being arbitrary, namely they hold uniformly for (x,y) but for u uniformly in the wide sence.

THEOREM *Under the above conditions there exists Lipschitz continuous (on $\bar{\Omega}$) weak solution for the problem* (1), (2) *in the Sobolev space* $\dot{W}_2^1(\Omega)$.

To prove the theorem we make the elliptic regularization of the problem (1), (2).

Let us consider the equations

$$L_\varepsilon u_\varepsilon=\{f(x,y,u_\varepsilon,u_{\varepsilon x},u_{\varepsilon y})+\varepsilon\}\ u_{\varepsilon yy}+u_{\varepsilon xx}+a(x,y,u_\varepsilon,u_{\varepsilon x},u_{\varepsilon y})=0 \qquad (3)$$

with boundary condition

$$u_\varepsilon|_{\partial\Omega} = 0$$

for any sufficiently small positive ε. For the solution u_ε, apriori estimates, the bounds of u_ε, Du_ε on the boundary and Du_ε on whole domain, being independent of ε and of the solutions u_ε, will be given in §§4, 5 and 6, respectively. Consequently there will exist $\{u_{\varepsilon_n}\}$ such that $\{u_{\varepsilon_n}\}$ converges weakly the solution of (1) in $\overset{\circ}{W}{}_2^1(\Omega)$ satisfying the Lipschitz condition ($\varepsilon_n \to 0$). A precise procedure of the above will be given in §7.

3. Maximum principle

We present in this section a number of maximum principle.

LEMMA 1. *(E. Hopf) Let G be a domain in* R^n *and*

$$Lu = \sum_{i,j=1}^{n} a_{ij}(x)u_{x_i x_j} + \sum_{i=1}^{n} b_i(x)u_{x_i} - c(x)u \text{ in } G,$$

Where the coefficients are all continuous in G and $x = (x_1, \dots, x_n)$.

Assume that

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geqq 0 \text{ for any real vector } \xi = (\xi_1, \dots, \xi_n) \neq 0,\ c(x) \geqq 0$$

and that $Lu > 0$ *for* $u \in C^2(G)$, *then* $u(x)$ *does not attain to positive maximum at the interior point of G.*

LEMMA 2. *(Serrin [11]) Let G be a domain in* R^n *and*

$$L(x, u, Du) = \sum_{i,j=1}^{n} a_{ij}(x, u, Du)\, u_{x_i x_j} - a(x, u, Du) \text{ in } G,$$

where the coefficients are continuous and continuously differentiable with their arguments except x.

Furthermore assume that

$$\sum_{i,j=1}^{n} a_{ij}(x, u, p_1, \dots, p_n)\xi_i\xi_j > 0$$

for any $x, u, p_1, \dots, p_n$ *and real vector* $\xi \neq 0$,

and that $u \in C^2(G)$. *Let* $w(x)$ *be a function in* $C^2(G)$ *such that*

$$L(w + b) \equiv L(x, w + b, Dw)\ (w + b) < 0 \text{ in } G$$

for all constant $b > 0$ *and lim sup* $(u - w) \leqq 0$ *as one approaches any point of the boundary. Then* $u \leqq w$ *in G.*

By a slight modification of the lemma, for L in lemma 2 we have

LEMMA 3. *Let* $u, w \in C^2(G) \cap C^0(\bar{G})$

$L(w+b) > 0$ *for any constant* b $(|b| < \max_G |w| + \max_G |u|)$,

$L(u) = 0$

then the funcion $w-u$ *doen't attain to positive maximum at the interior point of* G.

PROOF OF LEMMA 3. The proof is a slight modification of [11] p. 426.

Now by hypothesis

$\sum a_{ij}(x, w+b, Dw) w_{x_i x_j} - a(x, w+b, Dw) > 0,$

this inequality being valid for all x in G and all real $b(|b| < \max_{\bar{G}} |w| + \max_{\bar{G}} |u|)$. Consequently it will hold for any function $b(x)$. Choosing $b = u - w$, we have

$\sum a_{ij}(x, u, Dw) w_{x_i x_j} - a(x, u, Dw) > 0.$

Subtracting the above inequality from $L(u) = 0$,
we obtain

$\sum a_{ij}(x, u, Du)\ (w-u)_{x_i x_j} + \textit{linear combination of } D(w-u) > 0.$

This means that $w-u$ does not attain to positive maximum at an interior point of G by virtue of lemma 1.

