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ON LINE GEOETRY

Toshiyuki MAEBASHI

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Introduction. The concepts of ruled surfaces, linear congruences and linear complex in classical differential geometry have some ambiguity, which we can see, e. g. in "Treatise" by L. P. Eisenhart or in "Vorlesungen" by W. Blaschke. In this note we wish to clear up this ambiguity in the frame-work of modern differential geometry, It has been known since J. Plücker that the straight lines in affine 3-space constitute a "Raum" of dimension 4. Actually it is isomorphic to the total space of a universal vector bundle over the Grassmann variety of planes through the origin of affine 3-space. We denote this smooth manifold of dimension 4 by X. The parallel translations to the origin define a Gaussian map: $X \longrightarrow RP(2)$ where $RP(2)$ denotes the real projective plane.

Let us go over to the definitions of ruled surfaces and linear congruences. Take the example of ruled surfaces. They are not surfaces in the strict sense, that is., not two dimensional manifolds, as we can see, e. g. in tangent surfaces. We can preferably define them as a kind of maps: a curve in X times the line of reals $\longrightarrow$ affine 3-space. This leads to the following definitions of ruled surfaces and linear congruences: By ruled surfaces (resp. linear congruences) we understand curves (resp. surfaces) in X which are transversal to the fibres of the Gaussian map. Actually we can assign to these curves (resp. surfaces), denoted Y, differentiable maps (natural in the geometrical sense): $Y \times \boldsymbol{R} \longrightarrow$ affine 3-space, which do not degenerate on a dense open subset of $Y \times \boldsymbol{R}$.

There is no adequate Riemannian metric in X, but we can introduce a positive Finslerian metric with geometric meaning, which may be useful in the differential geometric research of linear congruences. We want to note here that there have been not so many excellent examples of Finsler spaces in history.

1. The 3-dimensional euclidean space is a homogeneus space which the euclidean group $\boldsymbol{E}(3)$ acts on transitively. The subgroup of translations is isomorphic to the additive group $\boldsymbol{R}^3$ canonically. It is normal in $\boldsymbol{E}(3)$ and $\boldsymbol{E}(3)$ is isomorphic to the semi-direct product of $O(3)$ by $\boldsymbol{R}^3$. Hence we can write each element of $\boldsymbol{E}(3)$ as (A, T) with $A \in O(3)$ and $T \in \boldsymbol{R}^3$. The pair (A, T) acts on euclidean 3-space $\boldsymbol{R}^3$ by

(1) $$P \longmapsto AP+T$$

where P and T are supposed to be written in the form of column vectors.

Let $l, m, n,$ be the direction cosines of a straight line L and $P_0(x_0, y_0, z_0)$ a point on L. Write D for ${}^t(l, m, n)$ where ^{t}denotes the transpose. Then the equation of L is given by

(2) $$P=P_0+Du$$

with u a parameter ranging over $\boldsymbol{R}$. The euclidean displacement (1) sends L to a line:

$$P=(AP_0+T)+ADu.$$

By this action the set of lines is $\boldsymbol{R}^3$ becomes a homogeneous space on which $\boldsymbol{E}(3)$ acts transitively. The isotropy subgroup at the z-axis is

$$\left\{\left(\begin{pmatrix} B & O \\ O & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ c \end{pmatrix}\right) \middle| \ B \in O(2) \text{ and } c \in \boldsymbol{R}\right\}$$

and it is isomorphic to the product of $O(2)$ and $\boldsymbol{R}$. Therefore the variety of lines is $\boldsymbol{R}^3$ can be considered to be the homogeneous space $\boldsymbol{E}(3)/O(2)\times\boldsymbol{R}$. We denote it by X. Then $\dim X=\dim \boldsymbol{E}(3)-2=4$.

