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AN APPLICATION OF FUHRKEN'S REDUCTION TECHNIQUE

Hiromitsu OKAZAKI

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In his paper [2] (cf. also Bell and Slomson [1]), G. Fuhrken obtained several results concerning the existence of models for sentences in first order languages with an additional quantifier. In this paper we concern ourselves with the language L_Q which is formed by adding, to first order language L with equality, an additional quantifier (Qx) with the interpretation "for almost all". We shall show that a Fuhrken type theorem holds for our L_Q as well.

§ 1. The language L_Q

Consider the first order logic L with equality and countable lists of predicate symbols, of function symbols and of individual constant symbols. We form the language L_Q by adding to L a new quantifier (Qx) which is read "for almost all x". Thus L_Q has the three quantifiers $(\exists x)$, $(\forall x)$ and (Qx). The set of *formulas* of L_Q is the least Φ which contains all atomic formulas of L and has the property: If $\phi, \psi \in \Phi$ and y is a variable, then $\neg\phi$, $\phi \vee \psi$, $\phi \wedge \psi$, $(\exists y)\phi$, $(\forall y)\phi$, $(Qy)\phi \in \Phi$. We make the convention that $\phi(v_1, \ldots, v_n)$ denotes a formula of L_Q whose free variables form a subset of $\{v_1, \ldots, v_n\}$. *Sentences* are formulas without free variables. By a *weak model* for L_Q (Keisler [3]) we mean a pair $(\mathfrak{A}, \mathfrak{F})$ such that $\mathfrak{A}$ is a model for the first order language L and $\mathfrak{F}$ is a set of subsets of the universe $|\mathfrak{A}|$ of $\mathfrak{A}$, i.e. $F \subseteq S(|\mathfrak{A}|)$. The notion that an n-tuple $\langle a_1, \ldots, a_n \rangle$, $a_i \in |\mathfrak{A}|$. satisfies a formula $\phi(v_1, \ldots, v_n)$ of L_Q in $(\mathfrak{A}, \mathfrak{F})$ is defined by induction on the complexity of ϕ, and denoted by

$$(\mathfrak{A}, \mathfrak{F}) \models \phi[a_1, \ldots, a_n].$$

The (Qx) clause in the definition is:

$$(\mathfrak{A}, \mathfrak{F}) \models (Qv)\psi(v)\ [a_1, \ldots, a_n]$$

if and only if

$$\{b \in |\mathfrak{A}| \mid (\mathfrak{A}, \mathfrak{F}) \models \psi[b, a_1, \ldots, a_n]\} \in \mathfrak{F}.$$

The other cases in the definition are familar ones for L.

§ 2. Fuhrken's reduction technique

We associate with each formula ϕ of L_Q a formula ϕ' of L_Q, defined by the following recursion:

(A) If ϕ is atomic, $\phi'=\phi$ and $(\neg\phi)'=\neg\phi$;
(B) $(\neg\neg\phi)'=\phi'$;
(C) $(\phi\wedge\psi)'=\phi'\wedge\psi'$;
(D) $(\neg(\phi\wedge\psi))'=(\neg\phi)'\vee(\neg\psi)'$;
(E) $(\phi\vee\psi)'=\phi'\vee\psi'$;
(F) $(\neg(\phi\vee\psi))'=(\neg\phi)'\wedge(\neg\psi)'$;
(G) $((\exists v)\phi(v))'=(\exists v)(\phi(v))'$;
(H) $(\neg(\exists v)\phi(v))'=(\forall v)(\neg\phi(v))'$;
(I) $((\forall v)\phi(v))'=(\forall v)(\phi(v))'$;
(J) $(\neg(\forall v)\phi(v))'=(\exists v)(\neg\phi(v))'$;
(K) $((Qv)\phi(v))'=(Qv)(\phi(v))'$;
(L) $(\neg(Qv)\phi(v))'=(Qv)(\neg\phi(v))'$.

For each set Σ of sentences of L_Q, let

$$\Sigma'=\{\sigma'\mid\sigma\in\Sigma\}.$$

The following lemma is trivial.

