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A HOMOTOPY CLASSIFICATION IN SCHEME THEORY

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Introduction. Let T be a finite CW complex. We denote by $K(T)$ the Grothendieck group of the classes of complex vector bundles over T. We further write Z, B_U for the integers with the discrete topology, the classifying space of the infinite unitary group respectively. Then the K-theoretic version of the homotopy classification theorem is given by the statement of the existence of a natural bijection:

$$K(T) \xrightarrow{\sim} [T, B_U \times Z]$$

where $[T, B_U \times Z]$ denotes the set of homotopy classes of maps of T into $B_U \times Z$.

The purpose of this paper is to present a scheme-theoretic analogue of the above classification theorem in topology. Let S be an arbitrary scheme. Let T be an irreducible regular affine scheme over S. We denote by $K^0(T)$ (resp. $K_0(T)$) the Grothendieck group of isomorphism classes of locally free O_T-Modules of finite rank (resp. of isomorphism classes of coherent O_T-Modules). Let i: $K^0(T) \longrightarrow K_0(T)$ be the homomorphism which sends the class in $K^0(T)$ of a locally free O_T-Module to that in $K_0(T)$. Suppose i be an isomorphism. This is the case if T is Spec A with A a Dedekind domain or if T is a regular noetherian scheme with an ample invertible O_T-Module $\mathscr{L}$ where $\mathscr{L}$ is called ample if for any coherent O_T-Module $\mathscr{F}$ there exists an epimorphism: $O_T{}^P \longrightarrow \mathscr{F} \otimes \mathscr{L}^{\otimes q}$ for positive integers p, q. Let $\mathscr{E}^0$ be the direct sum of countably infinite copies of O_S. Recall that $\boldsymbol{Grass}_n(\mathscr{E}^0)$ stands for the Grassmannian of degree n which is defined by $\mathscr{E}^0$ [1]. In analogy with Borel-Serre [2], we can define a rational S-homotopy of S-morphisms. For the precise definition see § 2. We denote by $[T, \boldsymbol{Grass}_n(\mathscr{E}^0)]$ the set of rational S-homotopy classes of S-morphism: $T \longrightarrow \boldsymbol{Grass}_n(\mathscr{E}^0)$. Let $[\mathscr{E}]$ be the class in $K^0(T)$ of a locally free O_T-Module $\mathscr{E}$ of rank m. The elements of the form $[\mathscr{E}]-m$ generates a subgroup of $K^0(T)$. We write it as $\tilde{K}(T)$. Then our result can be stated as follows.

Theorem. There exists a natural bijection

$$\varinjlim_n [T, \boldsymbol{Grass}_n(\mathscr{E}^0)] \xrightarrow{\sim} \tilde{K}(T).$$

Especially we can take Spec $\mathbf{Z}$, Spec A with A a Dedekind domain for S, T respectively. Then we have the following

Corollary.

$$\text{The class number of } A = \text{Card } [T, \boldsymbol{P}(\mathscr{E}^0)]$$

where $\boldsymbol{P}(\mathscr{E}^0)$ is the projective bundle over Spec $\mathbf{Z}$, defined by $\mathscr{E}^0$ (c. f. [1], I, 9, 7, 5).

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1. The Grassmannians. Let S be a scheme and $\mathscr{E}$ a quasicoherent O_S-Module. Let T be an S-scheme with the structural morphism f_T. $\mathscr{E}_{(T)}$ stands for the inverse image of $\mathscr{E}$ by f_T and $Grass_n(\mathscr{E}_{(T)})$ for the set of locally free quotients of $\mathscr{E}_{(T)}$ with rank n. Then the functor: $T \longmapsto Grass_n(\mathscr{E}_{(T)})$ is represented by $\boldsymbol{Grass}_n(\mathscr{E})$, i. e. the Grassmannian of degree n, defined by $\mathscr{E}$. Furthermore there exists a Module $\mathscr{G}$ over $\boldsymbol{Grass}_n(\mathscr{E})$ called the fundamental Module over $\boldsymbol{Grass}_n(\mathscr{E})$ in such a way that the functor isomorphism

$$Hom_S(T, \boldsymbol{Grass}_n(\mathscr{E})) \xrightarrow{\sim} Grass_n(\mathscr{E}_{(T)}) \tag{1}$$

is given by the map that sends $g \in Hom_S(T, \boldsymbol{Gras}_n(\mathscr{E}))$ to $g^*(\mathscr{G})$.

