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ON THE SECOND REDUCTION THEOREM OF P. FONG

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Introduction

Here we refer to the Theorem 2D of Fong [7] as the Second reduction Theorem (while the First reduction Theorem means the Theorem 2B of the same paper). We begin with a brief outline of the character correspondences given in that Theorem.

Let G be a finite group and H a normal p'-subgroup of G. Let T be an irreducible complex representation of G and X an irreducible component of T_H. Assume that X is G-stable. Then as is well known, T is the tensor product of two projective representations; $T=\hat{T}\otimes\hat{X}$, where $\hat{T}$ is a projective representation of $\bar{G}=G/H$ and $\hat{X}$ is a projective representation of G such that $\hat{X}_H=X$ (Clifford [4]).

Let Ω be an algebraic number field containing the $|G|$-th roots of unity. By Reynolds [13], we may assume that $\hat{T}$ and $\hat{X}$ are written in Ω and that the 2-cocycle, say α, of $\bar{G}$ arising from the projective representation $\hat{T}$ takes e-th roots of unity as values, where e is the order of α (computed in the field of complex numbers). Let $\bar{X}$ and $\mathfrak{X}$ be p-modular representations induced by X and $\hat{X}$ respectively.

Using α, we construct a central extension

$$1\longrightarrow Z\longrightarrow\hat{\hat{G}}\longrightarrow\bar{G}\longrightarrow 1$$

, where Z is a cyclic group of order e.

Then $\hat{T}$ is lifted to a representation of $\hat{\hat{G}}$. We proceed on fixed coset representatives of Z in $\hat{\hat{G}}$. Since $p\nmid|H|$, the induced modular representation $\bar{X}$ is irreducible. If M is an irreducible modular constituent of T, then M_H contains $\bar{X}$ as a component. Since $\mathfrak{X}$ is an irreducible projective representation of G such that $\mathfrak{X}_H=\bar{X}$, we have $M=\hat{M}\otimes\mathfrak{X}$ for some projective representation $\hat{M}$ of $\bar{G}$, which has 2-cocycle $\bar{\alpha}$.

Furthermore, from this and that e is prime to p (Fong [7] pp. 274), it follows that if T and T' belong to the same p-block of G, then they determine the same 2-cocycle α. In particular, $\hat{T}'$ is lifted to a representation of $\hat{\hat{G}}$.

From the aboves and from that (*without restrictions on the characteristic of Ω and the order of H*)

(*) *$T \leftrightarrow \hat{T}$ is a 1-1 correspondence between $Irr(\Omega G|X)$ and $Irr(\Omega\hat{\hat{G}})_\alpha$, where the latter is a set of the (non-equivalent) irreducible representations of $\hat{\hat{G}}$ which have 2-cocycle α when regarded as projective representations of G.*

, we may now conclude that with respect to the correspondences $T \leftrightarrow \hat{T}$ and $M \leftrightarrow \hat{M}$, the decomposition numbers are same; $d_{T,M} = d_{\hat{T},\hat{M}}$. Incidently, it follows that they occur between the p-block of G and that of $\hat{\hat{G}}$ to which T and $\hat{T}$ belong respectively.

The above (*) was obtained by Clifford [4]. Since then, several authors has obtained it in their study of endomorphism rings of induced G-modules. On the other hand, Dade has shown it in a most native way using the language of Clifford systems (Dade [6] Prop. 8.10).

The aim of this parer is to interpret the Second reduction Theorem from the Dade's theory of Clifford systems. It will turn out that the correspondences given above are induced from a certain block ideal isomorphism over a local ring. Though the theory of Clifford systems is considerably deep, we need only an introductory part of it. So in the first section, we shall summarize necessary results with proofs.

In the final section, we shall show a result which will refine Satz 7 of Huppert [10] as well as Theorem 2.1 of Hamernik and Michler [9].

Recently, Cliff [3] has extended the above (*) to a correspondence between indecomposable (modular) representations. We shall refer to his result for the completeness.

