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Lumped mass approximation to the nonlinear bending of elastic plates

Tetsuhiko MIYOSHI

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Preface

In this paper we study some approximate properties of a mixed finite element scheme applied to a boundary value broblem arising in the analysis of nonlinear bending of elastic plates.

The method presented here is an ultimate extension of the one which has been analyzed in the author's paper [3], and the outline of the theory was already announced in [4]. The purpose of the present paper is thus to give the details of the theory.

Especially, we want to give a rigorous proof about the existence and convergence of the approximate solutions. Our scheme can be regarded also as a generalized finite difference scheme more clearly than the usual finite element approximation, so that the usual Hilbert space approach meets a considerable difficulty for our scheme. One way to avoid this difficulty is to devide the analysis into two steps. First we study the consistent mass scheme, which was the one treated in [3], and then we regard our scheme as an approximate scheme to this. In the last section we report a numerical example obtained by this method.

1. Approximate scheme

Consider a thin elastic plate of arbitrary shape subjected to a lateral load g. Let Ω be a bounded region of x_1, x_2 – plane which represents the shape of the plate. Then the system of equations

$$(1.1)\qquad \begin{cases} \Delta^2 f = -[w, w] \\ \Delta^2 w = [f, w] + g \end{cases}$$

is known as a mathematical model of the nonlinear bending of this plate, where w and f correspond to the normal deflection and Airy's stress function respectively. Here, $[f, w]$ denotes the following quadratic term:

$$[f, w] = D_{11}fD_{22}w + D_{22}fD_{11}w - 2D_{12}fD_{12}w.$$

Our aim is to solve the system (1.1) under the boundary condition

$$w=\frac{dw}{dn}=f=\frac{df}{dn}=0 \quad \text{on } \partial\Omega,$$

where $\partial\Omega$ denotes the boundary of Ω, and n is the unit outward normal to $\partial\Omega$. We assume through the present paper that the boundary $\partial\Omega$ and the load g are sufficiently smooth, so that the equation (1.1) has a sufficiently smooth solution (see, for instance, [1], [2] about the existence and the smoothness of the solution).

In order to set up our scheme we introduce two important spaces L_2 and H. Let $W_\alpha^k(\Omega)$ (k: positive integer, $\alpha>1$) be the Sobolev space of functions. Let $\mathring{W}_2^1(\Omega)$ be the completion of the space of all C^∞-functions with support in Ω in the norm

$$|u|_1^2=\sum_{|\alpha|=1}\int|D^\alpha u|^2dx_1dx_2.$$

L_2 is the product space $\mathring{W}_2^1\times L_2(\Omega)\times L_2(\Omega)\times L_2(\Omega)$ with the norm

$$\|W\|_{L_2}^2=|w|_1^2+\sum_{i\leq j}\|W_{ij}\|_{L_2}^2,$$

and H is its subspace $\mathring{W}_2^1\times W_2^1\times W_2^1\times W_2^1$, but the norm is changed to

$$\|W\|_H^2=|w|_1^2+\sum_{i\leq j}\|W_{ij}\|_{W_2^1}^2,$$

where $W=(w, W_{11}, W_{12}, W_{22})$.

Let us define a bilinear form:

$$L(W,\Phi)=\sum_{i\leq j}\{(D_jw, D_i\Phi_{ij})_{L_2}+(W_{ij},\Phi_{ij})_{L_2}\}+\sum_{i,j}(D_iW_{ij}, D_j\phi)_{L_2}$$

for $W,\Phi\in H$, where $W_{12}=W_{21}$, $\Phi_{12}=\Phi_{21}$.

DEFINITION. A pair $(F, W)\in H\times H$ is called a weak solution of the equation (1.1), if

$$\text{(1.2)}\qquad \begin{cases} L(F,\Phi)=([W,W],\phi)_{L_2} & \text{for all } \Phi\in H,\\ L(W,\Phi)+([F,W],\phi)_{L_2}+(g,\phi)_{L_2}=0 & \text{for all } \Phi\in H.\end{cases}$$

This definition is, as stated in [3], based on the following inequality assured by the Sobolev's imbedding theorem.

$$|([F,W],\phi)_{L_2}|\leq c\|F\|_{L_2}\|W\|_H|\phi|_1.$$

An abstract representation of (1.2) introduced in [3] is as follows. Let the operators L, C and B be defined by

$$L(W,\Phi)=(LW,\Phi)_H \qquad \text{for all } \Phi\in H,$$
$$([F,W],\phi)_{L_2}=(C(F,W),\Phi)_H \qquad \text{for all } \Phi\in H,$$
$$(g,\phi)_{L_2}=(Bg,\Phi)_H \qquad \text{for all } \Phi\in H.$$

Since L is invertible on $H_{1+\varepsilon}=(W^2_{1+\varepsilon}\cap \mathring{W}^1_2)\times\{0\}\times\{0\}\times\{0\}(\varepsilon>0)$ and $C(F,W)$ belongs to H_{1+1} for any $F,W\in H$, we have the following single operator equation as an abstract version of (1.2).

(1.3) $$W+C(W)+L^{-1}Bg=0,$$

where

$$C(W)=L^{-1}C(L^{-1}C(W,W),W).$$

In solving the equation (1.3), we give up to seek solutions at which singular phenomena (like bifurcation, snap through) occur.

To define the singularity more precisely, take $W_0, W_1\in H$ and set $Z=W_1-W_0$. Then we can write

$$C(W_1)-C(W_0)=C'_{(W_0)}Z+D(W_0,Z),$$

where

$$C'_{(W_0)}Z=L^{-1}C(L^{-1}C(W_0,W_0),Z)+2L^{-1}C(L^{-1}C(W_0,Z),W_0)$$

and $D(W_0,Z)$ is a nonlinear term of third order nonlinearlity in Z. The operator $C'_{(W_0)}$ defined on H can be extended to whole L_2 as a compact operator [3]. In what follows we fix a solution W_0 of (1.3) and assume that the equation

(1.4) $$KZ=(I+C'_{(W_0)})Z=0$$

has no nontrivial solution. In this sense, the solution to be approximated by our scheme is well behaved one.

