Kumamoto J. Sci, (Math.)
Vol. 12, 52~55 march (1977)

ON THE RANK OF A CERTAIN CURVATURE TENSOR OF A SASAKIAN MANIFOLD

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(Received November 29, 1976)

1. The Riemannian curvature tensor R of a Riemannian manifold (M, g) is regarded as a self adjoint operator R: $\wedge^2 M_p \longrightarrow \wedge^2 M_p$ for each $p \in M$. C. Udriste [5] proved that if (M,g) is a Riemannian locally symmetric space of dimension n and if the curvature tensor R has the maximal rank $n(n-1)/2$, then (M, g) is of constant curvature. In the case of a Kählerian manifold, K. Satō [3] proved that if (M,J,g) is a Hermitian locally symmetric space of dimension $2n$ and if the Riemannian curvature tensor R has the maximal rank n^2, then (M,J,g) is a space of constant holomorphic sectional curvature. In this note, we consider the case of a Sasakian manifold.

2. Let M be a $(2n+1)$-dimensional Sasakian manifold with the structure tensors ϕ, ξ, η and g:

$$(1)\quad \begin{cases} \phi^2 X = -X + \eta(X)\xi \\ \eta(\xi) = 1 \\ g(\phi X, \phi Y) = g(X, Y) - \eta(X)\,\eta(Y) \\ (\nabla_X \phi) Y = \eta(Y) X - g(X, Y)\xi, \end{cases}$$

where ∇ is the Riemannian connection for g and X and Y are tangent vectors on M. Let U be a small open neighborhood of an arbitrary point x of M such that the induced Sasakian structure on U, denoted by the same letters, is regular. Let Π: $U \longrightarrow \bar{U} = U/\xi$ be a local fibering, and let $(J, \bar{g})$ be the induced Kählerian structure on $\bar{U}$. Let R and $\bar{R}$ be the curvature tensors constructed by g and $\bar{g}$, respectively. For a vector field $\bar{X}$ on $\bar{U}$, we denote its horizonal lift (with respect to the connection form η) by $\bar{X}^*$. Then we have, for any vector fields $\bar{X}$, $\bar{Y}$ and $\bar{Z}$ on $\bar{U}$,

$$(2)\qquad (\bar{\nabla}_{\bar{X}} \bar{Y})^* = \nabla_{\bar{X}^*} \bar{Y}^* + d\eta(\bar{X}^*, \bar{Y}^*)\xi,$$

$$(3)\qquad (\bar{R}(\bar{X}, \bar{Y})\bar{Z})^* = R(\bar{X}^*, \bar{Y}^*)\bar{Z}^* + g(\phi\bar{Y}^*, \bar{Z}^*)\phi\bar{X}^* - g(\phi\bar{X}^*, \bar{Z}^*)\phi\bar{Y}^* - 2g(\phi\bar{X}^*, \bar{Y}^*)\phi\bar{Z}^*,$$

where $\bar{\nabla}$ is the Riemannian connection for $\bar{g}$ (cf. Ogiue [2]). Making use of these formulas, we get

$$(4)\qquad ((\bar{\nabla}_{\bar{V}}\bar{R})(\bar{X},\bar{Y})\bar{Z})^* = -\phi^2[(\nabla_{\bar{V}^*}R)(\bar{X}^*,\bar{Y})\bar{Z}^*]$$

for any vector fields $\bar{X},\bar{Y},\bar{Z}$ and $\bar{V}$ on $\bar{U}$ (Takahashi [4]).

A Sasakian manifold is said to be a locally ϕ-symmetric space if

$$(5)\qquad \phi^2[\nabla_V R)(X,Y)Z]=0$$

holds for any horizontal vectors X,Y,Z and V, where a horizontal vector means that it is horizontal with respect to the connection form η of the local fibering; namely, a horizontal vector is nothing but a vector which is orthogonal to ξ. By the definition of a locally ϕ-symmetric space and (4), we get

LEMMA 1 (Takahashi [4]). *A Sasakian manifold is locally ϕ-symmetric if and only if each Kählerian manifold, which is a base space of a local fibering, is a Hermitian locally symmetric space.*