1. Basic materials of this section will be found in Dade [6]. Let G be a finite group. Let $\mathfrak{o}$ be a local domain with quotient field Ω and residue field k. We consider a graded Clifford system $\mathfrak{A}^0$, $\{\mathfrak{A}_g^0 \mid g \in G\}$ over $\mathfrak{o}$. Hence it holds that

(1) $\mathfrak{A} = \oplus \sum_{g \in G} \mathfrak{A}_g^0$

(2) $\mathfrak{A}_g^0 \mathfrak{A}_h^0 = \mathfrak{A}_{gh}^0$ *for all* $g, h \in G$

(3) $1 \in \mathfrak{A}_1^0$.

Here we assume that $\mathfrak{A}_g^0$ is free of finite rank over $\mathfrak{o}$ for all $g \in G$. Let $\mathfrak{C}^0$ and $\mathfrak{C}_g^0$ be the centralizers of $\mathfrak{A}_1^0$ in $\mathfrak{A}^0$ and in $\mathfrak{A}_g^0$ respectively. Then $\mathfrak{C}^0 = \oplus \sum_{g \in G} \mathfrak{C}_g^0$.

Let $\mathfrak{A} = \Omega \otimes \mathfrak{A}^0$ and $\bar{\mathfrak{A}} = k \otimes \mathfrak{A}^0$. Then both $\mathfrak{A}$, $\{\mathfrak{A}_g = \Omega \otimes \mathfrak{A}_g^0 \mid g \in G\}$ and $\bar{\mathfrak{A}}$, $\{\bar{\mathfrak{A}}_g = k \otimes \mathfrak{A}_g^0 \mid g \in G\}$ are graded Clifford systems over Ω and k respectively.

In case the that $\mathfrak{o}$ is a field, the following result has been proved by Dade. Also, the first statement below is classical.

THEOREM 1. *Assume* $\mathfrak{A}_1^0$ *is central separable over* $\mathfrak{o}$. *Then we have*

(1) $\mathfrak{C}^0 \otimes \mathfrak{A}_1^0 \longrightarrow \mathfrak{A}^0$

$\quad\;\; \cup \qquad\qquad\quad \cup$

$x \otimes y \longrightarrow xy$, *as* $\mathfrak{o}$-*algebras, under which* $\mathfrak{C}_g^0 \otimes \mathfrak{A}_1^0$ *is mapped onto* $\mathfrak{A}_g^0$

(2) $\mathfrak{C}_g^0$ *is rank one over* $\mathfrak{o}$ *and* $\mathfrak{C}_g^0\, \mathfrak{C}_h^0 = \mathfrak{C}_{gh}^0$ *for all* $g, h \in G$

(3) $\Omega \otimes \mathfrak{C}^0 \cong \mathfrak{C} = C_{\mathfrak{A}}(\mathfrak{A}_1)$ and $k \otimes \mathfrak{C}^0 \cong \overline{\mathfrak{C}} = C_{\overline{\mathfrak{A}}}(\overline{\mathfrak{A}}_1)$.

PROOF. The first statement is clear from Theorem 3.1 and Theorem 3.3 of Auslander and Goldman [2]. Clearly we have $\mathfrak{C}_g^0\, \mathfrak{C}_h^0 \subset \mathfrak{C}_{gh}^0$. On the other hand, from $\mathfrak{A}_g^0\, \mathfrak{A}_h^0 = \mathfrak{A}_{gh}^0$ we have $\mathfrak{C}_g^0\, \mathfrak{C}_h^0 \otimes \mathfrak{A}_1^0 = \mathfrak{C}_{gh}^0 \otimes \mathfrak{A}_1^0$, whence we get $\mathfrak{C}_g^0\, \mathfrak{C}_h^0 = \mathfrak{C}_{gh}^0$ since $\mathfrak{A}_1^0$ is free over $\mathfrak{o}$. Since $\mathfrak{C}_1^0 = \mathfrak{o}$ by the assumption, we get the first half of (2) by the same method as in the proof of Lemma 14.2 of Dade [6]. Since $\mathfrak{A}_1^0$ is projective over its enveloping algebra, say Δ^0, we have a natural isomorphism $k \otimes \mathfrak{C}^0 \cong k \otimes \mathrm{Hom}_{\Delta^0}(\mathfrak{A}_1^0, \mathfrak{A}^0) \cong \mathrm{Hom}_{\overline{\Delta}}(\overline{\mathfrak{A}}_1, \overline{\mathfrak{A}}) = \overline{\mathfrak{C}}$, where $\overline{\Delta} = k \otimes \Delta^0$. This completes the proof.