Let $\mathscr{E}^0$ be the direct sum $O_S \oplus O_S \oplus \ldots$ of countably infinite copies of O_S. Let $\mathscr{Q} \in Grass_n((\mathscr{E}^0)_{(T)})$. Then

$$\underbrace{O_T \oplus \ldots \oplus O_T}_{i \text{ copies}} \oplus \mathscr{Q}$$

can be considered naturally as an element of $Grass_{n+i}(\mathscr{E}^0{}_{(T)})$ where i runs through $1, 2, \ldots$. Hence we have an inclusion: $Grass_n(\mathscr{E}^0{}_{(T)}) \subset Grass_{n+i}(\mathscr{E}^0{}_{(T)})$. This inclusion corresponds to a closed immersion $f_{n,n+i}$: $\boldsymbol{Grass}_n(\mathscr{E}^0) \longrightarrow \boldsymbol{Grass}_{n+i}(\mathscr{E}^0)$. In addition $(\boldsymbol{Grass}_n(\mathscr{E}^0), f_{n,n+i})$ $(n, i = 1, 2, \ldots)$ constitutes a direct system of S-schemes.

2. Rational S-homotopy. Let Z be an indeterminate. Let $S[Z]$ be the spectre of the symmetric Algebra $O_S[Z]$ of O_S. We write (Z) (resp. $(1-Z)$) for the $O_S[Z]$-Ideal generated by Z (resp. $1-Z$). The corresponding projection

$$O_S[Z] \longrightarrow O_S[Z]/(Z) \xrightarrow{\sim} O_S$$
$$(\text{resp. } O_S[Z] \longrightarrow O_S[Z]/(1-Z) \xrightarrow{\sim} O_S)$$

induces an S-morphism $S \longrightarrow S[Z]$, denoted by t_1 (resp. t_2). Let $\xi_i = 1_T \times t_i$ for $i=1,2$ where 1_T denotes the identity of T. If we identify $T \times_S S$ with T, then ξ_i can be viewed as morphisms: $T \longrightarrow T[Z]$.

Let T' be an S-scheme. Let f_1, f_2: $T \longrightarrow T'$ be S-morphisms. By a rational S-homotopy from f_1 to f_2 we understand an S-morphism h: $T[Z] \longrightarrow T'$ such that

$$f_i = h \circ \xi_i \tag{2}$$

where $i=1,2$. Let $R\{f_1, f_2\}$ be the relation ≪there exists a rational S-homotopy from f_1 to f_2≫. Then this relation is reflexive and symmetric. Hence the relation ≪there exist an integer $n>0$ and a sequence $(g_i)_{0 \leq i \leq n}$ of S-morphisms: $T \longrightarrow T'$ such that $g_0 = f_1, g_n = f_2$ and $R\{g_1, g_{i+1}\}$ for $i=0, \ldots, n-1$≫is an equivalence relation (cf. [3], §6, Exercise 9). We call the class of f_1 mod this equivalence relation the rational S-homotopy class of f_1. We denote it by $[f_1]_{S-rat}$ or simply $[f_1]$. f_1 is said to be rationally S-homotopic to f_2 if f_1 is eqnivalent to f_2 with respect to this equivalence relation.

3. $K^0(T)$. Let $K^0(T)$ be the Grothendieck group of isomorphism classes of locally free O_T-Modules of finite type. Recall that there is a homomorphism called rank,
rk: $K^0(T) \longrightarrow C(T;Z)$, where $C(T;Z)$ is the abelian group of all continuous maps of T into Z with the discrete topology. Let us denote the kernel of rk by $\tilde{K}(T)$.

We write $[T, \boldsymbol{Grass}_n(\mathscr{E}^0)]_{S-rat}$ or simply $[T, \boldsymbol{Grass}_n(\mathscr{E}^0)]$ for the set of rational S-homotopy classes of S-morphisms: $T \longrightarrow \boldsymbol{Grass}_n(\mathscr{E}^0)$. Let us define maps

$$\iota_{n,n+i}\colon \quad [T, \boldsymbol{Grass}_n(\mathscr{E}^0)] \longrightarrow [T, \boldsymbol{Grass}_{n+i}(\mathscr{E}^0)]$$

by $\iota_{n,n+i}([f]) = [f_{n,n+i} \circ f]$ where f is any S-morphism: $T \longrightarrow \boldsymbol{Grass}_n(\mathscr{E}^0)$. Then $([T, \boldsymbol{Grass}_n(\mathscr{E}^0)], \iota_{n,n+i})$ is a direct system of sets. We denote by ι_n the canonical map of $[T, \boldsymbol{Grass}_n(\mathscr{E}^0)]$ into $\varinjlim_n [T, \boldsymbol{Grass}_n(\mathscr{E}^0)]$.

