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PROBLEMS IN H-SPACES AND NONASSOCIATIVE ALGEBRAS

J. P. HOLMES and A. A. SAGLE

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1. Introduction. In this paper we present a central problem and partial solution for the local analysis of an analytic H-space analogous to the local analysis of a Lie group by the Campbell-Hausdorff theorem. Thus (M, m) is an *analytic H-space* if M is an analytic manifold and $m: M\times M\rightarrow M$ is an analytic function so that there exists $e\in M$ with $m(e, x)=m(x, e)=x$ for all $x\in M$; [8]. Thus following Lie group theory we first consider the analysis of (M, m) near e by using a local H-space. A *local analytic H-space* is a triple (M, E, m) where M is an analytic manifold, E is an open set containing the point e, and m: $E\times E\rightarrow M$ is an analytic function satisfying $m(e,x)=m(x,e)=x$ for all $x\in E$. In this paper all (local) H-spaces are assumed to be analytic. For the analysis near e of (M, E, m) choose a coordinate function f with domain a neighborhood of e so that $f(e)=0$. There is a neighborhood D of 0 in R^n so that m is represented in terms of f by $f(m(x,y))=V(f(x), f(y))$ where $V: D\times D\rightarrow R^n$ is analytic. Thus (R^n, D, V) is a local H-space locally isomorphic with (M,E,m) at e. The function V has a Taylor's series representation at $(0,0)\in D$ as

$$V(x,y)=\sum_{k=0}^{\infty}\frac{1}{k!}\ \mathrm{V}^k(x,y)^k$$

where $V^k=V^k(0,0)$ is the kth derivative of V at $(0,0)$; see [3,4]. The derivative V^k is a symmetric k-linear function on $(R^n\times R^n)^k$ to R^n and it was shown in [4] that $V(0, 0)=0$, $V^1(x, y)=x+y$ and $V^2(x, y)^2=\alpha(x, y)$ is bilinear on R^n. Thus, an algebra $(R^n, +, \alpha)$ can be associated with the local H-space (M, E, m) relative to the coordinate function f. In the case of a Lie group G, f can be chosen to be a canonical coordinate and the algebra $(R^n, +, \alpha)$ is the Lie algebra associated with G. From the Campbell-Hausdorff theorem for a Lie group [6], each higher derivative $V^k(x, y)^k$ is a specific homogeneous polynomial of degree k given in terms of the algebra determined by $V^2(x, y)^2$. Thus the derivatives V^1 and V^2 determine the higher derivatives V^k and consequently determine V. This gives rise to the following general problem of giving a local classification of H-spaces.

MAIN PROBLEM. What conditions on the local H-space (M, E, m) and the coordinate function f imply there exists an integer N so that the terms $V^k(x, y)^k$ for $k \leq N$ determine the terms $V^n(x, y)^n$ for $n > N$ and consequently determine V. By "determine" we mean that for every $x, y \in R^n$, $V^n(x,y)^n$ for $n > N$ is in the subsystem of the algebraic structure $(R^n; V^1, V^2, \ldots, V^N)$ generated by x and y.

2. Partial Solution. We consider the above problem for power-associative analytic H-spaces with $N=2$. Thus suppose (M, E, m) is a local analytic H-space and x is in E. Let $x^0 = e$ and if n is a positive integer for which x^{n-1} is in E, let $x^n = m(x, x^{n-1})$. Then (M, E, m) is *power-associative* in case $x^{m+n} = m(x^n, x^m)$ whenever x^n and x^m is in E and x^{n+m} exists. Examples arise from Lie groups and the multiplicative structure of alternative algebras e.g. the 7-sphere S^7 obtained from the Cayley numbers of norm 1. Analogous to canonical coordinates for Lie groups we have the following results with the proof appearing in [2].

THEOREM 1. *Let (M, E, m) be a local analytic H-space.*

(i) *Let f be a coordinate function at e which induces the local H-space (R^n, D, V) as above. If the derivatives $V^k(sx, tx)^k = 0$ for $k \geq 2$, $s, t \in R$ and $x \in R^n$, then (M, E, m) is power-associative and $V(sx, tx) = (s+t)x$.*

(ii) *Conversely, if (M, E, m) is power associative, then there exists a coordinate function f at e which induces the local H-space (R^n, D, V) satisfying $V(0,0)=0$ and $V^k(sx, tx)^k = 0$ for $k \geq 2$, $s, t \in R$ and $x \in R^n$. In this case $V(sx, tx) = (s+t)x$ and the associated algebra $(R^n, +, \alpha)$ is anti-commutative (where $\alpha(X, Y) = V^2(X, Y)^2$).*

We apply these results to the construction of power-associative local analytic H-spaces from power-associative algebras in a manner analogous to the construction of the general linear group $GL(m, R)$ of non-singular $m \times m$ matrices from the algebra M_m of all $m \times m$ matrices. Thus let $(A, +, \cdot)$ be an n-dimensional power-associative algebra over the real field R which has a two-sided multiplicative identity element 1. With A as the underlying manifold, the local analytic H-space (A, A, m) is power-associative; where $m(x, y) = x \cdot y$ in A; we use $\cdot$ to denote m. Hence we may choose a canonical coordinate function f as in Theorem 1 for $(A, A, \cdot)$. We show that f is the local inverse function of the exponential function, E, computed in A by $E(x) = \sum_{k=0}^{\infty} 1/k!\ x^k$ as for $GL(n, R)$. Consequently the equation $f(m(x, y)) = V(f(x), f(y))$ of section 1 can be put in more familiar form

$$E(x) \cdot E(y) = E(V(x, y))$$

to compute the Taylor's series for V. We call V the canonical coordinate representation for m.