Now let H be the plane through the origin and perpendicular to a line L. We write P for the intersection of H and L. Then the map:

(3) $$L \longmapsto (H, P)$$

is a diffeomorphism of X onto the total space of the universal vector bundle over the Grassmann variety of planes through the origin. Hence we can see again that X is a smooth manifold of dimension 4. Let us consider the Gaussian map π of X onto real projective plane $\boldsymbol{RP}(2)$. This is the map which assigns to each line of X the parallel line through the origin. Then $(X, \pi, \boldsymbol{RP}(2))$ is a fibre bundle with the typical fibre $\boldsymbol{R}^2$. Let u_0, u_1, u_2 be the homogeneous coodinates of $K \in \boldsymbol{RP}(2)$. We set

$$v_i=\{K \mid u_i \neq 0\} \qquad (i=0,1,2).$$

Let us take the example of $i=2$. Then

$$v_2 \ni K \longmapsto \left(\frac{u_0}{u_2}, \frac{u_1}{u_2}\right) \in \boldsymbol{R}^2$$

is a chart of $\boldsymbol{RP}(2)$. We write

$$u=\frac{u_0}{u_2},\ v=\frac{u_1}{u_2}.$$

The equation $z=-ux-vy$ gives the plane perpendicular to K. We denote it by H_K. In this way we can assign to any triple $(K,x,y)\in V\times R^2$ a point on H_K: $(x,y,-ux-vy)$. We write it by $P(K,x,y)$. Then by the diffeomorphism (3) a line, denoted $L(K,x,y)$, corresponds to $(H_K,P(K,x,y))$. Hence we have a map: $K\times R^2 \longrightarrow \pi^{-1}(V_2)$. We write h_2 for the inverse of this bijection. h_2 is a chart on X which gives local triviality of the bundle $(X,\pi,RP(2))$. We can define charts h_0,h_1 in the same way. Thus we obtain "*a system of local decomposition*" $\{(V_i,h_i)\ i=0,1,2\}$.

2. Metrics. Let us now consider two functions on $X\times X$. Let $(L,L')\in X\times X$. Then the first function d_1 is defined by

$$d_1(L,L')=\text{the spherical distance between } (L) \text{ and } (L').$$

The second function is

$$d_2(L,L')=\text{the set-theoretical distance between } L \text{ and } L' = \min\{\overline{PP'}\mid P\in L \text{ and } P'\in L'\}.$$

Using the coodinates u,v,x,y we can express these functions as follows.

$$d_1(L,L')=\text{Arc cos}\frac{1+uu'+vv'}{\sqrt{1+u^2+v^2}\ \sqrt{1+u'^2+v'^2}}$$

and, supposing $L'\in\pi^{-1}(L)$,

$$d_2(L,L')=\frac{1}{c}\times \text{absolute value of}\begin{vmatrix} x-x' & y-y' & z-z' \\ u & v & 1 \\ u' & v' & 1\end{vmatrix}$$

where $z=-ux-vy, z'=-u'x'-v'y'$,

$$c=\sqrt{\begin{vmatrix} u & v \\ u' & v'\end{vmatrix}^2+\begin{vmatrix} u & 1 \\ u' & 1\end{vmatrix}^2+\begin{vmatrix} v & 1 \\ v' & 1\end{vmatrix}^2}.$$

Let $T(X)$ be the tangent vector bundle of X and $S(X)$ the sub-bundle of $T(X)$ which is constituted by the vectors tangent to the fibres of π. We now define

$$(4)\qquad ds^2=d_1(L,L+dL)^2+d_2(L,L+dL)^2.$$

Hence we can consider ds as a function defined over $T(X)$. Actually it gives a positive Finslerian metric to X. It is interesting to note that the square of this function is not differentiable. To see that we express it in terms of coodinates. For examle we have

$$ds^2=\left\{d\left(\frac{1}{\sqrt{1+u^2+v^2}}\right)\right\}^2+\left\{d\left(\frac{u}{\sqrt{1+u^2+v^2}}\right)\right\}^2$$
$$+\left\{d\left(\frac{v}{\sqrt{1+u^2+v^2}}\right)\right\}^2+$$
$$\frac{\begin{vmatrix} dx & dy & dz \\ u & v & 1 \\ du & dv & 0 \end{vmatrix}^2}{du^2+dv^2+(udv-vdu)^2}\ .$$

Let S be a submanifold in X. We denote by $T(S)$ (resp. $T(\pi)$) the tangent vector bundle of S (resp. the bundle of the vectors tangent to the fibres of π.) Then we have natural inclusions

$$T(S) \hookrightarrow T(X), \quad T(\pi) \hookrightarrow T(X),$$

S is called transversal to π if and only if the above natural inclusions give rise to an isomorphism:

$$T(S)\oplus T(\pi) \xrightarrow{\sim} T(X).$$

Suppose S be transversal to π. Then we can induce on S a positive Finslerian metric from that of X (4). It is a usual differentiable metric. Thus we can obtain excellent examples of Finslerian manifolds in the usual sense.