Now, let H be a normal subgroup of G and $X \in Irr(\Omega H)$. We assume that Ω is a splitting field for H and X is G-stable. Then $\mathfrak{A}_1 = \Omega H/(0: X)$ is a central separable algebra over Ω, being isomorphic to the full matrix algebra $M(n, \Omega)$, $n = dim_{\Omega} X$. From the assumption, we have $g(0: X)g^{-1} = (0: X)$ for all $g \in G$. We form a Ω-algebra $\mathfrak{A} = \Omega G/(0: X)\Omega G$ and let $\mathfrak{A}_{\bar{g}} = (\Omega Hg + (0: X)\, \Omega G)/(0: X)\, \Omega G$ for $\bar{g} = \bar{g} \in \bar{G} = G/H$. Then $\mathfrak{A}$, $\{\mathfrak{A}_{\bar{g}} \mid \bar{g} \in \bar{G}\}$ is a graded Clifford system over Ω. We have $\mathfrak{A} \cong \mathfrak{C} \otimes \mathfrak{A}_1$ by Theorem 1, where $\mathfrak{C} = C_{\mathfrak{A}}(\mathfrak{A}_1)$ as above. We choose a Ω-basis $\{\lambda_{\bar{g}}\}_{\bar{g} \in \bar{G}}$ such that $\lambda_{\bar{g}} \in \mathfrak{C}_{\bar{g}}$ and $\lambda_1 = 1$. Then we have

$$\lambda_{\bar{g}} \lambda_{\bar{h}} = \alpha(\bar{g}, \bar{h})\ \lambda_{\bar{g}\bar{h}} \text{ for some } \alpha(\bar{g}, \bar{h}) \in \Omega^* = \Omega - \{0\}.$$

Hence α is a 2-cocycle of $\bar{G}$. Let g' be the image of $g \in G$ by the natural map $\Omega G \to A$. Then there is a unique element n_g of $\mathfrak{A}_1$ such that $g' = \lambda_{\bar{g}} \otimes n_g$, so that we have $n_g n_h = \alpha(\bar{g}, \bar{h})^{-1} n_{gh}$ for all $g, h \in G$. In particular X affords a projective representation of G with a 2-cocyble β such that $\beta(g, h) = \alpha(\bar{g}, \bar{h})^{-1}$ for all $g, h \in G$. Thus we get the following well-known results.

COROLLARY 1. *There is a* 1-1 *correspondence*

$$\begin{array}{ccc} Irr(\Omega G \mid X) & \longleftrightarrow & Irr(\mathfrak{C}) \\ \cup & & \cup \\ M & \longleftrightarrow & \hat{M} \end{array}$$

such that $M \cong \hat{M} \otimes X$ *as* $\mathfrak{A} = \mathfrak{C} \otimes \mathfrak{A}_1$ *-modules.*

PROOF. Clear since $\mathfrak{A}_1$ is isomorphic to $M(n, \Omega)$.

COROLLARY 2. *Let us denote by the same symbol X the representation afforded by the ΩH-module X. Let T be an irreducible representation of G whose restriction to H contains X as a component.*

Then T is the tensor product of two projective repersentations; $T=\hat{T}\otimes\hat{X}$, where $\hat{T}$ is a projective representation of $\bar{G}$ with α as its 2-cocycle and $\hat{X}$ is a projective representation of G such that $\hat{X}_H=X$.

COROLLARY 3. *Let α be as above. Then there is a simple ΩG-module M such that $M_H=X$ if and only if α is cohomologous to the identity in $H^2(\bar{G}, \Omega^*)$.*

PROOF. "if part" is clear from Corollary 1 and that $\mathfrak{C}\cong\Omega\bar{G}$. Suppose there is a simple ΩG-module M such that $M_H=X$, then we have $dim_\Omega\ \hat{M}=1$. In other words, the twisted group ring $\mathfrak{C}$ has a one dimensional representation μ: $\mathfrak{C}\longrightarrow\Omega$, whence it follows that $\alpha(\bar{g},\bar{h})=\mu(\bar{g})\ \mu(\bar{h})\ \mu(\bar{g}\bar{h})^{-1}\sim 1$, where $\mu(\bar{g})=\mu(\lambda_{\bar{g}})$.

