Kumamoto J. Sci. (Math.)
Vol. 12, 23-27 (1976)

A REMARK ON THE INDUDTION THEOREM FOR MODULAR CHARACTERS

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(Received October 31, 1975)

Let G be a finite group and $\mathfrak{E}(G)$ be the family of elementary subgroups of G. Brauer's induction theorem with its converse given by Green [3] reads as: Let $\mathfrak{H}$ be a family of subgroups of G. Then the followings are equivalent to each other.

(A) For any $E\in\mathfrak{E}(G)$, some member $H\in\mathfrak{H}$ contains a subgroup which is conjugate (in G) to E.

(B) Any character of G can be expressed as a linear combination of characers induced from characters of the subgroups $H\in\mathfrak{H}$ with rational integral coefficients.

(C) A class function ψ on G is a generalized character of G if the restriction $\psi_{|_H}$ of ψ to H is a generalized character, for every $H\in\mathfrak{H}$.
Further, an analogue of the induction theorem was established for modular characters for p, p a prime number (R. Brauer [1], Theorem 5 and J. P. Serre [5], Theorem 39).

Recently, W. F. Reynolds, in §7 of [4], has worked relating to the above. We shall describe the outline of his results in order to show our aim exactly. Let $\mathscr{V}_G$ be the character ring of G over the field C of comlex numbers and $\mathscr{I}_G$ be the character ring of G over the ring Z of rational integers. For a subset $\mathscr{X}$ of $\mathscr{V}_G$ and for a subset S of G, we denote $\{\theta\in\mathscr{X}\mid\theta(g)=0 \text{ if } g\notin S\}$ by $\mathscr{X}(S)$. Let π be a fixed set of prime numbers. We put $V_G=\mathscr{V}_G(G_{\pi'})$, $P_G=\mathscr{I}_G(G_{\pi'})$ and $M_G=\{\theta\in V_G\mid(\theta,\zeta)_G\in Z \text{ for all } \zeta\in P_G\}$, where $G_{\pi'}$ is the set of all π'-elemets of G and where $(\theta,\zeta)_G$ means the usual inner product $1/|G|\sum_{g\in G}\theta(g)\overline{\zeta(g)}$.

I$_0$ (Theorem 7.3 in [4]) *Let $\mathfrak{E}_0(G,\pi)$ be the subfamily of $\mathfrak{E}(G)$ which consists of all $E=\langle x\rangle\times Q$ with $x\in G_{\pi'}$. If $\mathfrak{H}$ is a family of subgroups of G, the followings are equivalent.*

(a_0) $\mathfrak{E}_0(G,\pi)\underset{G}{\leqslant}\mathfrak{H}$, *that is, any $E\in\mathfrak{E}_0(G,\pi)$ is conjugate to a subgroup contained in some $H\in\mathfrak{H}$.*

(b_0) $\{\sum_{H\in\mathfrak{G}}(\zeta_H)^G \mid \zeta_H\in M_H\}=M_G.$

(c_0) $\{\theta\in V_G \mid \theta|_H\in P_H$ *for all* $H\in\mathfrak{H}\}=P_G.$

$\mathbf{II}_0$ (Theorem 7.4 in [4] *Let* $\mathfrak{F}_0(G,\pi)$ *be a subfamily of* $\mathfrak{E}_0(G,\pi)$*, which consists of all* $E\in\mathfrak{E}(G)$ *with* $E\subseteq G_{\pi'}$*, and let* $\mathfrak{H}$ *be a family of subgroups of* G *such that*