4. A bound of $|u_\varepsilon|$.

We consider the Dirichlet problem

$$L_\varepsilon u_\varepsilon = \{f(x, y, u_\varepsilon, Du_\varepsilon) + \varepsilon\} u_{\varepsilon yy} + u_{\varepsilon xx} + a(x, y, u_\varepsilon, Du_\varepsilon) = 0 \qquad (3)$$

$$u_\varepsilon|_{\partial\Omega} = 0 \qquad (4)$$

LEMMA 4. *Let the function* $a(x, y, u, p_1, p_2)$ *satisfy the condition* (3), *then the solutions* u_ε, *belong to* $C_2(\Omega) \cap C_0(\bar{\Omega})$, *of the Dirichlet problem* (3), (4) *satisfy the following inequality*

$$|u_\varepsilon(x, y)| \leq \underset{(x,y)\in\bar{\Omega}}{Max} |a(x, y, 0, 0, 0)| / C_2$$

PROOF. By M_1 denote the right hand side of the above inequality. Let

$w(x,y) = M_1$. By the use of the mean value theorem

$$L_\varepsilon M_1 = a(x,y,M_1,0,0) = M_1\, a_u(x,y,\theta M_1,0,0) + a(x,y,0,0,0)$$

holds for some $\theta(0<\theta<1)$. By virtue of the condition (3)

$$L_\varepsilon M_1 \leqq -M_1 C_2 + \underset{\bar{\Omega}}{Max} \mid a(x,y,0,0,0) \mid < 0.$$

Applying lemma 2, we have $u_\varepsilon \leqq M_1$. Similarly we obtain $-u_\varepsilon \geqq -M_1$. This completes the proof.

5. Boundary estimates of $\mid Du_\varepsilon \mid$

Here we state, on the boundary of Ω, the bound of the derivatives of the solutions u_ε for the Dirichlet problem (3), (4).

LEMMA 5. *Let the functions f and a satisfy the conditions* (3), (5), (6) *and one of the conditions* (7) *or* (7)′ *and the solution $u_\varepsilon \in C_2(\Omega) \cap C_1(\bar{\Omega})$, then*

$$\underset{(x,y)\in\partial\Omega}{Max} |Du_\varepsilon| \leqq M_2$$

where M_2 is some positive constant independent of u_ε and ε.

PROOF. Let S be a closed small portion of the boundary $\partial\Omega$.
i) First assume that S doesn't contain the point $(n,(0,1)) = \pm 1$. Furthermore assume that S is represented by the equation

$$x = \psi(y), \quad h \leqq y \leqq k.$$

Let $\varphi(y)$ be a non-negative twice continuously differentiable function such that

$$\varphi(y) = \beta: \ h+\delta \leqq y \leqq k-\delta, \quad \varphi(y) = 0: \ y<h \ or \ y>k,$$

where δ is a sufficiently small positive constant and $\beta>0$.

For the function $w = e^{\alpha(-x+\psi(y)+\varphi(y))}$ and for any constant b, $|b| \leqq 2M_1 + 2$, M_1 being a constant which appeared in the proof of lemma 4, we have

$$\begin{aligned} L_\varepsilon(w+b) &= \{f(x,y,w+b,Dw)+\varepsilon\}\, w_{yy} + w_{xx} + a(x,y,w+b,Dw) \\ &= \{f(x,y,w+b,-\alpha w,\alpha w(\varphi_y+\psi_y))+\varepsilon\}\, \alpha^2 w(\varphi_y+\psi_y)^2 \\ &\quad -(f+\varepsilon)\alpha w(\varphi_{yy}+\psi_{yy}) + \alpha^2 w + a(x,y,w+b,-\alpha w,\alpha w(\varphi_y+\psi_y)). \end{aligned}$$

Let $e^{\alpha\beta} = M_1 + 2$.