LEMMA 1. *If $\neg\phi$ is a subformula of a sentence of Σ', ϕ is an atomic formula.*

LEMMA 2. *Let $(\mathfrak{A},\mathfrak{F})$ be a weak model and $\mathfrak{F}$ be an ultrafilter over $|\mathfrak{A}|$. Then*

$$(\mathfrak{A},\mathfrak{F})\models\Sigma \text{ iff } (\mathfrak{A},\mathfrak{F})\models\Sigma'.$$

PROOF. Let $\phi(v_1,\ldots,v_n)$ be a subformula of a sentence of Σ and $a_1,\ldots,a_n$ be elements of $|\mathfrak{A}|$. We shall show

(1) $$(\mathfrak{A},\mathfrak{F})\models\phi[a_1,\ldots,a_n] \text{ iff } (\mathfrak{A},\mathfrak{F})\models\phi'[a_1,\ldots,a_n]$$

by induction on the construction of ϕ. The only non-trivial step is (L). Assume that (1) holds for the formula $\neg\psi(v,v_1,\ldots,v_n)$ and $a_1,\ldots,a_n\in|\mathfrak{A}|$. Let

(2) $$T=\{c\in|\mathfrak{A}|\ |(\mathfrak{A},\mathfrak{F})\models \neg\psi[c,a_1,\ldots,a_n]\}.$$

Then by the induction hypothesis

$$T=\{c\in|\mathfrak{A}|\ |(\mathfrak{A},\mathfrak{F})\models (\neg\psi)'[c,a_1,\ldots,a_n]\}.$$

By the definition of satisfaction

$$(\mathfrak{A},\mathfrak{F})\models (Qv)\,(\neg\psi(v))'[a_1,\ldots,a_n] \text{ iff } T\in\mathfrak{F}.$$

On the other hand,

$$(\mathfrak{A},\mathfrak{F})\models \neg\,(Qv)\psi(v)\,[a_1,\ldots,a_n]$$

if and only if

$$\{c\in|\mathfrak{A}|\ |(\mathfrak{A},\mathfrak{F})\models \psi[c,a_1,\ldots,a_n]\}\notin\mathfrak{F}.$$

This means by (2)

$$(\mathfrak{A},\mathfrak{F})\models \neg(Qv)\psi(v)\,[a_1,\ldots,a_n] \text{ iff } T\in\mathfrak{F}.$$

Therefore by the definition of (L) and above results

$$(\mathfrak{A},\mathfrak{F})\models \neg(Qv)\psi(v)\,[a_1,\ldots,a_n]$$

if and only if

$$(\mathfrak{A},\mathfrak{F})\models (\neg(Qv)\psi(v))'[a_1,\ldots,a_n].$$ **q. e. d.**

We let L^* be the language obtained from L by adding a new binary predicate letter H. We associate with each formula ϕ of L_Q a formula ϕ^* of L^*, defined by the following recursion:

(1°) If ϕ is an atomic formula, $\phi^*=\phi$;
(2°) $(\neg\phi)^*=\neg\phi^*$;
(3°) $(\phi\wedge\psi)^*=\phi^*\wedge\psi^*$;
(4°) $(\phi\vee\psi)^*=\phi^*\vee\psi^*$;
(5°) $((\exists v)\phi(v))^*=(\exists v)\,(\phi(v))^*$;
(6°) $((\forall v)\phi(v))^*=(\forall v)\,(\phi(v))^*$;
(7°) $((Qv)\phi(v))^*=(\exists u)\,(\forall v)\,\{H(v,u)\rightarrow(\phi(v))^*\}$,

where u is the first variable occuring after all those variable in $(\phi(v))^*$ in the list of the variables.

Next we associate with each Σ of sentences L_Q a set Σ^* of sentences of L^*. Σ^* consists of all the following sentences:

(a) All the sentences σ^* for $\sigma \in \Sigma$;

(b) $(\forall v_0)(\exists v_1)H(v_1, v_0)$;

(c) $(\forall v_0)(\forall v_1)(\exists v_2)(\forall v_3)\ \{H(v_3, v_0) \wedge H(v_3, v_1) \leftrightarrow H(v_3, v_2)\}$.