(d_0) $\mathfrak{F}_0(G,\pi)\underset{G}{\leqslant}\mathfrak{H}.$

Then

(e_0) $\{\sum_{H\in\mathfrak{H}}(\theta_H)^G \mid \theta_H\in P_H\}=P_G.$

(f_0) $\{\zeta\in V_G \mid \zeta|_H\in M_H$ *for all* $H\in\mathfrak{H}\}=M_G.$

Let x be a π-element of G and $i^{\pi}_{x,G}$ be the bijection of $V_{C_G(x)}$ onto $\mathscr{V}_G(S(x))$ defined by $(i^{\pi}_{x,G}\alpha)(xy)=\alpha(y)$, $\alpha\in V_{C_G(x)}$, $y\in(C_G(x))_{\pi'}$, where $S(x)$ is the π-section containing x. For a complete system of representatives $x_1, x_2, \cdots$ for the conjugate classes of π-elements in G, we put

$$\mathscr{P}_G=\oplus\sum_m i^{\pi}_{x_m,G}(P_{C_G(x_m)}),\quad \mathscr{M}_G=\oplus\sum_m i^{\pi}_{x,G}(M_{C_G(x_m)}).$$

I (Theorem 7.1 in [4]) *If* $\mathfrak{H}$ *is a family of subgroups of* G*, the followings are equivalent.*

(a) $\mathfrak{E}(G)\underset{G}{\leqslant}\mathfrak{H}.$

(b) $\{\sum_{H\in\mathfrak{H}}(\zeta_H)^G \mid \zeta_H\in\mathscr{M}_H\}=\mathscr{M}_G.$

(c) $\{\theta\in\mathscr{V}_G \mid \theta|_H\in\mathscr{P}_H$ *for all* $H\in\mathfrak{H}\}=\mathscr{P}_G.$

II (Theorem 7.2 in [4]) *Let* $\mathfrak{F}(G,\pi)$ *be a subfamlly of* $\mathfrak{E}(G)$*, which consists of all* $E=\langle x\rangle\times Q$ *with* $Q\subseteq G_{\pi'}$ *and let* $\mathfrak{H}$ *be a family of subgroups of* G *such that*

(d) $\mathfrak{F}(G,\pi)\underset{G}{\leqslant}\mathfrak{H}.$

Then

(e) $\{\sum_{H\in\mathfrak{H}}(\theta_H)^G \mid \theta_H\in\mathscr{P}_H\}=\mathscr{P}_G,$

(f) $\{\zeta\in\mathscr{V}_G \mid \zeta|_H\in\mathscr{M}_H$ *for all* $H\in\mathfrak{H}\}=\mathscr{M}_G.$

Our aim is to give an improvement of $\mathbf{II}_0$ and $\mathbf{II}$, which has the same form as $\mathbf{I}_0$ and $\mathbf{I}$:

$\mathbf{II}'_0$ *Let* $\mathfrak{H}$ *be a family of subgroups of* G. *The followings are equivalent.*

(d_0) $\mathfrak{F}_0(G,\pi) \underset{G}{\leqslant} \mathfrak{H}$.

(e_0) $\{\sum_{H\in\mathfrak{H}}(\theta_H)^G \mid \theta_H \in P_H\} = P_G$.

(f_0) $\{\zeta\in V_G \mid \zeta_{|_H} \in M_H$ *for all* $H\in\mathfrak{H}\} = M_G$.

$\mathbf{II}'$ *Let* $\mathfrak{H}$ *be a family of subgroups of* G. *The followings are equivalent.*

(d) $\mathfrak{F}(G,\pi) \underset{G}{\leqslant} \mathfrak{H}$.

(e) $\{\sum_{H\in\mathfrak{H}}(\theta_H)^G \mid \theta_H \in \mathscr{P}_H\} = \mathscr{P}_G$.

(f) $\{\zeta\in \mathscr{V}_G \mid \zeta_{|_H} \in \mathscr{M}_H$ *for all* $H\in\mathfrak{H}\} = \mathscr{M}_G$.