Now this section will be devoted to the definition of a natural surjection φ: $\tilde{K}(T) \longrightarrow \varinjlim_n [T, \boldsymbol{Grass}_n(\mathscr{E}^0)]$. We need the following proposition for it.

Proposition 1. Let $\mathscr{Q}_1, \mathscr{Q}_2 \in Grass_n(\mathscr{E}^0{}_{(T)})$. Let f_1 (resp. f_2) be the morphism: $T \longrightarrow \boldsymbol{Grass}_n(\mathscr{E}^0)$ which corresponds to $\mathscr{Q}_1$(resp. $\mathscr{Q}_2$) by (1). If $\mathscr{Q}_1$ and $\mathscr{Q}_2$ are isomorphic, then f_1 and f_2 are rationally S-homotopic.

Proof. Let q_1 (resp. q_2) be the projection of $\mathscr{E}^0 \oplus \mathscr{E}^0$ onto the first factor (resp. the second factor). q_i give rise to closed immersions $\boldsymbol{Grass}_n(q_i)$: $\boldsymbol{Grass}_n$

$(\mathscr{E}^0) \longrightarrow \boldsymbol{Grass}_n(\mathscr{E}^0 \oplus \mathscr{E}^0)$ (cf. [1], I, 9, 7), where $i=1, 2$. For brevity let us write $\boldsymbol{q}_i$ for $\boldsymbol{Grass}_n(q_i)$ in what follows. Then the first step of the proof is to show that $\boldsymbol{q}_1 \circ f_1$ is rationally S-homotopic to $\boldsymbol{q}_2 \circ f_2$.

For that let us take an isomorphism γ: $\mathscr{Q}_2 \xrightarrow{\sim} \mathscr{Q}_1$ once and for all. Let g_1 (resp. g') be the projection of $\mathscr{E}^0{}_{(T)}$ onto $\mathscr{Q}_1$ (resp. $\mathscr{Q}_2$). We write g_2 instead of $\gamma \circ g'$. Then $\mathscr{Q}_i = \mathscr{E}_{(T)}/Ker\ g_i$ where $i=1, 2$. We write U for the S-scheme $T \times_S S[Z]$ where Z is an indeterminate. Let f_U be the structural morphism of U and p the projection of U onto the first factor T. Note that Z, $1-Z$ can be viewed as elements of $\Gamma(U, O_U)$. Let us now define α: $\mathscr{E}^0{}_{(U)} \oplus \mathscr{E}^0{}_{(U)} \longrightarrow p^*(\mathscr{Q}_1)$ by

$$\alpha = (1-Z)p^*(g_1) \circ f_U{}^*(q_1) + Zp^*(g_2) \circ f_U{}^*(q_2).$$

In fact α is an epimprphism as easily seen, whence

$$\mathscr{E}^0{}_{(U)} \oplus \mathscr{E}^0{}_{(U)}/Ker\ \alpha \xrightarrow{\sim} p^*(\mathscr{Q}_1).$$

Since $p^*(\mathscr{Q}_1)$ is locally free of rank n, we have $\mathscr{E}^0{}_{(U)} \oplus \mathscr{E}^0{}_{(U)}/Ker\ \alpha \in Grass_n(\mathscr{E}^0{}_{(U)} \oplus \mathscr{E}^0{}_{(U)})$. In consequence an S-morphism h: $U \longrightarrow \boldsymbol{Grass}_n(\mathscr{E}^0 \oplus \mathscr{E}^0)$ corresponds to it by (1). Thus we obtain the commutative diagram:

$$\begin{array}{ccc} T & \xrightarrow{\ \xi_i\ } & T \times S[Z] \\ {\scriptstyle f_i}\downarrow & & \downarrow{\scriptstyle h} \\ \boldsymbol{Grass}_n\ (\mathscr{E}^0) & \xrightarrow[\ \boldsymbol{q}_i\ ]{} & \boldsymbol{Grass}_n\ (\mathscr{E}^0 \oplus \mathscr{E}^0). \end{array}$$

Equivalently we can say that h is a rational S-homotopy from $\boldsymbol{q}_1 \circ f_1$ to $\boldsymbol{q}_2 \circ f_2$.