By the use of conditions (5) and (6), we obtain

$$L_\varepsilon(w+b)>o(\alpha^2)+\alpha^2+\alpha o(\alpha)+o(\alpha^2)>0 \tag{5.1}$$

for sufficiently large α.

By G and S^* we denote the set $\{(x,y)\colon \psi(y)<x<\varphi(y)+\varphi(y)\}$ and the portion of S corresponds to $x=\psi(y)$ $(h+\delta\leqq y\leqq k-\delta)$, respectively.

Furthermore, by σ we denote the part of the boundary of G represented by $x=\psi(y)+\phi(y)$.

On S^*, $w-u_\varepsilon=e^{\alpha\beta}=M_1+2$, and on σ,

$$w-u_\varepsilon|_\sigma\leqq w|_\sigma+\max_{\bar G}|u_\varepsilon(x)|=1+\max_{\bar G}|u_\varepsilon(x)|\leqq 1+M_1<e^{\alpha\beta}.$$

By the inequality (5.1) one can apply lemma 3 to obtain that the function $w-u$ doesn't attain to positive maximum in G. With the aid of the above consideration it reaches to the maximum on all point of S^*, because of $w-u_\varepsilon|_{S-S^*}=e^{\alpha\varphi}<e^{\alpha\beta}$ and $w-u_\varepsilon|_\sigma<e^{\alpha\beta}$.

Hence $\dfrac{\partial w}{\partial n}-\dfrac{\partial u_\varepsilon}{\partial n}\leqq 0$. Similarly $\dfrac{\partial w}{\partial n}+\dfrac{\partial u_\varepsilon}{\partial n}\leqq 0$ holds. Consequently, on S^*, $\left|\dfrac{\partial u_\varepsilon}{\partial n}\right|\leqq$ constant being independent of u_ε and ε.

ii) Second, consider the case S contains the points $(n,(0,1))=\pm 1$.

Without loss of generality we may consider in the case S doesn't contain the points $(n,(0,1))=-1$. Assume that $S\colon y=\psi(x)$, $h\leqq x\leqq k$.

Let φ be a non-negative twice continuously differentiable function such that

$$\varphi(x)=\beta,\ h+\delta\leqq x\leqq k-\delta,\ \varphi(x)=0,\ x<h \ or\ x>k,$$

δ and β being the same as in i).

By G and S^* denote the set $\{(x,y)\,|\,\psi(x)<y<\varphi(x)+\psi(x)\}$ and the portion of S corresponds to $y=\psi(x)$, $h+\delta\leqq x\leqq k-\delta$, respectively. And by σ we denote the part of the boundary of G represented by $y=\varphi(x)+\psi(x)$.

Let $w(x,y)=e^{\alpha(-y+\varphi(x)+\psi(x))}$, then

$$\begin{aligned}L_\varepsilon(w+b)=&\{f(x,y,w+b,w(\varphi_x+\psi_x),-\alpha w)+\varepsilon\}\alpha^2 w+\alpha^2 w(\varphi_x+\psi_x)^2\\&+\alpha w(\varphi_{xx}+\psi_{xx})+a(x,y,w+b,-\alpha w(\varphi_x+\psi_x),-\alpha w).\end{aligned}$$

If we assume the all conditions of this lemma except (7)$'$, we can obtain $L_\varepsilon(w+b)>0$ for sufficiently large α similarly to the case i).

In the case the condition (7)′ is assumed, it holds

$$L_\varepsilon(w+b) = (f+\varepsilon)\alpha^2 w+\alpha^2 w(\varphi_x+\psi_x)^2+\alpha w(\varphi_{xx}+\psi_{xx})+a(x,y,w+b,0)$$
$$-a_{p_2}(x,y,w+b,\theta Dw)\alpha e^{\alpha(-y+\varphi+\psi)}$$
$$+a_{p_1}(x,y,w+b,\theta Dw)\alpha e^{\alpha(-y+\varphi+\psi)}(\varphi_x+\psi_x)$$

for some θ, $0<\theta<1$. Here we have used the mean value theorem.