Finite element subspaces: Let $\Omega_h(h>0)$ be a triangulation of Ω. We assume Ω_h is a closed subregion of Ω satisfying the following conditions.

(1) Any vertex of a triangle does not lie part way along the side of another.

(2) Adjecent nodes on $\partial\Omega_h$ do not lie together in Ω. If two adjecent nodes on $\partial\Omega_h$ are both on $\partial\Omega$, then the boundary $\partial\Omega$ must be nonconcave between them. If p, q and r are serial nodes on $\partial\Omega_h$, being q in Ω, then the boundary $\partial\Omega$ must contain a concave part between p and r. The length of the perpendicular from q to the line segment connecting the boundary nodes on $\partial\Omega$ does not exceed $O(h^2)$

as $h\to 0$, being h the largest side length of all triangles in Ω_h.

(3) The ratio of the smallest and the largest sides of all triangles of Ω_h is bounded below by a positive constant as $h\to 0$.

(4) There is a closed subregion Ω_h' of Ω_h which is composed of square meshes of side length $\bar{h}$, and the number of grids in $\Omega_h-(\Omega_h')_{\text{interior}}$ is of order $O(h^{-1})$ as $h\to 0$. Each square in Ω_h' is triangulated by the diagonal of north-east direction.

Let $\{\hat{\phi}_p\}$ be the piecewise linear finite element basis belonging to $W_2^1(\Omega_h)$ and satisfying $\hat{\phi}_p=1$ at the node p and $=0$ at all other nodes. For rigorous theoretical treatment of our method, we extend $\hat{\phi}_p$ to the skin $\Omega-\Omega_h$ and regard it as a function in the space $W_2^1(\Omega)$—this extension is unnecessary for the actual computation. If p lies in the interior of the domain Ω_h, there is no problem. In this case it is only necessary to define $\hat{\phi}_p\equiv 0$ in $\Omega-\Omega_h$. When p is on $\partial\Omega_h$, there will be many ways of extension. One way, and possiblly most convenient one will be such extension as to satisfy $d\hat{\phi}/d\nu=0$, being ν the direction of the perpendicular to the line segment connecting the two boundary nodes on $\partial\Omega$. In what follows, $\{\hat{\phi}_p\}$ denotes the piecewise linear finite element basis extended to whole Ω in this way.

Corresponding to each $\hat{\phi}_p$ we define a piecewise constant function $\bar{\phi}_p$ as follows. Let $T_{p,k}$ $(k=1,2,\ldots, K_p)$ be the set of all triangles in Ω_h with vertex p. Let $Q_{p,k}$ $(\subset T_{p,k})$ be the quadrilateral obtained by connecting the vertex p, middle points of the two sides containing p and the center of the gravity of $T_{p,k}$. Then $\bar{\phi}_p$ is the characteristic function of the region

$$U_p=\sum_{k=1}^{K_p} Q_{p,k}.$$

Let be defined

$\hat{S}_0$: subspace of $\mathring{W}_2^1(\Omega)$ spanned by $\{\hat{\phi}_p;\ p\in\Omega_h-\partial\Omega_h\}$,

$\hat{S}_1$: subspace of $W_2^1(\Omega)$ spanned by $\{\hat{\phi}_p;\ p\in\Omega_h\}$,

$\bar{S}$: linear space spanned by $\{\bar{\phi}_p;\ p\in\Omega_h\}$,

$\hat{H}$: $=\hat{S}_0\times\hat{S}_1\times\hat{S}_1\times\hat{S}_1$ (subspace of H).

Finite element schemes : Let us introduce the following bilinear form on $\hat{H}\times\hat{H}$.

$$\bar{L}(\hat{W},\hat{\Phi})=\sum_{i\leq j}\{(D_j\hat{W},D_i\hat{\Phi}_{ij})_{L_2}+(\bar{W}_{ij},\bar{\Phi}_{ij})_{L_2}\}$$
$$+\sum_{i,j}(D_i\hat{W}_{ij},D_j\hat{\phi})_{L_2},$$

where $\overline{W}_{ij}$ denotes the function which belongs to $\bar{S}$ and agrees with $\hat{W}_{ij}$ at all nodes in Ω_h (we define $\overline{W}_{ij}\equiv 0$ outside of Ω_h).

The approximating scheme analyzed in [3] is

$$(C)\quad \begin{cases} L(\hat{F},\hat{\Phi}) = ([\hat{W},\hat{W}],\hat{\phi})_{L_2} & \text{for all } \hat{\Phi}\in\hat{H}, \\ L(\hat{W},\hat{\Phi}) + ([\hat{F},\hat{W}],\hat{\phi})_{L_2} + (g,\hat{\phi})_{L_2}=0 & \text{for all } \hat{\Phi}\in\hat{H}, \end{cases}$$

where $(\hat{F},\hat{W})\in\hat{H}\times\hat{H}$.

The scheme proposed here is

$$(L)\quad \begin{cases} \bar{L}(\hat{F},\hat{\Phi}) = ([\overline{W},\overline{W}],\bar{\phi})_{L_2} & \text{for all } \hat{\Phi}\in\hat{H}, \\ \bar{L}(\hat{W},\hat{\Phi}) + ([\bar{F},\overline{W}],\bar{\phi})_{L_2} + (g,\bar{\phi})_{L_2}=0 & \text{for all } \hat{\Phi}\in\hat{H}. \end{cases}$$

where $[\bar{F},\overline{W}]=\bar{F}_{11}\overline{W}_{22}+\bar{F}_{22}\overline{W}_{11}-2\bar{F}_{12}\overline{W}_{12}$. Note that this scheme is exactly the 13-points finite difference scheme in the interior of Ω'_h. Our problem in this paper is to study whether this scheme can give reasonable solutions. Since this equation can not be treated under the same frame work as for (C) — because $\bar{L}(\ ,\ )$ etc. are not well defined in H —, we have to change the point of view in this case.