Some preliminaly results

For the completeness, we refer some results. Let ε_G^π be the characteristic function of the subset $G_{\pi'}$ of G. It is easy to see from Frobenius' theorem (C. W. Curtis and I. Reiner [2], Corollary (41.9)) that $|G|_\pi \varepsilon_G^\pi \in P_G$, where $|G|_\pi$ is the π-part of the order $|G|$ of G. Since $|G|_\pi \mathscr{I}_G \varepsilon_G^\pi \subseteq P_G \subseteq \mathscr{I}_G \varepsilon_G^\pi$, the Z-rank of P_G is equal to that of $\mathscr{I}_G \varepsilon_G^\pi$. We have

$$\mathscr{V}_G = C\otimes_Q(Q\otimes_Z \mathscr{I}_G) = C\otimes_Q[(Q\otimes_Z\mathscr{I}_G)\,\varepsilon_G^\pi] \oplus C\otimes_Q[(Q\otimes_Z\mathscr{I}_G)(1-\varepsilon_G^\pi)].$$

Since $C\otimes_Q[(Q\otimes_Z\mathscr{I}_G)\varepsilon_G^\pi]$ and $C\otimes_Q[(Q\otimes_Z\mathscr{I}_G)(1-\varepsilon_G^\pi)]$ are contained in $\mathscr{V}_G\varepsilon_G^\pi$ and $\mathscr{V}_G(1-\varepsilon_G^\pi)$, respectively, we have $V_G = \mathscr{V}_G\varepsilon_G^\pi = C\otimes_Q[(Q\otimes_Z\mathscr{I}_G)\varepsilon_G^\pi]$. Hence we obtain the following: P_G is an algebra and a free Z-module of rank $l(G,\pi)$, where $l(G,\pi)$ is the number of conjugate classes of π'-elements in G. Any Z-basis $\{\Phi_j \mid 1\leqslant j\leqslant l(G,\pi)\}$ of P_G is a C-basis of V_G. (M. Suzuki [6], Theorem 2) Therefore, for each $j\,(1\leqslant j\leqslant l(G,\pi))$, there exists $\phi_j\in V_G$ such that $(\phi_j, \Phi_h)_G = \delta_{jh}\,(1\leqslant h\leqslant l(G,\pi))$, and $\{\phi_j \mid 1\leqslant j\leqslant l(G,\pi)\}$ form a Z-basis of M_G. For a family $\mathfrak{H}$ of subgroups of G, we put $P_{\mathfrak{H}}^G = \{\sum_{H\in\mathfrak{H}}(\theta_H)^G \mid \theta_H\in P_H\}$ and put ${}_GM_{\mathfrak{H}} = \{\zeta\in V_G \mid \zeta_{|_H} \in M_H$ for all $H\in\mathfrak{H}\}$, then we have $\{\zeta\in V_G \mid (\zeta,\theta)_G\in Z$ for all $\theta\in P_{\mathfrak{H}}^G\} = {}_GM_{\mathfrak{H}}$. From the above and the invariant factor theorem for modules we can see that (e_0) and (f_0) ((e) and (f)) are equivalent.

Proof of II_0' To prove II_0', we have to show (e_0) yields (d_0). We assume (e_0) for a family $\mathfrak{H}$ of subgroups of G. We can write $|G|_\pi \varepsilon_G^\pi = \sum_{H \in \mathfrak{H}} (\theta_H)^G$, $\theta_H \in P_H$. Let $E \in \mathfrak{F}_0(G, \pi)$ and $E = \langle x \rangle \times Q$, where Q is a q-group and $\langle x \rangle$ is q'-group for some prime number q, $q \notin \pi$. Since $|G|_\pi \varepsilon_G^\pi(x) = |G|_\pi$, a π-number, and every $(\theta_H)^G(x)$ is an algebraic integer, there exists $H \in \mathfrak{H}$ such that $(\theta_H)^G(x)$ is not divisible by q. Let $y_1, y_2, \cdots$ be a complete system of representatives for the double cosets of G with respect to $C_G(x)$, H. We have $(\theta_H)^G(x) = \sum_{x^{y_j} \in H} |C_G(x)|/|C_G(x) \cap H^{y_j^{-1}}| \; \theta_H(x^{y_j})$. Since every $\theta_H(x^{y_j})$ is an algebraic integer, there exists y_j such that $x^{y_j} \in H$ and that $|C_G(x)|/|C_G(x) \cap H^{y_j^{-1}}|$ is not divisible by q. Hence, some conjugate of any q-Sylow subgroup of $C_G(x)$ is contained in $C_G(x) \cap H^{y_j^{-1}}$, so Q^u is contained in $H^{y_j^{-1}}$ for some $u \in C_G(x)$. Thus we see $(\langle x \rangle \times Q)^u = \langle x \rangle \times Q^u \subseteq H^{y_j^{-1}}$. This completes the proof.