COROLLARY 4. *Assume that there is a simple ΩG-module $\hat{X}$ such that $\hat{X}_H=X$. Then there is a 1-1 correspondence*

$$\begin{array}{ccc} Irr\ (\Omega G\,|\,X) & \longleftrightarrow & Irr\ (\Omega\bar{G}) \\ \uplus & & \uplus \\ M & \longleftrightarrow & \hat{M} \end{array}$$

such that $M\cong\hat{M}\otimes\hat{X}$ as ΩG-modules.

2. In what follows, we shall use the following notation; p is a fixed prime number. Q is a Sylow p-subgroup of G. Ω is an algebraic number field containing the $|G|$-th roots of unity. Let $\mathfrak{p}$ be a prime divisor of p in Ω with $\mathfrak{o}$ the ring of $\mathfrak{p}$-integers and k the residue field. If M is an indecomposable kG-module, then we denote by $vx_G(M)$ a vertex of M.

In this section we assume that

(1) $G\triangleright H$, $p\nmid |H|$

(2) $X\in Irr(\Omega H)$ and $I(X)=G$, where $I(X)$ denotes the inertial group of X in G.

We apply arguments in the preceding section. Let e be the order of the 2-cohomology class of α. We know that e is prime to p (and divides $[G: H]$, see Fong [7]). Also by the well-known argument due to Schur (see Curtis and Reiner [5] § 53), there is a 2-cocycle α' equivalent to α, such that

(1) $\alpha'(\bar{g},\bar{h})^e=1$ *and* $\alpha'(\bar{g},\bar{1})=\alpha'(1,\bar{h})=\bar{1}$ *for all* $g, h\in G$.

(2) $\alpha'(\bar{g},\bar{h})=1$, *for all* $g, h\in Q$.

Considering α as a 2-cocycle of G, we construct a central extension

$$1\longrightarrow Z\longrightarrow\hat{G}\overset{f}{\longrightarrow}G\longrightarrow 1$$

, where Z is a cyclic group of order e.

We see easily

(3) *there exists* $\hat{H}\lhd\hat{G}$ *such that* $\hat{H}\overset{f}{\cong}H$ *and* $\hat{H}\cap Z=1$.

(4) Q *may be identified with a Sylow p-subgroup of* $\hat{G}$.

Since Ω is also a splitting field for $\hat{G}$ (Reynolds [13]), we have

(5) *there is simple* $\Omega\hat{G}$*-module* $\hat{X}$ *such that* $\hat{X}_{\hat{H}}=X$, X *being regarded as* $\hat{H}$*-module through* f.

Let $X^0(\hat{X}^0$ resp.) be an $\mathfrak{o}\hat{H}$-lattice ($\mathfrak{o}\hat{G}$-lattice resp.) such that $X=\Omega\otimes X^0$ ($\hat{X}=\Omega\otimes\hat{X}^0$ resp.). Since $p\nmid|H|$, X^0 is uniquely determined up to $\mathfrak{o}\hat{H}$-isomorphisms and we have $\mathrm{Hom}_{\hat{H}}(X^0,X^0)=\mathfrak{o}$. Furthermore, $(0:X^0)$ is a two sided (direct) component of $\mathfrak{o}\hat{H}$ and $\mathfrak{A}_1^0=\mathfrak{o}\hat{H}/(0:X^0)$ is central separable over $\mathfrak{o}$, being isomorphic to the full matrix algebra $M(n,\mathfrak{o})$, $n=$rank of X^0 over $\mathfrak{o}$. Hence it follows that $\mathfrak{A}^0=\mathfrak{o}\hat{G}/(0:X^0)$ $\mathfrak{o}\hat{G}$ is a two sided component of $\mathfrak{o}\hat{G}$ and is a direct sum of blocks of $\mathfrak{o}\hat{G}$; $\mathfrak{A}^0=\oplus\sum_{i=1}^{r}\mathfrak{B}_i$.