In [3] we represented the system (C) in the form

$$\hat{L}\hat{W} + PC(\hat{L}^{-1}PC(\hat{W},\hat{W}),\hat{W}) + PBg=0,$$

where $\hat{L}=PLP$, being P the projection $H\rightarrow\hat{H}$. Let us regard the equation (C) as an *original equation* defined only on $\hat{H}$, then we can write this as

$$(1.5)\qquad \hat{L}\hat{W} + \hat{C}(\hat{L}^{-1}\hat{C}(\hat{W},\hat{W}),\hat{W}) + \hat{B}g=0.$$

In this expression, the operators $\hat{L}$, $\hat{C}$ and $\hat{B}$ work, of course, only in the space $\hat{H}$, and have the same characters as the previous ones.

The bilinear form $\bar{L}(\ ,\ )$ is now well defined on $\hat{H}\times\hat{H}$ and, by the same reason as for $\hat{L}$, can be represented by a bounded operator, say by $\bar{L}$:

$$\bar{L}(\hat{F},\hat{\Phi}) = (\bar{L}\hat{F},\hat{\Phi})_{\hat{H}} \qquad \text{for all } \hat{\Phi}\in\hat{H}.$$

In order to represent the nonlinear term, we have to prove the following inequality.

LEMMA 1. Assume $p\geqslant 2$, then for any $\hat{u}$ in $\hat{S}_1$ holds

$$(1.6)\qquad \|\hat{u}\|_{L_p(\Omega)}\leq c_1\|\bar{u}\|_{L_p(\Omega)}\leq c_2\|\hat{u}\|_{L_p(\Omega)},$$

where c_1, c_2 are constants independent of $\hat{u}, h, p$.

PROOF. The first inequality follows from the facts that

$$\|\hat{u}\|_{L_p(\Omega_\delta)} \leqq ch^{\frac{1}{p}} \|\bar{u}\|_{L_p(\Omega_h)} \qquad (\Omega_\delta = \Omega - \Omega_h),$$

$$\|\hat{u}\|_{L_p(\Omega_h)} \leqq c\|\bar{u}\|_{L_p(\Omega_h)}$$

for $p>1$. To prove the second one, we recall the inverse inequality

$$\underset{e}{Max}\, |\hat{u}| \leqq \frac{c}{h} \|\hat{u}\|_{L_2(e)},$$

where e is any triangular or curved element. Squaring the both sides and integrating on e we have

$$\int_e \hat{u}^2 \leqq mes(e)^{\frac{p-2}{p}} \Big[\int_e |\hat{u}|^p\Big]^{\frac{2}{p}}.$$

Substituting this result into the above inequality, we have

$$\underset{e}{Max}\, |\bar{u}|^p \leqq ch^{-p}\, mes(e)^{\frac{p-2}{2}} \int_e |\hat{u}|^p$$

$$\leqq ch^{-2} \int_e |\hat{u}|^p.$$

Therefore we have

$$\int_e |\bar{u}|^p \leqq c\underset{e}{Max} |\bar{u}|^p h^2 \leqq c\int_e |\hat{u}|^p,$$

which proves the second inequality.

REMARK. More straight but not elemetally proof is the use of Sobolev's integral inequality. In this case, the condition $p \geqslant 2$ can be replaced by $p>1$.

Thanks to this lemma, we have the estimate for the nonlinear term:

$$|([\bar{F}, \bar{W}], \bar{\phi})_{L_2}| \leqq c\|\hat{F}\|_{L_2} \|\hat{W}\|_H |\hat{\phi}|_1.$$

Therefore, just the same as in the consistent case, for fixed $\hat{F}$ and $\hat{W}$ there is a unique $\bar{C}(\hat{F}, \hat{W}) \in \hat{H}$ such that

$$([\bar{F}, \bar{W}], \bar{\phi})_{L_2} = (\bar{C}(\hat{F}, \hat{W}), \hat{\Phi})_{\hat{H}} \qquad \text{for all } \hat{\Phi} \in \hat{H}.$$

Since the invertibility of $\bar{L}$ is easily proved, our equation (L) too can be expressed by the similar form as for (C):

$$(1.7) \qquad \bar{L}\hat{W} + \bar{C}(\bar{L}^{-1}\bar{C}(\hat{W}, \hat{W}), \hat{W}) + \hat{B}g = 0.$$

Once the discrete system is represented in this form, the analysis can be given by the similar method for (C). In what follows we shall describe the details of this process.

2. Some results from linear problems

Let us consider the equation

$$(2.1)\qquad \bar{L}(\hat{W},\hat{\Phi})=(g,\bar{\phi})_{L_2} \qquad \text{for all } \hat{\Phi}\in\hat{H}.$$

This equation is, of course, the lumped approximation to the original equation $\Delta^2 w=g$. Now, first we want to prove the following theorem.

THEOREM 1. Let w be the exact solution of the Dirichlet problem of the biharmonic equation for the loard term $g\in L_2(\Omega)$ and $\hat{W}$ the solution of (2.1). Then we have the following error estimate.

$$|w-\hat{w}|_1,\ \|D_{ij}w-\hat{W}_{ij}\|_{L_2}\leqq ch^{\frac{1}{2}}\|g\|_{L_2}.$$

PROOF. We use the notations $w_{ij}, \hat{w}, \hat{w}_{ij}$ and $\bar{w}_{ij}$ to express $D_{ij}w$, consistent interpolate of w, consistent interpolate of $D_{ij}w$ and lumped interpolate of $D_{ij}w$, respectively.