For sufficiently small δ_1, we can assume that

$$|\varphi_x+\psi_x|, |\varphi_{xx}|<\delta_1.$$

In fact, the above is obtained for sufficiently small S and β. Thus we obtain, by the conditions,

$$L_\varepsilon(w+b)>-\alpha e^{\alpha(-y+\varphi+\psi)}\delta_1-\textit{positive const.}$$
$$+\alpha e^{\alpha(-y+\varphi+\psi)}(C_5-C_6C_5\delta_1)>0$$

for sufficiently large α and small δ_1. Here we used tha fact that $\partial\Omega$ is convex.

Similarly to i), we obtain $\left|\frac{\partial u_\varepsilon}{\partial n}\right|\leq const.$ on S^*. Dividing the boundary into small pieces to apply the conclusions in i) and ii), we assert that $\partial u_\varepsilon/\partial n$ are bounded on $\partial\Omega$. As the derivatives of u_ε for tangential direction to $\partial\Omega$ are zero, the proof is end.

6. Estimates of $|Du_\varepsilon|$ on whole domain

Here we show that the derivatives of the solutions u_ε for the problem (3), (4) are bounded by a constant independent of u_ε and ε.

$$L_\varepsilon u_\varepsilon = \{(f(x,y,u_\varepsilon,u_{\varepsilon x},u_{\varepsilon y})+\varepsilon\}u_{\varepsilon yy}+u_{\varepsilon xx}+a(x,y,u_\varepsilon,u_{\varepsilon x},u_{\varepsilon y})=0 \quad (3)$$

$$u_\varepsilon|_{\partial\Omega}=0 \quad (4)$$

for small $\varepsilon>0$.

LEMMA 6. *Let us assume the conditions of* §2 *and that the solutions* u_ε *of our problem belong to class* $C^3(\Omega)\cap C^1(\bar{\Omega})$.
Then we have

$$\max_{(x,y)\in\bar{\Omega}}|Du_\varepsilon|\leq M_3$$

where M_3 *is some constant independent of* u_ε *and* ε, *but it depends on* M_1, M_2 *appearing in lemmas* 5, 6.

PROOF. i) First we show the estimate

$$\max_{\bar{\Omega}} |u_{\varepsilon y}(x, y)| \leqq M_3 \tag{6.1}$$

For brevity, throughout this section, we shall denote u_ε by u. Differentiating the equation (3) with respect to y, we obtain

$$\begin{aligned}(f_y+f_u u_y+f_{p_1}u_{xy}+f_{p_2}u_{yy})u_{yy}+(f+\varepsilon)u_{yyy}+u_{xxy} \\ = -(a_y+a_u u_y+a_{p_1}u_{xy}+a_{p_2}u_{yy}).\end{aligned} \tag{6.2}$$

Denote u_y^2 by v, namely $v \equiv u_y^2$. Multiplying (6.2) by $2u_y$, we have

$$\begin{aligned}Lv &\equiv (f+\varepsilon)v_{yy}+v_{xx} \\ &= 2(f+\varepsilon)u_{yy}{}^2+2u_{xy}{}^2-(f_{p_1}u_{yy}+a_{p_1})v_x-(f_{p_2}u_{yy}+a_{p_2})v_y \\ &\quad -(f_y v_y+f_u u_y v_y)-2(a_y u_y+a_u v)\end{aligned}$$

where the argument of f, a and their derivatives is (x, y, u, u_x, u_y). Now we estimate the last term of the above. By the use of conditions (3) and (4) and of $|u_\varepsilon| \leqq M_1$

$$\begin{aligned}-2(a_y u_y+a_u v) &\geqq 2v\left(-a_u-\frac{|a_y|}{|u_y|}\right) \\ &\geqq 2v\left(C_2-\frac{|a_y|}{|u_y|}\right) > C_2 v\end{aligned}$$

holds for $v > G_1$, where G_1 is a sufficiently large constant. Hence

$$L'v \equiv Lv+(f_{p_1}u_{yy}+a_{p_1})v_x+(f_{p_2}u_{yy}+a_{p_2})v_y+f_y v_y+f_u u_y v_y > 0$$
$$\text{for } v > G_1.$$

Applying lemma 1, we see that $\max_{\bar{\Omega}} v \leqq G_1$ or, that $\max_{\bar{\Omega}} v$ is less than $\max_{\partial\Omega} v$ whose estimate has been obtained by lemma 5. This implies (6.1).

ii) Next, we shall estimate $\max_{\bar{\Omega}} |u_{\varepsilon x}(x, y)|$.