LEMMA 3. *Let Σ be a countable set of sentences of L_Q and $\mathfrak{A}$ be an infinite structure. The following conditions are equivalent.*

(i) *There exists an ultrafilter $\mathfrak{F}$ over $|\mathfrak{A}|$ such that $(\mathfrak{A}, \mathfrak{F}) \models \Sigma'$.*

(ii) *There exists a binary relation R on $|\mathfrak{A}|$ such that $(\mathfrak{A}, R) = \Sigma'^*$.*

PROOF. (i)⇒(ii). Suppose that $\mathfrak{F}$ is an ultraflilter over $|\mathfrak{A}|$ such that $(\mathfrak{A}, \mathfrak{F}) \models \Sigma'$. Let $\mathscr{A}$ be the collection of all those set $S \in \mathfrak{F}$ such that

$$S = \{b \in |\mathfrak{A}| \ \ |(\mathfrak{A}, \mathfrak{F}) \models \psi[b, a_1, \ldots, a_n]\}$$

for some subformula $(Qv)\psi(v, v_1, \ldots, v_n)$ of a sentence in Σ' and some sequence $a_1, \ldots, a_n \in |\mathfrak{A}|$. Let $\mathscr{B}$ be the set of all finite intersections of the elements of $\mathscr{A}$. $\mathscr{B}$ is a subset of $\mathfrak{F}$ since $\mathfrak{F}$ is a filter. Clearly $\overline{\overline{\mathscr{B}}} \leq \overline{\overline{|\mathfrak{A}|}}$. Let f be a map of $|\mathfrak{A}|$ onto $\mathscr{B}$ and R be the binary relation defined on $|\mathfrak{A}|$ by

$$<b, a> \in R \text{ if } b \in f(a).$$

From the construction of R it is clear that $(\mathfrak{A}, R)$ is a model of sentences (b) and (c) above. We shall show that $(\mathfrak{A}, R)$ is a model of other sentences of Σ'^* by proving that for all subformula $\phi(v_1, \ldots, v_n)$ of sentences in Σ' and all $a_1, \ldots, a_n \in |\mathfrak{A}|$,

(3) $\qquad (\mathfrak{A}, \mathfrak{F}) \models \phi[a_1, \ldots, a_n]$ iff $(\mathfrak{A}, R) \models \phi^*[a_1, \ldots, a_n]$.

The proof is by induction on the number of longical symbols in $\phi(v_1, \ldots, v_n)$. Cleary (3) holds for the atomic formulas and for the formula $\neg\phi, \phi \wedge \psi, \phi \vee \psi$, $(\exists v)\phi$ and $(\forall v)\phi$ whenever (3) holds for ϕ and ψ. Assume that (3) holds for $\psi(v, v_1, \ldots, v_n)$ and $(\mathfrak{A}, \mathfrak{F}) \models (Qv)\psi(v)[a_1, \ldots, a_n]$. Let

(4) $\qquad S = \{a \in |\mathfrak{A}| \ \ |(\mathfrak{A}, \mathfrak{F}) \models \psi[a, a_1, \ldots, a_n]\}$,

then $S \in \mathfrak{F}$. There exists $a \in |\mathfrak{A}|$ such that

(5) $\qquad S = f(a) = R``\{a\}$.

On the other hand by the hypothesis, for $c \in S$

(6) $\qquad (\mathfrak{A}, \mathfrak{F}) \models \psi[c, a_1, \ldots, a_n]$ iff $(\mathfrak{A}, R) \models \psi^*[c, a_1, \ldots, a_n]$.

By (4), (5) and (6), it follows that

$$(\mathfrak{A}, \boldsymbol{R}) \models (\exists u)(\forall v)\{H(v,u) \to (\psi(v))^*\}[a_1, \ldots, a_n],$$

that is

$$(\mathfrak{A}, \boldsymbol{R}) \models ((Qv)\psi(v))^*[a_1, \ldots, a_n].$$

The converse direction of (3) is proved similarly.