We start from the following identities.

$$(2.2)\qquad \begin{aligned} &\sum_{i,j}\{(D_jw,D_i\hat{\Phi}_{ij})+(\bar{w}_{ij},\bar{\Phi}_{ij})\}\\ &\qquad +\sum_{i,j}\{(w_{ij},\hat{\Phi}_{ij})-(\bar{w}_{ij},\bar{\Phi}_{ij})\}\\ &=\sum_{i,j}\{(D_j\hat{w},D_i\hat{\Phi}_{ij})+(\bar{W}_{ij},\bar{\Phi}_{ij})\} \qquad \text{for any } \hat{\Phi}\in\hat{H}, \end{aligned}$$

$$(2.3)\qquad \begin{aligned} &\sum_{i,j}\{(D_iw_{ij},D_j\hat{\phi})+(g,\hat{\phi})\}\\ &=\sum_{i,j}\{(D_i\hat{W}_{ij},D_j\hat{\phi})+(g,\bar{\phi})\} \qquad \text{for any } \hat{\phi}\in\hat{S}_0. \end{aligned}$$

Set $e^*=w-\hat{w}, \hat{e}=\hat{w}-\hat{w}$ and $\hat{E}_{ij}=\hat{W}_{ij}-\hat{w}_{ij}$. We have, by (2.2),

$$\begin{aligned} &\sum_{i,j}\{(D_je^*,D_i\hat{\Phi}_{ij})\}\\ &\qquad +\sum_{i,j}\{(w_{ij},\hat{\Phi}_{ij})-(\bar{w}_{ij},\bar{\Phi}_{ij})\}\\ &=\sum_{i,j}\{(D_j\hat{e},D_i\hat{\Phi}_{ij})+(\bar{E}_{ij},\bar{\Phi}_{ij})\} \qquad \text{for any } \hat{\Phi}\in\hat{H}. \end{aligned}$$

Therefore, the error is the sum of the three quantities:

$$\sum_{i,j}(\bar{E}_{ij},\bar{E}_{ij})=E_1+E_2+E_3,$$

where

$$E_1=\sum_{i,j}(D_je^*, D_i\hat{E}_{ij}),$$
$$E_2=\sum_{i,j}\{(w_{ij}, \hat{E}_{ij})-(\bar{w}_{ij}, \bar{E}_{ij})\},$$
$$E_3=-\sum_{i,j}(D_j\hat{e}, D_i\hat{E}_{ij}).$$

Estimate of E_1: This quantity is just the α_h in [3], and we have obtained

$$|E_1|\leqq ch^{\frac{1}{2}}\|g\|_{L_2}\|\hat{E}\|_{L_2}.$$

Estimate of E_2: Let p be a grid point in $\Omega_h'-\partial\Omega_h'$ and S_p be the square centered at p and of side lenth $2\bar{h}$. Then by the Lemma 4 in [3], we have

$$|(D_{ij}w, \hat{\phi})-w_{ij}(p)(1, \hat{\phi}_p)|\leqq ch^2\sum_{|\alpha|=3,4}\|D^\alpha w\|_{L_2(S_p)}. \tag{2.4}$$

On the other hand, for any basis $\hat{\phi}_p$ the following estimate holds.

$$|(D_{ij}w, \hat{\phi}_p)-w_{ij}(p)(1, \hat{\phi}_p)|\leqq ch^2\|g\|_{L_2}. \tag{2.5}$$

Denoting by $\sum^{(1)}$ and $\sum^{(2)}$ the sums over the node points in $\Omega_h'-\partial\Omega_h'$ and in $\Omega_h-(\Omega_h'-\partial\Omega_h')$ respectively, we have

$$\begin{aligned}|E_2|&\leqq \sum_{i,j}|\sum_p{}^{(1)}ch^2\|w\|_{W_2^4(S_p)}\hat{E}_{ij}(p)|\\&\quad+\sum_{i,j}|\sum_p{}^{(2)}ch^2\|g\|_{L_2}\hat{E}_{ij}(p)|\\&\leqq ch^{\frac{1}{2}}\|g\|_{L_2}\|\hat{E}\|_{L_2}.\end{aligned}$$

In this calculation we used the relation

$$O(\sum_p\hat{E}_{ij}(p)^2h^2)=O(\|\hat{E}_{ij}\|^2_{L_2(\Omega_h)})=O(\|\bar{E}_{ij}\|^2_{L_2(\Omega_h)}),$$

which is an obvious consequence of the definitions of the basis functions.

Estimate of E_3: By (2.3) we have

$$\begin{aligned}|E_3|&=|\sum_{i,j}(D_i[\hat{w}_{ij}-w_{ij}], D_j\hat{e})+(g, \bar{e})-(g, \hat{e})\\&\leqq\sum_{i,j}|\hat{w}_{ij}-w_{ij}|_1\,|\hat{e}|_1+ch\|g\|_{L_2}|\hat{e}|_1.\end{aligned} \tag{2.6}$$

To estimate $|\hat{e}|_1$ we give attention to the identity.

$$(D_i\hat{w}, D_i\hat{\Phi}_{ii})+(\bar{W}_{ii}, \bar{\Phi}_{ii})=0.$$

Subtracting $(D_i\hat{w}, D_i\hat{\Phi}_{ii})+(\bar{w}_{ij}, \bar{\Phi}_{ii})$ from the both sides of this identity we have

$$(D_i\hat{e}, D_i\hat{e}) + (\bar{E}_{ii}, \bar{e})$$
$$= -\sum_p \{(\bar{w}_{ii}, \bar{\Phi}_{ii}(p)) + (D_i\hat{w}, D_i\hat{\Phi})_{ii}(p))\} e_p.$$
$$\leqq ch^{\frac{1}{2}} \|g\|_{L_2} |\hat{e}|_1.$$