As in the field case, $\mathfrak{A}^0$, $\{\mathfrak{A}_{\bar{g}}^0\,|\,\bar{g}\in\mathfrak{G}=\hat{G}/\hat{H}\}$ is a graded Clifford system over $\mathfrak{o}$, where $\mathfrak{A}_{\bar{g}}^0=(\mathfrak{o}\hat{H}g+(0:X^0)\mathfrak{o}\hat{G})/(0:X^0)\mathfrak{o}\hat{G}$, for $g\in\hat{G}$. By Theorem 1, we have $\mathfrak{A}^0\cong\mathfrak{C}^0\otimes\mathfrak{A}_1^0$. Moreover from Corollary 3 and (5), we conclude that $\mathfrak{C}=\Omega\otimes\mathfrak{C}^0$ is isomorphic to the group ring $\Omega\mathfrak{G}$, whence we get $\mathfrak{C}^0\cong\mathfrak{o}\mathfrak{G}$.

LEMMA 1. *Let $\mathfrak{R}$ and $\mathfrak{S}$ be $\mathfrak{o}$-orders. Let X be an $\mathfrak{S}$-lattice such that* $\mathrm{Hom}_{\mathfrak{S}}(X,X)=\mathfrak{o}$. *Then, if Y and Y' are $\mathfrak{R}$-lattices, the map* $\varphi\to\varphi\otimes 1$: $\mathrm{Hom}_{\mathfrak{R}}(Y,Y')\to\mathrm{Hom}_{\mathfrak{S}\otimes\mathfrak{R}}(Y\otimes X, Y'\otimes X)$ *is an $\mathfrak{o}$-isomorphism.*

PROOF. See the proof of Lemma (51.2) [5].

We remark that $\{\mathfrak{B}_1,\cdots,\mathfrak{B}_r\}$ is the set of blocks of $\mathfrak{o}\hat{G}$ which cover the block ideal $\mathfrak{A}_1^0$ of $\mathfrak{o}\hat{H}$. We remark also that since $p\nmid|Z|$, $\mathfrak{o}G$ is isomorphic to a two sided component of $\mathfrak{o}\hat{G}$ and hence, if $\mathfrak{B}$ is a block of $\mathfrak{o}G$ which covers $\mathfrak{A}_1^0$, then $\mathfrak{B}=\mathfrak{B}_i$ for some i $(1\leq i\leq r)$.

Now we have arrived at the Second reduction Theorem of Fong.

THEOREM 2 (Fong [7]) *Let $\mathfrak{B}$ be a block of $\mathfrak{o}G$ which covers the block ideal $\mathfrak{A}_1^0$ of $\mathfrak{o}H$ and let $\mathfrak{G}=\hat{G}/\hat{H}$.*

I. *there is a block $\hat{\mathfrak{B}}$ of $\mathfrak{o}\mathfrak{G}$ such that $\mathfrak{B}\cong\hat{\mathfrak{B}}\otimes\mathfrak{A}_1^0$ as $\mathfrak{o}$-algebras.*

The diagram

$$\begin{array}{ccc} \mathfrak{o}\hat{G} & \xrightarrow{f} & \mathfrak{o}G \\ \downarrow & & \\ \mathfrak{A}^0 & & \downarrow \\ \downarrow & & \\ \hat{\mathfrak{B}}\otimes\mathfrak{A}_1^0 & \longrightarrow & \mathfrak{B} \end{array}$$

is commutative, where the vertical maps are projections.

Hence the way of regarding a $\mathfrak{B}$*-module as a* $\hat{\mathfrak{B}}\otimes\mathfrak{A}_1^0$*-module through the isomorphism above is compatible with the one of regarding an* $\mathfrak{o}G$*-module as an* $\mathfrak{o}\hat{G}$*-module through* f.

II. *there is a* 1-1 *correspondence*

$$\begin{array}{ccc} Irr\,(\Omega\otimes\mathfrak{B}) & \longleftrightarrow & Irr\,(\Omega\otimes\hat{\mathfrak{B}}) \\ \Psi & & \Psi \\ V & \longleftrightarrow & \hat{V} \end{array}$$

, *such that* $V\cong\hat{V}\otimes\hat{X}$ *as* $\Omega\hat{G}$*-modules, where* $\hat{X}$ *is a fixed* $\Omega\hat{G}$*-module satisfying* (5).