Proof of II' Let x be an arbitrarily given π-element of G. For $\tilde{H} \in \mathfrak{F}_0(C_G(x), \pi)$ and for $\alpha_{\tilde{H}} \in P_{\tilde{H}}$, we have $\alpha_{\tilde{H}}^{\langle x \rangle \times \tilde{H}} \in P_{\langle x \rangle \times \tilde{H}}$ and

$$(1) \qquad i^\pi_{x,G}(\alpha_{\tilde{H}}^{C_G(x)}) = (i^\pi_{x, \langle x \rangle \times \tilde{H}}(\alpha_{\tilde{H}}^{\langle x \rangle \times \tilde{H}}))^G.$$

Let H be a subgroup of G which contains x. For $\alpha_H \in P_{C_H(x)}$, we have

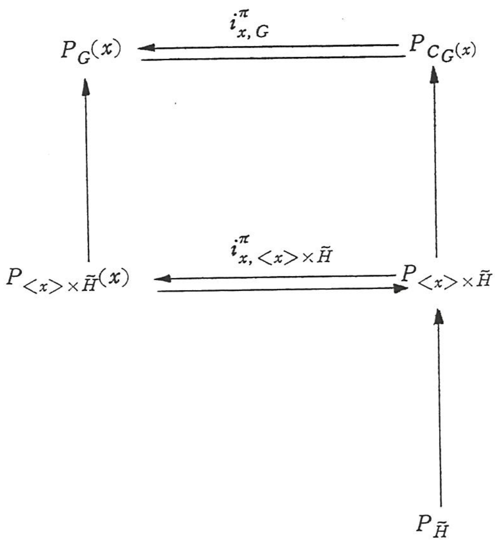
$$(2) \qquad (i^\pi_{x,H}(\alpha_H))^G = i^\pi_{x,G}((\alpha_H)^{C_G(x)}).$$

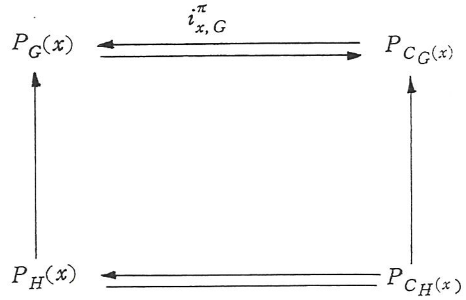
We put $P_G(x) = i^\pi_{x,G}(P_{C_G(x)})$, $P_{\mathfrak{H}}^G(x) = \{\sum_{H \in \mathfrak{H}} (\theta_H)^G \mid \theta_H \in P_H(x)\}$ for a family $\mathfrak{H}$ of subgroups of G which contains x. From (1) and (2) and II_0' we obtain the following: *Let $\mathfrak{H}$ be a family of subgroups of G which contains x. The followings are equivalent.*

(d_x) $\{\langle x \rangle \times \tilde{H} \mid \tilde{H} \in \mathfrak{F}_0(C_G(x), \pi)\} \underset{C_G(x)}{\leqslant} \mathfrak{H}$.

(e_x) $P_{\mathfrak{H}}^G(x) = P_G(x)$.

From this we can see easily that (d) and (e) are equivalent. This finishes the proof of II'.





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