Therefore we have

$$(2.7) \qquad |\hat{e}|_1 \leqq ch^{\frac{1}{2}} \|g\|_{L_2} + c\|\bar{E}\|_{L_2}.$$

Substituting this result into (2.6) we get

$$|E_3| \leqq ch^{\frac{3}{2}} \|g\|^2_{L_2} + ch\|g\|_{L_2}\|\bar{E}\|_{L_2}.$$

By these three estimates on E_i, we finally get the following quadratic inequality about $\|\bar{E}\|$.

$$\sum_{i,j} (\bar{E}_{ij}, \bar{E}_{ij}) \leqq ch^{\frac{1}{2}}\|g\|_{L_2}\|\bar{E}\|_{L_2} + ch^{\frac{3}{2}}\|g\|^2_{L_2}.$$

The second estimate of the theorem is obtained by solving this inequality and considering that the lumped mass interpolate has the accuracy of $O(h)$. The first one follows from (2.7)

Let us rewrite the equation (1.7), by operating L^{-1}, as follows.

$$(2.8) \qquad \hat{W} + \bar{C}(\hat{W}) + \bar{L}^{-1}\hat{B}g = 0.$$

Let be defined

$$\bar{C}'_{(\hat{W})}\hat{Z} = \bar{L}^{-1}\bar{C}(\bar{L}^{-1}\bar{C}(\hat{W}, \hat{W}), \hat{Z}) + 2\bar{L}^{-1}\bar{C}(\bar{L}^{-1}\bar{C}(\hat{W}, \hat{Z}), \hat{W}).$$

Then we can write

$$(2.9) \qquad \bar{C}(\hat{W}_1) - \bar{C}(\hat{W}_0) = \bar{C}'_{(\hat{W}_0)}(\hat{W}_1 - \hat{W}_0) + \bar{D}(\hat{W}_0, \hat{W}_1 - \hat{W}_0),$$

where $\bar{D}(\hat{W}_0, \hat{Z})$ is a nonlinear operator of third order defined by

$$(2.10) \qquad \bar{D}(\hat{W}_0, \hat{Z}) = 2\bar{L}^{-1}\bar{C}(\bar{L}^{-1}\bar{C}(\hat{W}_0, \hat{Z}), \hat{Z}) + \bar{L}^{-1}\bar{C}(\bar{L}^{-1}\bar{C}(\hat{Z}, \hat{Z}), \hat{W}_0 + \hat{Z}).$$

Let $\hat{C}'_{(\hat{W})}$ be the linear operator derived from $\hat{C}(\hat{W})$ and has the same form as for $\bar{C}'_{(\hat{W})}$. Let W_0 be a solution of (1.3) — exact solution of our problem — and $\hat{W}_0 \in \hat{H}$ be its interpolate.

A fundamental theorem in [3] is that if h is sufficiently small then the operator $I + \hat{C}'_{(\hat{W}_0)}$ is invertible on $\hat{H}$ and holds

$$(2.11) \qquad \sup_{\hat{W} \in \hat{H}} \|I + \hat{C}'_{(\hat{W}_0)})^{-1}\hat{W}\|_{L_2} / \|\hat{W}\|_{L_2} \leqq c < \infty \qquad \text{as } h \to 0.$$

In what follows we want to show this is the case for the operator $I+\bar{C}'_{(\hat{W}_0)}$, that is, that if the equation (1.4) has no nontrivial solution, then the equation

$$(2.12) \qquad \bar{K}\hat{Z}\equiv(I+\bar{C}'_{(\hat{W}_0)})\,\hat{Z}=\hat{G} \qquad \hat{G}\in\hat{H}$$

has a unique solution. In order to prove this we shall prepare the following lemma.

LEMMA 2.

(1) Let w be a function in W_2^1. Then holds

$$|(\hat{u}-\bar{u},w)_{L_2}|\leq ch\|\hat{u}\|_{L_2}\|w\|_{W_2^1}.$$

(2) Let $\hat{U}=\bar{L}^{-1}\bar{C}(\hat{V},\hat{W})$ and $0<\varepsilon<1$ $(\hat{V},\hat{W}\in\hat{H})$. Then holds

a. $\|\hat{U}\|_{L_2}\leq c_\varepsilon\|[\bar{V},\bar{W}]\|_{L_{1+\varepsilon}}$,

b. $\|\hat{U}\|_{L_2}\leq c_\varepsilon\|\hat{V}\|_{\hat{H}}\|\hat{W}\|_{L_2}$,

c. $\|\hat{U}\|_{L_2}\leq c|\hat{V}|_{max}\|\hat{W}\|_{L_2}$,

d. $\|[\bar{V},Z]\|_{L_{1+\varepsilon}}\leq ch^{\frac{-2\varepsilon}{1+\varepsilon}}\|\hat{V}\|_{L_2}\|Z\|_{L_2} \qquad (Z\in L_2)$,

where $|V|_{max}=max\,(max|v|,\ max|V_{ij}|)$ and c_ε is a constant dependent of ε but not on h and the functions.