III. (Cliff [3]) *there is a* 1-1 *correspondence*

$$\begin{array}{ccc} Ind\,(\bar{\mathfrak{B}}) & \longleftrightarrow & Ind\,(\bar{\mathfrak{b}}) \\ \Psi & & \Psi \\ M & \longleftrightarrow & \hat{M} \end{array}$$

, *such that* $M\cong\hat{M}\otimes\mathfrak{X}$ *as* $k\hat{G}$*-modules, where* $\bar{\mathfrak{B}}=k\otimes\mathfrak{B}$, $\bar{\mathfrak{b}}=k\otimes\hat{\mathfrak{B}}$ and $\mathfrak{X}=\hat{X}^0/\mathfrak{p}\hat{X}^0$.

M is irreducible (projective resp.) if and only if $\hat{M}$ is irreducible (projective resp.).

Moreover we have $vx_G(M)\hat{H}/\hat{H}=vx_{\mathfrak{G}}(\hat{M})$ *(we assume* $vx_G(M)$ *is contained in* Q*).*

The same holds between $Ind^0(\mathfrak{B})$ *and* $Ind^C(\hat{\mathfrak{B}})$ *by replacing* $\mathfrak{X}$ *with* $\hat{X}^0$*, where* $Ind^0\,(\mathfrak{B})$ *is the subset of* $Ind\,(\mathfrak{B})$ *consisting of of* $\mathfrak{B}$*-lattices.*

IV. $\mathfrak{B}$ *and* $\hat{\mathfrak{B}}$ *have isomorphic defect groups, the same decomposition matrix and the same Cartan matrix.*

PROOF. First we shall show III for $\mathfrak{B}$-lattices. Let M be a $\mathfrak{B}$-lattice and let $\hat{M}=\mathrm{Hom}_{\mathfrak{A}_1^0}(\hat{X}^0, M)$. Since the operations of $\hat{\mathfrak{B}}$ and $\mathfrak{A}_1^0$ commute on M, we can naturally make $\hat{M}$ into a $\hat{\mathfrak{B}}$-module. Then $\hat{M}\otimes\hat{X}^0$ is a $\hat{\mathfrak{B}}\otimes\mathfrak{A}_1^0=\mathfrak{B}$-module and is isomorphic to M. In fact, the map $\varphi\otimes x\rightarrow\varphi(x)$ ($\varphi\in\hat{M}$, $x\in\hat{X}^0$) is a $\mathfrak{B}$-homomorphism $\hat{M}\otimes\hat{X}^0\rightarrow M$. Furthermore, we have $M_{\mathfrak{A}_1^0}\cong sX^0$ for some $s\geq 1$ and then $\hat{M}\otimes\hat{X}^0\cong s\,\mathrm{Hom}_{\mathfrak{A}_1^0}(X^0, X^0)\otimes X^0\cong sX^0\cong M$ (of course, the composite map coinsides with the map given above).

If F and F' are $\hat{\mathfrak{B}}$-lattices, then by Lemma 1 we have $\mathrm{Hom}_{\hat{\mathfrak{B}}}(F,F')\cong\mathrm{Hom}_{\mathfrak{B}}(F\otimes\hat{X}^0, F'\otimes\hat{X}^0)$. We see easily from this that $F\otimes\hat{X}^0$ is indecomposable if and only if F is and that $F\otimes\hat{X}^0\cong F'\otimes\hat{X}^0$ if and only if $F\cong F'$. Thus we have shown the first statement of III. The second one is clear.

To show the third one, let D be a p-subgroup of G (contained in Q) and $\hat{T}=Z\hat{H}D$. We note that ${}_GM$ is D-projective $\Longleftrightarrow M<\oplus\,\mathfrak{o}G\otimes_D M=\mathfrak{o}\hat{G}\otimes_{DZ}M\Longleftrightarrow{}_{\hat{G}}M$ is

DZ-projective$\Longleftrightarrow {}_{\hat{G}}M$ is D-projective.