Now let us choose an isomorphsim π: $\mathscr{E}^0 \oplus \mathscr{E}^0 \xrightarrow{\sim} \mathscr{E}^0$ once and for all. (Such an isomorphism certainly exists because $\mathscr{E}^0$ is the direct sum of countably infinite copies of O_S.) It induces an isomorphism $\boldsymbol{Grass}_n(\pi)$: $\boldsymbol{Grass}_n(\mathscr{E}^0) \xrightarrow{\sim} \boldsymbol{Grass}_n(\mathscr{E}^0 \oplus \mathscr{E}^0)$. The second step of the proof is to show that $\boldsymbol{Grass}_n(\pi)$ and $\boldsymbol{q}_i$ are rationally S-homotopic. Let us denote by f_Z the structural morphism of S-scheme $S[Z]$. We define

$$h'_i: (\mathscr{E}^0 \oplus \mathscr{E}^0)_{(S[Z])} \longrightarrow \mathscr{E}^0{}_{(S[Z])}$$

by

$$h'_i = (1-Z)\ f_Z{}^*(q_i) + Zf_Z{}^*(\pi),$$

where $i=1,2$. It is epic as easily seen. Then $\boldsymbol{Grass}_n(h'_i)$ are the required rational S-homotopy from $\boldsymbol{q}_i$ to $\boldsymbol{Grass}_n(\pi)$.

From the Second Step it follows that for each i $\boldsymbol{q}_i\circ f_i$ and $\boldsymbol{Grass}_n(\pi)\circ f_i$ are rationally S-homotopic. Thus $\boldsymbol{Grass}_n(\pi)\circ f_1$, $\boldsymbol{q}_1\circ f_1$, $\boldsymbol{q}_2\circ f_2$ and $\boldsymbol{Grass}_n(\pi)\circ f_2$ are mutually rationally S-homotopic. We can therefore conclude that f_1 and f_2 are rationally S-homotopic, since $\boldsymbol{Grass}_n(\pi)$ is an isomorphism. This completes the proof of the proposition.

From now on we assume that T is affine. Let us define the natural surjection

$$\varphi\colon\ \tilde{K}(T) \longrightarrow \varinjlim_{n}\ [T,\ \boldsymbol{Grass}_n(\mathscr{E}^0)].$$

Any element v of $\tilde{K}(T)$ can be written in the form $[\mathscr{E}]-m$ where m is a positive integer standing for the class of $O_T{}^m$ and $\mathscr{E}$ a locally free O_T-Module of rank m. Since $\mathscr{E}$ is projective, there is an O_T-Module $\mathscr{F}$ such that $\mathscr{E}\oplus\mathscr{F} \xrightarrow{\sim} O_T{}^r$ for some positive integer r. Hence $\mathscr{E}$ is isomorphic to some quotient of $\mathscr{E}^0{}_{(T)}$, i. e. $\mathscr{E}$ ε $Grass_n(\mathscr{E}^0{}_{(T)})$. Thus to $\mathscr{E}$ there corresponds $f_{\mathscr{E}}\colon T \longrightarrow \boldsymbol{Grass}_n(\mathscr{E}^0)$ by (1). Let $v=[\mathscr{E}']-m'$ be another form in which v is expressed. Then $\mathscr{E}\oplus O_T{}^S \xrightarrow{\sim} \mathscr{E}'\oplus O_T{}^{S'}$ for positive integers s, s' (cf. [4], p.347). Therefore, by Proposition 1, the S-morphism $f_{m,m+S}\circ f_{\mathscr{E}}$ corresponding to $\mathscr{E}\oplus O_T{}^S$ is rationally S-homotopic to the one $f_{m',m'+S'}\circ f_{\mathscr{E}'}$ corresponding to $\mathscr{E}'\oplus O_T{}^{S'}$. Hence $\iota_m([f_{\mathscr{E}}])=\iota_{m'}([f_{\mathscr{E}'}])$. In other words the element $\iota_m([f_{\mathscr{E}}])$ ε $\varinjlim_{n}$ $[T,\ \boldsymbol{Grass}_n\ (\mathscr{E}^0)]$ does not depend on the choice of ways of writing $v=[\mathscr{E}]-m$. So we define

$$\varphi(v)=\iota_m([f_{\mathscr{E}}]).$$

Remark. Suppose that T is Spec A for a noetherian integral domain A. Then we can take an O_T-Module of rank$\leqq N$ for $\mathscr{E}$ in the above where $N=\dim$ Specm A ([5], Theorem 1). Hence there exists a natural map $\varphi_N\colon \tilde{K}(T) \longrightarrow [T,\ \boldsymbol{Grass}_N(\mathscr{E}^0)]$ such that $\varphi=\iota_N\circ\varphi_N$.