PROOF. (1) We first prove that the inequality is valid for any $\hat{w}\in\hat{S}_1$. Let p be any node in Ω_h and $T_{p,k}$ and $Q_{p,k}$ be any triangle and quadrilateral associated with p. Expandnig $\hat{w}$ in each $T_{p,k}$ and using the fact

$$\sum_k(\hat{\phi}_p-\bar{\phi}_p,1)_{T_{p,k}}=0,$$

we have the following estimate.

$$|(\hat{\phi}_p-\bar{\phi}_p,\hat{w})|\leq ch^2\sum_k(\sum_i\|D_i\hat{w}\|_{T_{p,k}}+\|\hat{w}\|_{T_{p,k}}).$$

Therefore,

$$|(\hat{u}-\bar{u},\hat{w})|=|\sum_p u_p(\hat{\phi}_p-\bar{\phi}_p,\hat{w})|$$
$$\leq ch\|\hat{u}\|_{L_2}\|\hat{w}\|_{W_2^1},$$

which is our assertion in the case of finite element function $\hat{w}$. Let w be a function in W_2^1. Then approximation theory assures that there is a finite element function $\hat{w}\in\hat{S}_1$ such that

$$\| w-\hat{w} \|_{L_2} \leqq ch \| w \|_{W_2^1}, \qquad \| \hat{w} \|_{W_2^1} \leqq c \| w \|_{W_2^1}.$$

(we have to always take care of the extension of the finite element basis in applying the results of approximation theory. However, as pointed out in [3], this extension does not disturb the basic character of the approximation or interpolation.)

Combining this result with the above one we have

$$\begin{aligned} |(\hat{u}-\bar{u}, w)| &\leqq \|\hat{u}-\bar{u}\|_{L_2} \| w-\hat{w} \|_{L_2} + |(\hat{u}-\bar{u}, \hat{w})| \\ &\leqq ch \| \hat{u} \|_{L_2} \| w \|_{W_2^1}, \end{aligned}$$

which completes the proof of (1).

(2). The equation $\bar{L}^{-1}\hat{U} = \bar{C}(\hat{V}, \hat{W})$ is equivalent to

$$\begin{aligned} \sum_{i \leqq j} \{ (D_j \hat{u}, D_i \hat{\Phi}_{ij})_{L_2} + (\bar{U}_{ij}, \bar{\Phi}_{ij})_{L_2} \} + \sum_{i,j} (D_i \hat{U}_{ij}, D_j \hat{\phi})_{L_2} \\ = ([\bar{V}, \bar{W}], \bar{\phi})_{L_2} \qquad \text{for all } \hat{\Phi} \in \hat{H}. \end{aligned}$$

Therefore, by Lemma 1 and the Sobolev's imbedding theorem we have

$$\sum_{i,j} (\hat{U}_{ij}, \hat{U}_{ij})_{L_2} = ([\bar{V}, \bar{W}], \bar{u})_{L_2} \leqq c_\varepsilon \| [\bar{V}, \bar{W}] \|_{L_{1+\varepsilon}} |\hat{u}|_1.$$

At the same time we have $|\hat{u}|_1 \leqq c \|\hat{U}\|_{L_2}$. (a) is thus proved. To prove (b) we have to show

$$(2.13) \qquad \| [\bar{V}, \bar{W}] \|_{L_{1+\varepsilon}} \leqq c_\varepsilon \| \hat{V} \|_{\hat{H}} \| \hat{W} \|_{L_2}.$$

By Hörder's inequality,

$$\int |\bar{u}\bar{v}|^{1+\varepsilon} \leqq \left[\int (|\bar{u}|^{1+\varepsilon})^{\frac{2\varepsilon}{1+\varepsilon}} \right]^{\frac{1+\varepsilon}{2}} \left[\int (|\bar{v}|^{1+\varepsilon})^{\frac{2}{1-\varepsilon}} \right]^{\frac{1-\varepsilon}{2}}.$$

Therefore, again by Lemma 1 and the Sobolev's theorem, we have

$$\| \bar{u}\bar{v} \|_{L_{1+\varepsilon}} \leqq c_\varepsilon \| \hat{u} \|_{L_2} |\hat{v}|_1,$$

from which inequality (2.13) and thus (b) follows.

Since (c) will be evident, let us prove (d). By the inverse relation we have, as the Lemma 10 in [3],

$$\| \bar{u} \|_{L_{1+\varepsilon'}} \leqq ch^{-\varepsilon'/(2+\varepsilon')} \| \bar{v} \|_{L_2},$$

where $2+\varepsilon' = 2(1+\varepsilon)/(1-\varepsilon)$. Therefore, by Lemma 1

$$\| [\bar{V}, Z] \|_{L_{1+\varepsilon}} \leqq c \sum \| \bar{V}_{ij} \|_{L_{2+\varepsilon'}} \| Z_{kl} \|_{L_2} \leqq ch^{\frac{-\varepsilon'}{2+\varepsilon'}} \| \hat{V} \|_{L_2} \| Z \|_{L_2},$$

which completes the proof.

Now by making use of this lemma we have

THEOREM 2. Assume that the equation (1.4) has no nontrivial solution. Then if h is sufficiently small, there is a function $\hat{H}^*$ for any $\hat{G}\in\hat{H}$ such that $\|\hat{H}^*\|_{L_2}\leqq c\|\hat{G}\|_{L_2}$ and

$$(2.14)\qquad \|\bar{K}\hat{H}^*-\hat{G}\|_{L_2}\leqq q\|\hat{G}\|_{L_2}\qquad (0<q<1)$$

where q is independent of h and $\hat{G}$. Therefore $\bar{K}$ is invertible for such small h and the norm of its inverse is uniformly bounded as h tends to 0.

PROOF. We shall show that the inequality is satisfied by the function

$$\hat{H}^*=(I+\hat{C}'_{(\hat{W}_0)})^{-1}\hat{G}.$$

First we note that this function is well defined for sufficiently small h and the operator defining this function is uniformly bounded. In fact, this was one of the main theorem of the previous paper [3]. Furthermore, the existence of q satisfying the inequality (2.14) implies not only the existence of a solution of (2.12), but also the uniqueness of the solution, because the operator is working in a finite dimensional space.

