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# ON THE CONTINUITY OF ENTIRE FUNCTIONS

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**1.** An entire function is not necessarily uniformly continuous except when it is linear or it is restricted on compact subset. In this note we shall first be concerned with uniformly continuous entire function and show that if an entire function is uniformly continuous, then it is linear, see THEOREM 1. In the second place, suggested by this fact we shall try to estimate the continuity of holomorphic functions. Our result is THEOREM 2, which is a direct consequence of wellknown Bremermann's charaterization of a domain of holomorphy in $C^n$.

**2.** Let $f(z)$ be an entire function in $C^n$.

THEOREM 1. *An entire function is uniformly continuous if and only if it is linear.*

PROOF. It is sufficient to show the necessity. By assumption, for any positive $\varepsilon$ there exists a positive $\delta$ such that $|f(z)-f(w)|<\varepsilon$ *if* $\|z-w\|<\delta$, where $\|z-w\|=\max_{i=1}^{n}\{|z_i-w_i|\}$, $z=(z_1, z_2, \cdots, z_n)$ and $w=(w_1, w_2, \cdots, w_n)$. Choose $\delta'$ so that $0<\delta'<\delta$. Let us denote the polydisk $\{z: \|z\|=k\delta'\}$ by $D_k$, where $k$ is an integer. Then we have $|f(0)-f(z)|<k\varepsilon$, if $z\in D_k$. Hence, we obtain by Cauchy's inequality an estimate for the coefficients of the Taylior expansion $f(z)=\sum a_{\nu_1\nu_2\cdots\nu_n} z_1^{\nu_1} z_2^{\nu_2} \cdots z_2^{\nu_n}$ as follows.

$$\begin{aligned} |a_{\nu_1\nu_2\cdots\nu_n}| &\leqq \sup_{\|z\|\leq k\delta'} |f(z)|/(k\delta')^{\nu_1+\cdots+\nu_n} \\ &< (|f(0)|+k\varepsilon)/(k\delta')^{\nu_1+\cdots+\nu_n} \\ &= (\delta')^{-\nu_1-\cdots-\nu_n}(|f(0)|+k\varepsilon)/k^{\nu_1+\cdots+\nu_n}. \end{aligned}$$

Since $k$ is arbitrary, we have

$$a_{\nu_1\nu_2\cdots\nu_n}=0 \text{ if } \nu_1+\nu_2+\cdots+\nu_n\geqq 2.$$

Thus, $f(z)$ should be linear.

*REMARK.* Another proof is given by making use of the fact that $f(z)$ is uniformly continuous if and only if the family $\mathscr{G}=\{g_\alpha(z)\}$ where $g_\alpha(z)=f(z+\alpha)$,

$\alpha \in C^n$ is equicontinuous. We know that $\mathscr{G}$ is equicontinuous in $C^n$ if and only if all families $\frac{\partial \mathscr{G}}{\partial z_i}$, $i=1,2,\cdots,n$ where $\frac{\partial \mathscr{G}}{\partial z_i}=\left\{\frac{\partial g}{\partial z_i}: g\in\mathscr{G}\right\}$ are simultaneously bounded on every compact subset of $C^n$, see [3]. This means that all the first order derivatives of $f(z)$ should be bounded in $C^n$,i.e. that $f(z)$ should be linear.

**3.** Let $f(z)$ be a continuous function in a domain $D$ in $C^n$. In order to estimate the continuity of $f(z)$ we define a positive function $\delta_\varepsilon(z)$ in $D$ by $\delta_\varepsilon(z)=\sup\{\delta: |f(z)-f(w)|<\varepsilon \text{ if } \|z-w\|<\delta \text{ and } w\in D\}$. $\delta_\varepsilon(z)$ is then a continuous function with values in the extended space $(0,+\infty]$. The purpose of this section is to say more about $\delta_\varepsilon(z)$ when $f(z)$ is holomorphic. We have the following

THEOREM 2. *If $f(z)$ is holomorphic in a domain of holomorphy $D$ in $C^n$, then $-\log\delta_\varepsilon(z)$ is plurisubharmonic in $D$.*

PROOF. Consider the holomorphic function $F(z,w)=f(z)-f(w)$ in $D\times D$ in $C^{2n}$. Since by assumption $D$ is a domain of holomorphy, the product $D\times D$ is a domain of holomorphy in $C^{2n}$. Further, $\Delta_\varepsilon=\{(z,w)\in D\times D: |F(z,w)|<\varepsilon\}$ is also a domain of holomorphy.

Now, let $\mathfrak{a}=(\alpha,\beta)$ be a 2n-dimensional complex vector with norm 1. Let us define the real valued function $\delta_\mathfrak{a}(z,w)$ by $\delta_\mathfrak{a}(z,w)=\inf\{|t|: (z,w)+t\mathfrak{a}\notin \Delta_\varepsilon, t\in C\}$. Then, $-\log\delta_\mathfrak{a}(z,w)$ is wellknown function (Distanz Funktion) considered by Bremermann in his characterization of a domain of holomorphy, see [1], [2]. Since $\Delta_\varepsilon$ is a domain of holomorphy, after Bremermann's main result we know that $-\log\delta_\mathfrak{a}(z,w)$ is plurisubharmonic in $\Delta_\varepsilon$. Consider the case $\mathfrak{a}=(0,\beta)$ and the function $\delta_\mathfrak{a}(z,z)$ which is the restriction of $\delta_\mathfrak{a}(z,w)$ to the diagonal set of $D\times D$ that is holomorphically equivalent to $D$. The function $\delta_\mathfrak{a}(z,z)$ is by definition given by $\inf\{|t|: (z,z+t\beta)\notin\Delta_\varepsilon, t\in C\}$. This implies that $|f(z)-f(z+t\beta)|<\varepsilon$ if $|t|<\delta_\mathfrak{a}(z,z)$. Putting $w=z+t\beta$, $\|\beta\|=1$ we have that $|f(z)-f(w)|<\varepsilon$ if $\|z-w\|<\delta_\mathfrak{a}(z,z)$. Conversely, the supremum of $\delta$ such that $|f(z)-f(z+t\beta)|<\varepsilon$ if $|t|<\delta$ is equal to $\delta_\mathfrak{a}(z,z)$, $\mathfrak{a}=(0,\beta)$. $\beta$ being arbitrary, we see that $\delta_\varepsilon(z)=\inf_{\|\beta\|=1}\delta_\mathfrak{a}(z,z)$ where $\mathfrak{a}=(0,\beta)$. Consequently, we have $-\log\delta_\varepsilon(z)=\sup_{\|\beta\|=1}\{-\log\delta_\mathfrak{a}(z,z)\}$ where $\mathfrak{a}=(0,\beta)$. As the restriction of $-\log\delta_\mathfrak{a}(z,w)$ to the diagonal set of $D\times D$ $-\log\delta_\mathfrak{a}(z,z)$ is plurisubharmonic is $D$ and hence $-\log\delta_\varepsilon(z)$ is plurisubharmonic in $D$. The proof is completed.

If $D=C^n$ and $f(z)$ is uniformly continuous, then $-\log\delta_a(z)$ is bounded from above. This implies that $-\log\delta_\varepsilon(z)$ is constant. Conversely, if $-\log_\varepsilon(z)$ is constant, then $\delta_\varepsilon(z)$ is constant and it is obvious that $f(z)$ is uniformly continuous.

PROPOSITION 3. *An entire function is linear if and only if the function* $\delta_\varepsilon(z)$ *is constant.*

## References

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