Note. I think that it should be better to write down (4) as

$$ds^2=dl^2+dm^2+dn^2+$$
$$\frac{\begin{vmatrix} dx & dy & dz \\ l & m & n \\ dl & dm & dn \end{vmatrix}^2}{\begin{vmatrix} l & m \\ dl & dm \end{vmatrix}^2+\begin{vmatrix} m & n \\ dm & dn \end{vmatrix}^2+\begin{vmatrix} l & n \\ dl & dn \end{vmatrix}^2}\ ,$$

using the direction cosines l, m, n of $L\in X$.

3. Given a ruled surface, the line of striction will be determined in the following manner. Let L and $L+dL$ be two generators. Let P and $P+dP$ be the points of meeting of the common perpendicular with L and $L+dL$ respectively. As $L+dL$ approaches L, the point P goes to a limiting position called the central point of generator L. The locus of the central points is the line of striction of this ruled surface.

Now we modify this concept so as to be adapted for our theory. For that we use the charts in Section 1. We write U for $\pi^{-1}(V_2)$. $C^\infty(U)$ and $\Gamma(U, T(X))$ stand for the ring of C^∞-functions on U and the sets of sections over U. Then $\Gamma(U, T(X))$ can be considered as a $C^\infty(U)$-module. Let us introduce a quadratic form

$$dldx + dmdy + dndz$$

on $\Gamma(U, T(X))$ where

$$z = -ux - vy$$

$$l = \frac{u}{\sqrt{u^2+v^2+1}}, \quad m = \frac{v}{\sqrt{u^2+v^2+1}}$$

$$n = \frac{1}{\sqrt{u^2+v^2+1}}.$$

Further we write σ for the quotient of two quadratic forms:

$$-\frac{dldx+dmdy+dndz}{dl^2+dm^2+dn^2}.$$

Then σ is a function defined on a dense open submanifold of $T(X)$. We define ρ_2, as an adequate substitute of the lines of restriction, by

$$(5) \qquad \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \sigma \begin{pmatrix} l \\ m \\ n \end{pmatrix}$$

Hence this is a differentiable map of the above submanifold into affine 3-space. In exactly the same way we can define ρ_1 by use of the chart h_1. These two maps coincide with each other on the intersection of their domains of definition. It follows that we can obtain a differentiable map of the dense open submanifold (of $T(X)$) constituted by the tangent vectors which are transversal to π, into affine 3-space, by gluing these maps. We denote it by ρ.

4. Let $G_2(T(X))$ be the fibre bundle over X whose fibre over $x \in X$ is the Grassmann manifold of 2-dimensional planes through the origin of $T_x(X)$, the fibre over x of $T(X)$. The set of 2-dimensional planes $T_x(X)$ which are transversal to π constitutes the total space of a subbundle of $G_2(T(X))$, denoted $G_{2,\pi}(X)$, where x ranges over X. Suppose $Z \in G_{2,\pi}(X)$. Then there exist min $\{\sigma(b) \mid b \in Z\}$ and max $\{\sigma(b) \mid b \in Z\}$. By (5), to these values of σ there correspond two points of affine 3-space, denoted $P_1(Z)$ and $P_2(Z)$ respectively. Let us define

$$R(Z) = (P_1(Z) + P_2(Z))/2.$$

Then R is a differentiable map of $G_{2,\pi}(X)$ into affine 3-space.

Let S be 2-dimensional submanifold of X which is trasversal to π. Then $x \longmapsto R(T_x(S))$ is a differentiable map of S into affine 3-space. We may call this the indicatrix of S.

Department of Mathamatics
Kumamoto University