Rewrite

$$\bar{K}\hat{H}^*-\hat{G}=\bar{C}'_{(\hat{W}_0)}\hat{H}^*-\hat{C}'_{(\hat{W}_0)}\hat{H}^*=S_1+R_1+S_2+R_2,$$

where

$$S_1=\bar{L}^{-1}\bar{C}(\bar{L}^{-1}\bar{C}(\hat{W}_0,\hat{W}_0),\hat{H}^*)-L^{-1}C(L^{-1}C(W_0,W_0),\hat{H}^*),$$

$$R_1=L^{-1}C(L^{-1}C(W_0,W_0),\hat{H}^*)-\hat{L}^{-1}\hat{C}(\hat{L}^{-1}\hat{C}(\hat{W}_0,\hat{W}_0),\hat{H}^*),$$

$$S_2=2\bar{L}^{-1}\bar{C}(\bar{L}^{-1}\bar{C}(\hat{W}^0,\hat{H}^*),\hat{W}_0)-2L^{-1}C(L^{-1}C(W_0,\hat{H}^*),W_0),$$

$$R_2=2L^{-1}C(L^{-1}C(W_0,\hat{H}^*),W_0)-2\hat{L}^{-1}\hat{C}(\hat{L}^{-1}\hat{C}(\hat{W}_0,\hat{H}^*),\hat{W}_0).$$

The quantities R_1 and R_2 are exactly those estimated in [3], and they satisfy

$$\|R_1\|_{L_2}\leqq c_\varepsilon h^{\frac{1}{2}-\frac{2\varepsilon}{1+\varepsilon}}\|\hat{H}^*\|_{L_2},$$

$$\|R_2\|_{L_2}\leqq ch^{\frac{1}{2}}\|\hat{H}^*\|_{L_2},$$

where ε is an arbitraly positive constant and c_ε is a constant depending on ε but not on h and $\hat{H}^*$.

Estimate of S_2. Let us define

$$V = L^{-1}C(W_0, \bar{H}^*), \qquad \hat{V} = \bar{L}^{-1}\bar{C}(\hat{W}_0, \hat{H}^*)$$

In order to estimate the difference of these functions, we introduce the functions

$$Z_1 = L^{-1}C(W_0, \bar{H}^*), \qquad Z_2 = L^{-1}C(\overline{W}_0, \bar{H}^*).$$

By (1) of Lemma 2, we have

$$\|V - Z_1\|_{L_2} \leqq ch\|\hat{H}^*\|_{L_2}.$$

Therefore it is easy to prove

$$\|V - Z_2\|_{L_2} \leqq ch\|\hat{H}^*\|_{L_2}.$$

Now it is evident that $\hat{V}$ is the approximate solution to Z_2. Therefore, by the error estimate in the linear problem we have

$$\|Z_2 - \hat{V}\|_{L_2} \leqq ch^{\frac{1}{2}}\|\hat{H}^*\|_{L_2},$$

and thus

$$\|V - \hat{V}\|_{L_2} \leqq ch^{\frac{1}{2}}\|\hat{H}^*\|_{L_2}.$$

Since $S_2 = 2\bar{L}^{-1}\bar{C}(\hat{V}, \hat{W}_0) - 2L^{-1}C(V, W_0)$, we have

$$\begin{aligned}\|S_2\|_{L_2} &\leqq 2\|\bar{L}^{-1}\bar{C}(\hat{V}, \hat{W}_0) - L^{-1}C(\bar{V}, \overline{W}_0)\|_{L_2} \\ &\quad + 2\|L^{-1}C(\bar{V}, \overline{W}_0) - L^{-1}C(V, W_0)\|_{L_2} \\ &\leqq ch^{\frac{1}{2}}\|\hat{H}^*\|_{L_2}.\end{aligned}$$

Estimate S_1. The difference of the two functions

$$V = L^{-1}C(W_0, W_0) \qquad \hat{V} = \bar{L}^{-1}\bar{C}(\hat{W}_0, \hat{W}_0)$$

is easily estimated and we have

$$\|V - \hat{V}\|_{L_2} \leqq ch^{\frac{1}{2}}.$$

Let $\hat{\hat{V}}$ be the interpolate of V. Then by the error estimate in the linear problem we have

$$\begin{aligned}&\|L^{-1}C(\bar{\hat{V}}, \bar{H}^*) - \bar{L}^{-1}\bar{C}(\bar{\hat{V}}, \hat{H}^*)\|_{L_2} \\ &\qquad \leqq ch^{\frac{1}{2}}\|\hat{H}^*\|_{L_2},\end{aligned}$$

On the other hand, by (2) of Lemma 2, it holds

$$\|\bar{L}^{-1}\bar{C}(\hat{\bar{V}},\hat{H}^*)-\bar{L}^{-1}\bar{C}(\hat{V},\hat{H}^*)\|_{L_2}$$
$$\leqq c_\varepsilon\|[\bar{\bar{V}}-\bar{V},\bar{H}^*]\|_{L_{1+\varepsilon}}$$
$$\leqq c_\varepsilon h^{\frac{-2\varepsilon}{1+\varepsilon}}\|\bar{\bar{V}}-\bar{V}\|_{L_2}\|\hat{H}^*\|_{L_2}$$
$$\leqq c_\varepsilon h^{\frac{1}{2}-\frac{2\varepsilon}{1+\varepsilon}}\|\hat{H}^*\|_{L_2}$$

Therefore we get

$$\|S_1\|_{L_2}\leqq c_\varepsilon h^{\frac{1}{2}-\frac{2\varepsilon}{1+\varepsilon}}\|\hat{H}^*\|_{L_2}.$$

Summarizing all results we finally get

$$\|\bar{K}\hat{H}^*-\hat{G}\|_{L_2}\leqq c_\varepsilon h^{\frac{1}{2}-\frac{2\varepsilon}{1+\varepsilon}}\|\hat{G}\|_{L_2}.$$

for sufficiently small h. Taking ε sufficiently small, and then also h, we have

$$c_\varepsilon h^{\frac{1}{2}-\frac{2\varepsilon}{1+\varepsilon}}<1,$$

which is the desired inequality. The second half of the theorem follows from the Theorem 8 in [3].