Define $w = u_x{}^2$. Similarly to i) we obtain

$$\begin{aligned}Lw &\equiv (f+\varepsilon)w_{yy}+w_{xx} \\ &= 2(f+\varepsilon)u_{xy}{}^2+2u_{xx}{}^2-(f_{p_1}u_{yy}+a_{p_1})w_x \\ &\quad -(f_{p_2}u_{yy}+a_{p_2})w_y-2u_x u_{yy}(f_x+f_u u_x)-2u_x(a_x+a_u u_x)\end{aligned} \tag{6.3}$$

where the argument of f, a and their derivatives is (x, y, u, u_x, u_y).

Furthermore, we see that

$$L(vw) = wLv + vLw + 2\{(f+\varepsilon)v_y w_y + v_x w_x\}$$
$$L(u^2) = 2\{(f+\varepsilon)v+w\} - 2ua \tag{6.4}$$

Let d be a constant such that $\sqrt{x^2+y^2} \leqq d$ for all $(x, y) \in \bar{\Omega}$. And define

$$h(x, y) = vw + k_1 w + k_2 u^2 + e^{k_3(d+x)},$$

where k_1, k_2 and k_3 are positive constant determined later on.

For h we have

$$\begin{aligned}
Lh \equiv\ & (f+\varepsilon)h_{yy} + h_{xx} \\
=\ & L(vw) + k_1 Lw + k_2 Lu^2 + Le^{k_3(x+d)} \\
=\ & 2w\{(f+\varepsilon)u_{yy}{}^2 + u_{xy}{}^2\} + 2v\{f+\varepsilon)u_{xy}{}^2 + u_{xx}{}^2\} \\
&+ 2k_1\{(f+\varepsilon)u_{xy}{}^2 + u_{xx}{}^2\} + 2k_2\{(f+\varepsilon)v + w\} \\
&+ k_3^2 e^{k_3(x+d)} - f_{p_1}u_{yy} + a_{p_1})h_x - (f_{p_2}u_{yy} + a_{p_2})h_y \\
&- 2wu_y(a_y + a_u u_y) - 2vu_x(a_x + a_u u_x) \\
&- 2k_1 u_x(a_x + a_u u_x) + 2\{(f+\varepsilon)v_y w_y + v_x w_x\} \\
&- 2wu_y u_{yy} f_y - 2vu_x u_{yy} f_x - 4vwu_{yy} f_u \\
&- 2k_1 u_x u_{yy}(f_x + f_u u_x) - 2\mathrm{k}_2 ua \\
&+ 2k_2 uu_x(f_{p_1}u_{yy} + a_{p_1}) + 2k_2 uu_y(f_{p_2}u_{yy} + a_{p_2}) \\
&+ k_3 e^{k_3(x+d)}\ (f_{p_1}u_{yy} + a_{p_1}).
\end{aligned} \tag{6.5}$$

Here we used (6.3) and (6.4).

Our aim is to obtain

$$L'h = Lh + (f_{p_1}u_{yy} + a_{p_1})h_x + (f_{p_2}u_{yy} + a_{p_2})h_y > 0.$$

For this purpose let us estimate the terms of the above.

With the aid of the conditions (3), (4) and in mind that $|u_y|$ is bounded, having been already shown, it follows that

$$\begin{aligned}
&-2wu_y(a_y + a_u u_y) - 2vu_x(a_x + a_u u_x) - 2k_1 u_x(a_x + a_u u_x) \\
&\quad \geqq -C|p|^2 + k_1 C_2 |p|^2(1-\theta_2) > 0
\end{aligned} \tag{6.6}$$

for $w > G_2$, where k_1 and G_2 are sufficietly large constant.