4. Proof of the Theorem. Let $K_0(T)$ be the Grothendieck group of isomorphism classes of coherent O_T-Modules. Let $\mathscr{E}$ be any locally free O_T-Module of finite type. We denote by i the homomorphism: $K^0(T) \longrightarrow K_0(T)$ which sends the class in K^0 (T) of $\mathscr{E}$ to that in $K_0(T)$. Suppose i be an isomorphism.

Proposition 2. Let T' be an S-scheme. Suppose S-morphisms $f_1, f_2\colon T \longrightarrow T'$ be rationally S-homotopic. Then f_1 and f_2 induce the same homomorphihm: K_0 $(T') \longrightarrow K_0(T)$.

Proof. We take a rational S-homotopy from f_1 to f_2. Let p be the first projection of $U=T\times_S S[Z]$ as before. Then $p\circ\xi_i=1_T$ (for $\xi_i, 1_T$ see §2) for $i=1,2$. On the other hand p induces an isomorphism: $K_0(T)\xrightarrow{\sim}K_0(U)$ since U is a vector bundle over T with the structural morphism p ([6], Exposé IX, §1, Proposition 1.6). Hence ξ_i induce the same isomorphism $\xi_1'=\xi_2'$: $K_0(U)\xrightarrow{\sim}K_0(T)$. It follows from (2) that

$$f_1'=\xi_1'\circ h'=\xi_2'\circ h'=f_2'.$$

This completes the proof.

This proposition guarantees the injectivity of φ. In fact the inverse of φ can be constructed as follows. Let w be an arbitrary element of $\varinjlim_n [T, \mathbf{Grass}_n(\mathscr{E}^0)]$.

Then w can be written as $\iota_m([f])$ where m is a positive integer and f an S-morphism: $T\longrightarrow\mathbf{Grass}_m(\mathscr{E}^0)$. To f there corresponds $\mathscr{E}_f\in Grass_m(\mathscr{E}^0{}_{(T)})$ by (1). Suppose w can also be written in the form $\iota_{m'}([f'])$ where f' is an S-morphism: $T\longrightarrow\mathbf{Grass}_{m'}(\mathscr{E}^0)$. Then there exists some positive integer r ($\geqq max\ (m,m')$) such that $f_{m,r}\circ f$ is rationally S-homotopic to $f_{m',r}\circ f'$. It follows from *Proposition* 2 that $\mathscr{E}_f\oplus O_T{}^{r-m}\xrightarrow{\sim}\mathscr{E}_{f'}\oplus O_T{}^{r-m'}$. We therefore have $[\mathscr{E}_f]-m=[\mathscr{E}_{f'}]-m'$ in $K_0(T)$. This implies that $[\mathscr{E}_f]-m$ does not depend on the choice of expressions $w=\iota_m([f])$. We can now define the inverse of φ, denoted Ψ, by

$$\Psi(w)=[\mathscr{E}_f]-m.$$

5. Dedekind rings. Let A be a Dedekind ring and M an A-module of finite type. Then there exist a projective A-module of finite type P and a finite number of maximal ideals in A, say $(M_i)_{i=1,\dots,r}$, such that

$$M\xrightarrow{\sim}P\oplus\sum_{i=1}^{r}A/M_i^{n_i}$$

where $n_i (i=1,\dots,r)$ are positive integers. We moreover know that P is isomorphic to the direct sum of a free A-module and an ideal of A. We denote this ideal by I. Let $T=\operatorname{Spec} A$ as before. We write $C(A)$ for the ideal class group of A and $r(M)$ for the rank of M. Define a map γ: $K_0(T)\longrightarrow \mathbf{Z}\times C(A)$ by

$$\gamma([\tilde{M}])=(r(M), cl(I)+\sum_{i=1}^{r}n_i\ cl(M_i))$$

where $cl(I)$, $cl(M_i)$ denote the classes in $C(A)$ of I, M_i respectively. In fact γ is an isomorphism ([7], 4.7, Proposition 17), and $\gamma\circ i$: $K^0(T)\longrightarrow \mathbf{Z}\times C(A)$ is also an isomorphism (cf. [8]). Hence we have

$$K^0(T)\overset{i}{\xrightarrow{\sim}}K_0(T).$$

The isomorphism $\gamma \circ i$ induces

$$\tilde{K}(T) \xrightarrow{\sim} C(A).$$

Thus φ is bijective. Since dim Spec $A=1$ on the other hand, we can get the corollary in the introduction.

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