Let r be an arbitrary fixed real number, and let A be a tensor field of type (1,2) defined by

$$(6)\qquad A(X)Y = d\eta(X,Y)\xi + r\eta(X)\phi Y - \eta(Y)\phi X.$$

The M-connection $\tilde{\nabla}$ is by definition

$$(7)\qquad \tilde{\nabla}_X Y = \nabla_X Y + A(X)Y.$$

We see that the tensor fields ϕ, ξ, η, g and A are parallel with respect to the M-connection $\tilde{\nabla}$ (Kato-Motomiya [1]). Let $\tilde{R}$ be the curvature tensor of $\tilde{\nabla}$. Then we get

$$(8)\qquad \tilde{R}(X,Y)Z = R(X,Y)Z + B(X,Y)Z,$$

where

$$(9)\qquad \begin{aligned} B(X,Y)Z = {} & \eta(Z)\,\{\eta(X)Y - \eta(Y)X\} \\ & + g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y + 2rg(\phi X,Y)\phi Z \\ & + \{g(X,Z)\,\eta(Y) - g(Y,Z)\,\eta(X)\}\xi. \end{aligned}$$

Now, let $\bar{X}$ and $\bar{Y}$ be vector fields on $\bar{U}$. Then, taking account of (2) and (6), we get

$$(10)\qquad (\bar{\nabla}_{\bar{X}}\bar{Y})^* = \tilde{\nabla}_{\bar{X}^*}\bar{Y}^*.$$

Making use of (10), $[\bar{X},\bar{Y}]^* = [\bar{X}^*,\bar{Y}^*] - \eta[\bar{X}^*,\bar{Y}^*])\xi$ and $\nabla_\xi \bar{X}^* = \nabla_{\bar{X}^*}\xi = \phi\bar{X}^*$, we get

(11) $(\bar{R}(\bar{X},\bar{Y})\bar{Z})^* = \tilde{R}(\bar{X}^*,\bar{Y}^*)\bar{Z}^* + (1+r)\,\eta([\bar{X}^*,\bar{Y}^*])\phi\bar{Z}^*.$

Hence, if we consider the M-connection $\tilde{\nabla}$ for $r=-1$, we see that, for any tangent vectors $\bar{X}$, $\bar{Y}$ and $\bar{Z}$ at $\Pi(x)\in\bar{U}$,

(12) $(\bar{R}(\bar{X},\bar{Y})\bar{Z})^* = \tilde{R}(\bar{X}^*,\bar{Y}^*)\bar{Z}^*$

holds good. On the other hand, since ξ and g are parallel with respect to the M-connection $\tilde{\nabla}$, we see that $\tilde{R}(\xi,X)=0$ holds good for any tangent vector X on U. Thus we get the following lemma:

LEMMA 2 *The rank of the curvature tensors $\bar{R}$ and $\tilde{R}$ are the same.*

We know the following:

LEMMA 3 (Ogiue [2]). *A Sasakian manifold is of constant ϕ-holomorphic sectional curvature k if and only if each Kählerian manifold, which is a base space of a local fibering, is of constant holomorphic sectional curvature $k+3$.*

LEMMA 4 (K. Satō [3]). *Let M be a Hermitian locally symmetric space of dimemsion $2n$. Then the Riemannian curvature tensor of M has the maximal rank n^2 if and only if M is of constant holomorphic sectional curvature $k\neq 0$.*

Combining Lemmas 1, 2, 3 and 4, we get the following:

THEOREM. *Let M be a Sasakian locally ϕ-symmetric space of dimension $2n+1$. Then the curvature tensor $\tilde{R}$ for $r=-1$ has the maximal rank n^2 if and only if M is of constant ϕ-holomorphic sectional curvature $k\neq -3$.*

If we define a ϕ-holomorphic M-sectional curvature $M(X)$ by

(13) $M(X) = g(\tilde{R}(X,\phi X)\,\phi X, X)$

for a unit horizontal vector X, the above Theorem is equivalent the following:

THEOREM$'$. *Let M be a Sasakian locally ϕ-symmetric space of dimension $2n+1$. Then the curvature tensor $\tilde{R}$ for $r=-1$ has the maximal rank n^2 if and only if M is of constant ϕ-holomorphic M-sectionl curvature $m\neq 0$.*

PROOF. Taking account of (8) and (9), we see that M is of constant ϕ-holomorphic sectional curvature k if and only if M is of constant ϕ-holomorphic M-sectional curvature $k+3$. Hence we get the conclusion.

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