Furthermore, using the conditions (1), (2) and (4), and remembering that $|u|$ and $|u_y|$ are bounded, an easy computation shows that

$$\begin{aligned}
&|2(f+\varepsilon)v_y w_y| \leqq \mu_1(f+\varepsilon)wu_{yy}{}^2 + \nu_1(f+\varepsilon)u_{xy}{}^2 \\
&|2v_x w_x| \leqq \mu_2 wu_{xy}{}^2 + \nu_2 u_{xx}{}^2 \\
&|2wu_y u_{yy} f_y| \leqq \mu_3 fwu_{yy}{}^2 + \nu_3 w \\
&|2vu_x u_{yy} f_x| \leqq \mu_4 fwu_{yy}{}^2 + \nu_4
\end{aligned}$$

$$
\begin{aligned}
&|4vwu_{yy}f_u| \leqq \mu_5 fwu_{yy}{}^2+\nu_5 w \\
&|2k_1u_xu_{yy}f_x| \leqq k_1\mu_6 fwu_{yy}{}^2+k_1\nu_6 \\
&|2k_1wu_{yy}f_u| \leqq k_1\mu_7 fwu_{yy}{}^2+k_1\nu_7 w \\
&|2k_2ua| = k_2 o(|p|^2) \\
&|2k_2uu_xu_{yy}f_{p_1}| \leqq \mu_8 fwu_{yy}{}^2 o(1)+k_2^2\nu_8 \\
&|2k_2uu_xa_{p_1}| = k_2 o(|p|^2) \\
&|2k_2uu_yu_{yy}f_{p_2}| \leqq \mu_9 fwu_{yy}{}^2+k_2^2\nu_9 \\
&|2k_2uu_ya_{p_2}| = k_2 o(|p|^2) \\
&|k_3e^{k3(x+d)}a_{p_1}| = k_3e^{k3(x+d)}o(|p|),
\end{aligned}
\tag{6.7}
$$

for large w, where we used Cauchy-Schwarz's inequality and μ_1 μ_9 are the positive constants determined later on and ν_i depends only on μ_i.

In addition to the above, we obtain also

$$
\begin{aligned}
|k_3e^{k3(x+d)}f_{p_1}u_{yy}| &\leqq \frac{1}{2}k_3e^{k3(x+d)}f_{p_1}{}^2wu_{yy}{}^2+\frac{1}{2}k_3e^{k3(x+d)} \\
&= \frac{1}{2}k_3e^{k3(x+d)}fwu_{yy}{}^2o(1)+\frac{1}{2}k_3e^{k3(x+d)}
\end{aligned}
\tag{6.8}
$$

here we have used the conditions (1), (2). Recalling

$$L'h=Lh+(f_{p_1}u_{yy}+a_{p_1})h_x+(f_{p_2}u_{yy}+a_{p_2})h_y,$$

and inserting (6.6), (6.7), (6.8) into (6.5), we have

$$
\begin{aligned}
L'h \geqq\ & 2w\{(f+\varepsilon)u_{yy}{}^2+u_{xy}{}^2\}+2k_1\{(f+\varepsilon)u_{xy}{}^2+u_{xx}{}^2\} \\
&+ 2k_2w+k_3{}^2e^{k3(x+d)}-\mu_1(f+\varepsilon)wu_{yy}{}^2 \\
&-(\mu_3+\mu_4+\mu_5+k_1\mu_6+k_1\mu_7+\mu_9)\,fwu_{yy}{}^2 \\
&-(\mu_8+\frac{1}{2}k_3e^{k3(x+d)})fwu_{yy}{}^2o(1)-\mu_2wu_{xy}{}^2 \\
&-\nu_1(f+\varepsilon)u_{xy}{}^2-\nu_2u_{xx}{}^2-(\nu_3+\nu_5+k_1\nu_7)w \\
&-(3k_2+k_3e^{k3(x+d)})o(|p|^2) \\
&-(\nu_4+k_1\nu_6+k_2{}^2\nu_8+k_2{}^2\nu_9+\frac{1}{2}k_3e^{k3(x+d)}).
\end{aligned}
$$

Now we shall determine the constants. First, determine μ_1, μ_3, μ_4, μ_5 and μ_9 as

$$\mu_1+\mu_3+\mu_4+\mu_5+\mu_9<\frac{1}{2} \text{ and } \mu_2<1,$$

and k_1 as $2k_1>\nu_1$, ν_2.