Now, by Lemma 1, $\mathrm{Hom}_{\mathfrak{D}\hat{G}}(\hat{M}\otimes\hat{X}^0, (\mathfrak{D}\hat{G}\otimes_{\hat{T}}\hat{M})\otimes\hat{X}^0)=\mathrm{Hom}_{\mathfrak{A}^0}(\hat{M}\otimes\hat{X}^0, (\mathfrak{D}\mathfrak{G}\otimes_{\mathfrak{T}}\hat{M})\otimes\hat{X}^0)=\mathrm{Hom}_{\mathfrak{D}\mathfrak{G}}(\hat{M}, \mathfrak{D}\mathfrak{G}\otimes_{\mathfrak{T}}\hat{M})$, where $\mathfrak{T}=\hat{T}/\hat{H}$.

Therefore, $\hat{M}<\oplus\,\mathfrak{D}\mathfrak{G}\otimes_{\mathfrak{T}}\hat{M}\Longleftrightarrow\hat{M}\otimes\hat{X}^0<\oplus\,(\mathfrak{D}\hat{G}\otimes_{\hat{T}}\hat{M})\otimes\hat{X}^0\Longleftrightarrow\hat{M}\otimes\hat{X}^0<\oplus\,\mathfrak{D}\hat{G}\otimes_{\hat{T}}(\hat{M}\otimes\hat{X}^0)$.

Thus ${}_GM$ is D-projective $\Longleftrightarrow {}_{\hat{G}}M$ is D-projective $\Longleftrightarrow {}_{\hat{G}}M$ is $\hat{T}$-projective $\Longleftrightarrow {}_{\mathfrak{G}}\hat{M}$ is $\mathfrak{T}$-projective $\Longleftrightarrow {}_{\mathfrak{G}}\hat{M}$ is $D\hat{H}/\hat{H}$-projective, which implies that $vx_G(M)\hat{H}/\hat{H}=vx_{\mathfrak{G}}(\hat{M})$. This completes the proof of *III*.

The first statement of IV will follow from III by the Green's vertex theory. All others are clear from the argument so far.

3. In this section, we assume G is p-solvable. A part of the following Theorem had been obtained by Huppert [10]. Our proof is fairly routine in view of Serre's method used in the proof of the Fong-Swan Theorem (Serre [14] n^0 17.6).

Let $\nu(n)$ be the exponent of the highest p-power dividing an integer n.

THEOREM 3. *Let M be a simple kG-module. Then, there exist a subgroup T of G and a simple kT-module L such that*

(1) *$vx(M)$ is conjugage to a Sylow p-subgroup of T*

(2) *$p\nmid \dim_k L$*

(3) *$M\cong kG\otimes_T L$*

COROLLARY 5 (Hamernik and Michler [9]).
$\nu(\dim_k M)=a-\nu(|vx(M)|)$, where $a=\nu(|G|)$.

PROOF (of Theorem 3) We prove by the induction on $|G|$ and $\nu(|G|)$. We may clearly assume $O_p(G)=1$. Let $H=O_{p'}(G)$. Let X be a simple ΩH-module such that $\bar{X}$ is a component of M_H and $I=I(X)=I(\bar{X})$. There exists a simple kI-module M' such that $M=kG\otimes_I M'$. We see easily that $vx_G(M)=vx_I(M')$. Hence if $G>I$, the assertion follows from the induction on $|G|$.

Assume that $G=I$. Using the same notation as in the Theorem 2, we have $M\cong\hat{M}\otimes\hat{X}$ and $vx_G(M)\hat{H}/\hat{H}=vx_{\mathfrak{G}}(\hat{M})$. Since $O_p(\mathfrak{G})\neq 1$, the assertion holds for $k\mathfrak{G}$-module $\hat{M}$ by the induction on $\nu(|G|)$. So that, there exist a subgroup $\hat{T}$ of $\hat{G}$ containing both $\hat{H}$ and Z, and a simple $k\mathfrak{T}$-module $\hat{L}$, where $\mathfrak{T}=\hat{T}/\hat{H}$ such that

(1) $vx_{\mathfrak{G}}(\hat{M})$ is conjugate to a Sylow p-subgroup of $\mathfrak{T}$

(2) $p\nmid \dim_k \hat{L}$

(3) $\hat{M}\cong k\mathfrak{G}\otimes_{\mathfrak{T}}\hat{L}=k\hat{G}\otimes_{\hat{T}}\hat{L}$

Let $T=f(\hat{T})\cong\hat{T}/Z$. Then $vx(M)$ is conjugate to a Sylow subgroup of T.