(ii)⇒(i). Assume that $\boldsymbol{R}$ is a binary relation on $|\mathfrak{A}|$ such that $(\mathfrak{A}, \boldsymbol{R}) \models \Sigma'^*$. Since $(\mathfrak{A}, \boldsymbol{R})$ is a model of the sentences (b) and (c), $\{\boldsymbol{R}``\{a\} \mid a \in |\mathfrak{A}|\}$ has the finite intersection property. Let $\mathfrak{F}$ be an ultrafilter containing it. We shall prove that $(\mathfrak{A}, \mathfrak{F}) \models \Sigma'$ by showing that

(7) $$(\mathfrak{A}, \mathfrak{F}) \models \phi[a_1, \ldots, a_n] \text{ if } (\mathfrak{A}, \boldsymbol{R}) \models \phi^*[a_1, \ldots a_n]$$

for any subformula $\phi(v_1, \ldots, v_n)$ of some sentence in Σ' and $a, \ldots, a_n \in |\mathfrak{A}|$. Again the proof will be curried out by induction on the number of the logical symbols in ϕ. If ϕ is an atomic formula, clearly we have

$$(\mathfrak{A}, \mathfrak{F}) \models \phi[a_1, .., a_n] \text{ iff } (\mathfrak{A}, \boldsymbol{R}) \models \phi^*[a_1, \ldots, a_n].$$

If $\neg\psi(v_1, \ldots, v_n)$ is a subformula of a sentence of Σ', ψ is an atomic formula by the lemma 1. Therefore

$$(\mathfrak{A}, \mathfrak{F}) \models \neg\psi[a_1, \ldots, a_n] \text{ iff } (\mathfrak{A}, \boldsymbol{R}) \models (\neg\psi)^*[a_1, \ldots, a_n].$$

We shall show that (7) holds also for $(Qv)\psi(v, v_1, \ldots, v_n)$.
Suppose that

$$(\mathfrak{A}, \boldsymbol{R}) \models ((Qv)\psi(v))^*[a_1, \ldots, a_n].$$

This means

$$(\mathfrak{A}, \boldsymbol{R}) \models (\exists u)(\forall v)\{H(v,u) \to (\psi(v))^*\}[a_1, \ldots, a_n].$$

Then there exists $a \in |\mathfrak{A}|$ such that

$$\boldsymbol{R}``\{a\} \subseteq \{c \mid (\mathfrak{A}, \boldsymbol{R}) \models \psi^*[c, a_1, \ldots, a_n]\}.$$

By the induction hypothesis,

$$\boldsymbol{R}``\{a\} \subseteq \{c \mid (\mathfrak{A}, \mathfrak{F}) \models \psi[c, a_1, \ldots, a_n]\}.$$

From the definition of the ultrafilter $\mathfrak{F}, \boldsymbol{R}``\{a\} \in \mathfrak{F}$. Therefore

$$(\mathfrak{A}, \mathfrak{F}) \models (Qv)\psi[a_1, \ldots, a_n].$$

The other cases are trivial. This complete the proof of (7) and hence that of the lemma 3. **q. e. d.**

By the lemma 2 and the lemma 3, we have the following theorem.

THEOREM 1. *Let Σ be a countable set of sentences of L_Q and $\mathfrak{A}$ be an infinite structure. There exists an ultrafilter $\mathfrak{F}$ over $|\mathfrak{A}|$ such that $(\mathfrak{A}, \mathfrak{F}) \models \Sigma$ iff there exists a binary relation $\boldsymbol{R}$ on $|\mathfrak{A}|$ such that $(\mathfrak{A}, \boldsymbol{R}) \models \Sigma'^{*}$.*

The following theorem is proved similarly.

THEOREM 2. *Let Σ be a countable set of sentences of L_Q and $\mathfrak{A}$ be an infinite structure. There exists a filter $\mathfrak{G}$ over $|\mathfrak{A}|$ such that $(\mathfrak{A}, \mathfrak{G}) \models \Sigma$ iff there exists a binary relation $\boldsymbol{P}$ on $|\mathfrak{A}|$ such that $(\mathfrak{A}, \boldsymbol{P})) \models \Sigma^{*}$.*

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Department of Mathematics,
Faculty of Education,
Kumamoto University