3. Existence and convergence of the approximate solutions.

The operator $\bar{C}'_{(\hat{W})}$ is the derivative of $\bar{C}(\hat{W})$. Theorem 2 thus assures the applicability of the Newton's iteration which starts from $\hat{W}_0$, interpolate of the exact solution W_0. We shall employ an iterative process as follows.

Let us rewrite (2.8) as

$$\text{(3.1)}\qquad \hat{W}=R\hat{W}$$
$$\equiv\hat{W}_0-(I+\bar{C}'_{(\hat{W}_0)})^{-1}[\hat{E}+\bar{D}(\hat{W}_0,\hat{Z})],$$

where $\hat{E}$ is the residual given by substituting $\hat{W}_0$ into (2.8), $\bar{D}$ is the nonlinear term defined by (2.10) and $\hat{Z}=\hat{W}-\hat{W}_0$. The next lemma may be proved easily by introducing the function

$$L^{-1}C(L^{-1}C(W_0,W_0),\hat{W}_0).$$

LEMMA 3. Let $W_0\in H$ be a solution of the original problem and $\hat{W}_0\in\hat{H}$ be its interpolate. Then holds, for any $\varepsilon>0$

$$\text{(3.2)}\qquad \|\hat{E}\|_{L_2}\equiv\|\hat{W}_0+\bar{C}(\hat{W}_0)+\bar{L}^{-1}\hat{B}p\|_{L_2}\leqq c_\varepsilon h^{\frac{1}{2}-\frac{2\varepsilon}{1+\varepsilon}}.$$

Now we can prove the next concluding theorem. For the proof of this theorem it is only necessary to repeat the ones given for Theorem 14 and 15 in [3], because we have proved the Lemma 2, Theorem 2 and Lemma 3 which corresponds to the Lemma 9 and 10, Theorem 11 and 12, and Lemma 13 of [3], which were essential for the proof of the Theorem 14 and 15.

THEOREM 3. Let W_0 be a solution of (1.3). Assume that the linear, homogeneous equation (1.4) has no nontrivial solution at W_0. Consider the iteration

$$\hat{W}_n = R\hat{W}_{n-1} \qquad (n=1,2,..),$$

where $\hat{W}_0$ is the interpolate of W_0. Then, if h is sufficiently small there is a closed ball

$$S_\delta = \left\{\hat{W} \in \hat{H};\ \|\hat{W} - \hat{W}_0\|_{L_2} \leq \delta\right\},\ \delta = h^{\frac{1}{2} - \frac{3}{n+1}} (n \geqslant 11, \text{ integer})$$

such that

(A) $\|R\hat{W} - \hat{W}_0\|_{L_2} \leq \delta$ for all $\hat{W} \in S_\delta$,

(B) $\|R\hat{W}_1 - R\hat{W}_2\|_{L_2} \leq q \|\hat{W}_1 - \hat{W}_2\|_{L_2}$ $(0<q<1)$

for all $\hat{W}_1, \hat{W}_2 \in S_\delta$. Therefore, the iteration (3.3) defines a function which is a solution of the discrete equation (1.5).
This solution is unique in the δ-neighbourhood of $\hat{W}_0$.

REMARK. We may assert, pure-theoretically, that the order of convergence of this method is almost the same as in the biharmonic case. However, the numerical constants appearing in the proofs of Theorem 2 and 3 depend on the choise of the number n of Theorem 3 (this n reflects the ε in Lemma 2), and if we want large n then we have to make h small inevitablly. This means that the actual accuracy of the approximate solution obtained may not be better than that of biharmonic case. In fact, numerical experience shows that this is the case. Of course, if we use more complicated elements, then the convergence rate will increase corresponding to its degree. In fact, it is very interesting problem whether the elements of degree k can give the approximate solution of $O(h^{k-1-\varepsilon})$ in the error, since $O(h^{k-1})$ is the rate in the biharmonic problem.

APPENDIX. **Numerical examples**

Let Ω be a square plate of side length 1, thickness t, Poisson's ratio ν, and flexural rigidity D. Let f and w be the stress function and normal deflection, respectively. Then the functions $F=(t/D)f$ and $W=w/t$ satisfy the following equations.

$$\begin{cases} \Delta^2 F = -6(1-\nu^2)[W, W] \\ \Delta^2 W = [F, W] + g/Dt, \end{cases}$$

where g denotes given lateral force. We solved this system under the boundary condition $F=dF/dn=W=dW/dn=0$ to test the accuracy of the approximate solutions. The following table shows the values of $\hat{W}$ for various mesh size h (we devide the region into congruent right-angled equilateral triangles). For comparison, the table includes the values obtained by the Galerkin's method in the conforming trial functions $(1-\cos 2m\pi x)(1-\cos 2n\pi y)$.

The solutions has been obtained as the limit state of the non-linear vibration by adding the inertia term and damping term to the original equation. The discretizations of these terms are both by the central difference. The time mesh is taken small enough to ensure the stability of the *linear* scheme. This is effective for the small load, but for the large load the scheme becomes unstable unless the time mesh is made small.

		(nonlinear)	(linear)
(author)			
	$h=1/6$	1. 7 1	2. 5 4
	$h=1/8$	1. 6 3	2. 2 8
	$h=1/16$	1. 5 5	2. 0 9
(Galerkin's method)			
	1 term	1. 4 6	2. 0 5
	3 term	1. 4 3	1. 9 5

(Deflection at the center of the plate. $g/Dt=1600$, $\nu=0.3$)

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Department of Mathematics
Faculty of Science
Kumamoto Universtiy