We have $M \cong \hat{M} \otimes \mathfrak{X} \cong k\hat{G} \otimes_{\hat{T}} (\hat{L} \otimes \mathfrak{X}) = k\hat{G} \otimes_{\hat{T}} L$, where $L = \hat{L} \otimes \mathfrak{X}$ and $p \nmid dim_k L$. Since Z acts trivially on M, we get $M \cong kG \otimes_T L$. This completes the proof.

Appendix

Here we shall give a ring theoritical interpretation on the First reduction Theorem from a result of Morita [11].

Let H be a normal subgroup of G and ε_1 a central primitive idempotent of $\mathfrak{o}H$. Let G_1 be the stabilizer of ε_1, namely $G_1 = \{g \in G \mid g^{-1}\varepsilon_1 g = \varepsilon_1\}$ and let $\{g_1 = 1, g_2, \cdots, g_r\}$ be a set of coset representatives of G_1 in G; $G = \bigcup G_1 g_i$. We put $\varepsilon_i = g_i^{-1} \varepsilon_1 g_i$ and $\varepsilon = \varepsilon_1 + \cdots + \varepsilon_r$.

Then, the Theorem 2 of Morita [11] may be stated as follows.

THEOREM. *$\mathfrak{o}G\varepsilon$ is isomorphic to the full matrix ring $M(r, \mathfrak{o}G_1\varepsilon_1)$. If $p \nmid |H|$, then $\mathfrak{o}G_1\varepsilon_1 \cong \mathfrak{C} \otimes \mathfrak{o}H\varepsilon_1$, so that $\mathfrak{o}G\varepsilon \cong M(r, \mathfrak{C}) \otimes \mathfrak{o}H\varepsilon_1$ where $\mathfrak{C}$ is a certain twisted group ring of G_1/H over $\mathfrak{o}$.*

PROOF. We have that $\mathfrak{o}G\varepsilon = \oplus \sum_{i=1}^{r} \mathfrak{o}G\varepsilon_i$ with $\mathfrak{o}G\varepsilon_i \cong \mathfrak{o}G\varepsilon_1$ as left $\mathfrak{o}G$-modules for each i. Hence as is well known, we have $\mathfrak{o}G\varepsilon \cong M(r, E)$, where $E = \mathrm{End}_{\mathfrak{o}G}(\mathfrak{o}G\varepsilon_1) = \varepsilon_1 \mathfrak{o}G\varepsilon_1$. On the other hand, we observe that if $g \notin G_1$, then $\varepsilon_1 g \varepsilon_1 = g(g^{-1}\varepsilon_1 g)\varepsilon_1 = 0$. Therefore we have $\varepsilon_1 \mathfrak{o}G\varepsilon_1 = \varepsilon_1 \mathfrak{o}G_1\varepsilon_1 = \mathfrak{o}G_1\varepsilon_1$.

The second half has been proved in section 2 by applying the general result of section 2 (of course a direct proof is given in [11], which is more elementary).

From this Theorem, we know that if $\mathfrak{B}$ is a block of $\mathfrak{o}G$ which covers the block $\mathfrak{o}H\varepsilon_1$ of $\mathfrak{o}H$, then there is a unique block $\hat{\mathfrak{B}}$ of $\mathfrak{o}G_1\varepsilon_1$ such that $\mathfrak{B} \cong M(r, \hat{\mathfrak{B}})$ as $\mathfrak{o}$-algebras. Moreover the Morita equivalence (see *e. g.* Anderson and Fuller [1] § 22) sets up a 1-1 correspondence between the category of left $\hat{\mathfrak{B}}$-modules and that of left $\mathfrak{B}$-modules, which eventually sends each $\hat{\mathfrak{B}}$-module N onto the induced $\mathfrak{B}$-module $\mathfrak{o}G \otimes_{G_1} N$.

Of course this coinsides with the (ordinary or modular) character corresponde-nces given in (the generalization by Reynolds [12] of) the First reduction Theorem of Fong.

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