Hence, the coefficients of $wu_{xy}{}^2$, $fu_{yx}{}^2$ and $u_{xx}{}^2$ become positive.

Next, determine μ_6 and μ_7 as $k_1\mu_6+k_1\mu_7<\frac{1}{2}$,

then

$$\{1-(\mu_1+\mu_3+\mu_4+\mu_5+k_1\mu_6+k_1\mu_7)\}fwu_{yy}{}^2\geqq 0$$

Furthermore, determine k_2 as

$$k_2>\nu_3+\nu_5+k_1\nu_7,$$

μ_8 as $\mu_8<1$, and k_3 as

$$k_3{}^2e^{k3(x+d)}>\nu_4+k_1\nu_6+k_3{}^2\nu_8+k_2{}^2\nu_9+\frac{1}{2}k_3e^{k3(x+d)},$$

so that the coefficients of w is larger than k_2 and constant term become positive.

Finally choose G_3 sufficiently large, then for $w>G_3$

$$fwu_{yy}{}^2\geqq\left(\frac{1}{2}k_3e^{k3(x+d)}+\mu_8\right)fwu_{yy}{}^2o(1)$$

and

$$wk_2-(3k_2+k_3e^{k3(x+d)})\,o\,(|p|^2)>0,$$

because of $|p|^2=u_x{}^2+u_y{}^2<u_x{}^2+\text{const.}$ Let $G_4=max(G_2,G_3)$, then $L'h>0$ for $w>G_4$.

Applying lemma 1 for $L'h$, we have the estimate

$$\max_{\bar{\Omega}} h(x,y)\leqq\max_{\partial\Omega} h(x,y)+G_4.$$

Remembering the definition of h and that v is less than some constant, we completes the proof.

7. The proof of the theorem and examplse

Consider the problem (3), (4).

$$L_\varepsilon u_\varepsilon=\}(f(x,y,u_\varepsilon,Du_\varepsilon)+\varepsilon\}U_{\varepsilon yy}+u_{\varepsilon xx}+a(x,y,u_\varepsilon,Du_\varepsilon)=0 \text{ in } \Omega \qquad (3)$$

$$u_\varepsilon|_{\partial\Omega}=0 \qquad (4)$$

To guarantee the existence of the solutions for the above problem it is sufficient that some growth condition of the equation and that $a(x,y,u,0,0,0)$ $u<-bu^2+b_2$([7], p. 373) hold for some constants b_1 and b_2. Apriori estimates of u_ε have been obtained in lemmas 4, 5 and 6, so that there is no necessity to consider the growth condition. On the other hand, by the condition (3) and mean value theorem

$$a(x, y, u, 0, 0, 0)u = a_u(x, y, \theta u, 0, 0, 0)u^2 < -C_2 u^2,\ 0<\theta<1,$$

is valid.

Hence, applying the theorem of Ladyzhenskaya and Ural'tseva [7] p. 373, we obtain the solution u_ε belonging to $C^{2+\alpha}(\bar{\Omega})$.

By the well-known regularity theorem for elliptic equations, $u_\varepsilon \in C^{3+\alpha}(\Omega)$.

Next, consider the sequence $\{u_{\varepsilon_m}\}$ in $\mathring{W}_2^1(\Omega)$, here $\varepsilon_m \to 0$ as $m \to \infty$.

In virtue of lemmas 4 and 6, $\|u_\varepsilon\|_1 <$ const. where $\|\ \|_1$ is the norm of the Sobolev space $\mathring{W}_2^1(\Omega)$. Hence the sequence $\{u_{\varepsilon_m}\}$ has a weak convergent subsequence in the space, whose limit we denote by u. It is easily seen that u is a weak solution of the original problem (1), (2) and it satisfies the Lipschitz condition on $\bar{\Omega}$. This completes the proof of the theorem.

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