In this case we obtain $V^2(x, y)^2 = xy - yx = [x, y]$ the "commutator function" in $(A, +, \cdot)$.

An algebra $(A, +, \cdot)$ is said to be *alternative* provided it satisfies the identities $(x, x, y) = (y, x, x) = 0$ where $(x, y, z) = (xy)z - x(yz)$ is the "associator function". From [7] an alternative algebra is power-associative and the subalgebra, $A(x, y)$, of A generated by x, y and 1 is associative. Using this the following is proved in [2].

THEOREM 2. *If $(A, +, \cdot)$ is an alternative algebra and V is the canonical coordinate representation for $\cdot$ constructed above, then V^2 determines V in the sense that V^k, for $k>2$, is the specific homogeneous polynomial in the V^2 multiplication on A given by the Campbell-Hausdorff formula.*

Let A^- denote the algebra with vector space A and multiplication $[x, y] = V^2(x, y)^2$. With A an alternative algebra, A^- is a Malcev algebra [5,6]. If A is associative, then A^- is a Lie algebra and if A is the nonassociative algebra obtained from the Cayley numbers, then A^- is not Lie and is discussed in [5]. In the latter case the one-dimensional subspace $R1$ is the center of A^- and $\mathscr{A} = A^-/R1$ is a simple 7-dimensional Malcev algebra which can be regarded as the "tangent algebra" to the H-space S^7. The algebra $\mathscr{A}$ can be used to analyze S^7 in a manner analogous to the use of a Lie algebra in the analysis of a Lie group.

3. The converse. We now consider the converse of Theorem 2 where we assume V^2 determines the higher derivatives V^k for $k>2$ and show that with mild restrictions the underlying power-associative algebra $(A, +, \cdot)$ is alternative. Thus since $V^2(x, y)^2 = [x, y]$ we assume $V^3(x, y)^3$ is a homogeneous polynomial in $V^2(x, y)^2$; that is, we assume the condition

H: There are real numbers a, b such that
$$V^3(x, y)^3 = 3a[x, [x, y],] + 3b[y, [y, x],]$$
for all x, y in $(A, +, \cdot)$.

Since undetermined numbers a, b enter into the identities for A, we take up the slack by considering quasi-equivalent algebras.

DEFINITION. Let A be an algebra and let u, v be numbers so that $u + v = 1$ and $u - v \neq 0$. Let A^0 denote the algebra with vector space A and multiplication

$x \circ y = uxy + uyx$. A is *quasi-equivalent* to an algebra B in case there are numbers u, v as above so that $A^0 = B$ as algebras.

THEOREM 3. *Let $(A, +, \cdot)$ be a power-associative algebra with* 1 *and let $(A, A, \cdot)$ be the corresponding local analytic H-space. Let f be the canonical coordinate system which represents $(A, A, \cdot)$ by (R^n, D, V) as before where V satisfies condition H. Then,*

(i) *$(A, +, \cdot)$ or its complexification is quasi-equivalent to an algebra B satisfying $(y, x, x) = 0$ and $(x, x, y) = \lambda[x, [x, y]]$ for some complex number λ;* [1].

(ii) *If the power-associative algebra $(A, +, \cdot)$ contains an idempotent $e = e^2$ which is not in the center of $(A, +, \cdot)$, then $(A, +, \cdot)$ or its complexification is quasi-equivalent to an alternative algebra.*

(iii) *If the algebra $(A, +, \cdot)$ in* (i) *or its complexification is a semi-simple power-associative algebra, then $(A, +, \cdot)$ or its complexification is quasi-equivalent to an alternative algebra.*

Thus, in this last two cases we see that when a power-associative algebra $(A, +, \cdot)$ induces a local H-space and the canonical coordinate representation V of the multiplication is determined by V^1 and V^2, then $(A, +, \cdot)$ is essentially an alternative algebra.

4. Homomorphisms of local *H*-spaces. The analytic homomorphisms of local analytic H-spaces are studied using canonical coordinates with results similar to those of Lie groups.

THEOREM 4. *Let (M, E, m) be a power-associative local analytic H-space.*

(i) *If h is a continuous homomorphism of (M, E, m) which is differentiable at e, then h is analytic at e.*

(ii) *Let f be the canonical coordinate function which represents (M, E, m) as (R^n, D, V) and let h be an analytic homomorphism of (M, E, m) into itself. Then h can be represented near e by $h = f^{-1} \circ T \circ f$ where $T: R^n \rightarrow R^n$ is linear and satisfies $T(V^k(x, y)^k) = V^k(Tx, Ty)^k$ for k is a positive integer and $x, y \in R^n$. Conversely, any linear map T satisfying these equations induces an analytic homomorphism of (M, E, m) into itself.*

Thus these results give a solution to the Main Problem for $N = 2$. Now, what algebraic structures solve this problem for $N \geq 3$?

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University of Hawaii at Hilo
Hilo, HI 96720