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f-SYMMETRIC SPACES

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0. Introduction.

Let $M^{2n+1}(\phi, \xi, \eta, g)$ be a Sasakian manifeld. A tangent vector of M^{2n+1} is said to be a horizontal vector if it is orthogonal to ξ. Let ∇ be the Riemannian connection for g, and let R be the curvature tensor of ∇. If $\phi^2[(\nabla_V R)(X,Y)Z]=0$ holds for any horizontal vectors X, Y, Z and V, the Sasakian manifold in consideration is said to be a Sasakian locally ϕ-symmetric space. In the previous note [6], the present author introduced the above notion, and discussed about its fundamental properties. In this note, we introduce a notion of a locally f-symmetric space for a S-manifold, and discuss the similar arguments about it.

1. Preliminaries.

Let $M^{2n+1}(f, \xi_1, \xi_2, \eta^1, \eta^2, g)$ be a manifold with a metric f-structure with complemented frames:

$$(1.1)\quad \begin{cases} f^2X=-X+\Sigma\eta^\alpha(X)\xi_\alpha, \\ \eta^\alpha(\xi_\beta)=\delta^\alpha_\beta, \quad \alpha, \beta=1, 2, \\ f\xi_\alpha=0, \quad \eta^\alpha\circ f=0, \quad \alpha=1, 2, \\ g(fX, fY)=g(X, Y)-\Sigma\eta^\alpha(X)\eta^\alpha(Y). \end{cases}$$

It is easy to see that

$$(1.2)\quad g(X, \xi_\alpha)=\eta^\alpha(X), \quad \alpha=1, 2$$

holds good.

We consider the product space $M^{2n+2}\times E^2$, where E^2 is 2-dimensional Euclidean space with a coordinate system (x^1, x^2). If we put $\zeta_1=\partial/\partial x^1$ and $\zeta_2=\partial/\partial x^2$, a tangent vector $\tilde{X}$ of $M^{2n+2}\times E^2$ has a direct sum decomposition

$$(1.3)\quad \tilde{X}=X+a_1\zeta_1+a_2\zeta_2,$$

where X is a tangent vector of M^{2n+2} and a_1 and a_2 are real numbers. Let J be a tensor field of type (1,1) defined by

(1.4) $J\tilde{X}=fX+\sum a_\alpha \xi_\alpha-\sum \eta^\alpha(X)\zeta_\alpha.$

It is easy to see that J is an almost complex structure of $M^{2n+2}\times E^2$. The torsion tensor $\tilde{N}$ of the almost comlex structure J is given by

(1.5) $\tilde{N}(\tilde{X},\tilde{Y})=[J\tilde{X},J\tilde{Y}]-J[\tilde{X},J\tilde{Y}]-J[J\tilde{X},\tilde{Y}]+J^2[\tilde{X},\tilde{Y}].$

We say that the f-structure is normal if the torsion tensor $\tilde{N}$ of J vanishes.

For any tangent vectors X and Y of M^{2n+2}, we have the direct sum decomposition of $\tilde{N}(X,Y)$:

(1.6) $\tilde{N}(X,Y)=N_0(X,Y)+N_1(X,Y)\zeta_1+N_2(X,Y)\zeta_2$

according to (1.3), where

(1.7) $N_0(X,Y)=[fX,fY]-f[X,fY]-f[fX,Y]+f^2[X,Y]$
$+2\sum d\eta^\alpha(X,Y)\xi_\alpha,$

(1.8) $N_\alpha(X,Y)=-2d\eta^\alpha(fX,Y)-2d\eta^\alpha(X,fY)$
$=-(L_{fX}\eta^\alpha)(Y)+(L_{fY}\eta^\alpha)(X), \qquad \alpha=1, 2.$

Similarly, for any tangent vector X of M^{2n+2}, we get

(1.9) $\tilde{N}(X,\zeta_\alpha)=N_{\alpha,0}(X)+N_{\alpha,1}(X)\zeta_1+N_{\alpha,2}(X)\zeta_2, \qquad \alpha=1, 2.$

where

(1.10) $N_{\alpha,0}(X)=-(L_{\xi_\alpha}f)X, \qquad \alpha=1, 2,$

(1.11) $N_{\alpha,\beta}(X)=(L_{\xi_\alpha}\eta^\beta)(X), \qquad \alpha, \beta=1. 2.$

Moreover, we see that

(1.12) $\tilde{N}(\zeta_1,\zeta_2)=[\xi_1,\xi_2]$

holds good.

In the following, we restate the results of H. Nakagawa [4] using the intrinsic notations. It is trivial that if the f-structure is normal, then N_0 vanishes. To the contrary, we have

THEOREM 1.1 (H. Nakagawa [4]). *If N_0 vanishes identically, the f-structure in consideration is normal.*

This theorem follows from the following formulas:

(1.13) $N_0(\xi_1, \xi_2) = -[\xi_1, \xi_2]$,

(1.14) $N_0(fX, \xi_\alpha) = -N_{\alpha,0}(X) - \Sigma\eta^\gamma(X) f[\xi_\gamma, \xi_\alpha]$, $\alpha=1,2$,

(1.15) $\eta^\beta(N_{\alpha,0}(fX)) = -N_{\alpha,\beta}(X) + \Sigma\eta^\gamma(X)\eta^\beta([\xi_\gamma, \xi_\alpha])$, $\alpha, \beta=1,2$,

(1.16) $\eta^\alpha(N_0(X,fY)) + \Sigma\eta^\alpha(N_0(fX,\xi_\gamma))\eta^\gamma(Y) = -N_\alpha(X,Y)$, $\alpha=1,2$.

If a metric f-structure with complemented frames is normal and if

(1.17) $d\eta^\alpha(X, Y) = g(fX, Y)$, $\alpha=1, 2$,

holds for any tangent vectors X and Y, it is said to be a S-structure. A manifold with a S-structure is said to be a S-manifold. It is known that if a metric f-structure with complemented frames is a S-structure, then

(1.18) $L_{\xi_\alpha} g=0$, $\alpha=1, 2$,

(1.19) $d\eta^\alpha(X,Y) = (\nabla_X\eta^\alpha)(Y)$, $\alpha=1, 2$,

(1.20) $\nabla_X\xi_\alpha = fX$, $\alpha=1,2$

hold good, where ∇ is the Riemannian connection for g (D. E. Blair [1]). Making use of these formulas, we see that

(1.21) $$(\nabla_X f)Y = \Sigma\{\eta^\alpha(Y)X - g(X,Y)\,\xi_\alpha\} + \sum_{\alpha,\beta}\eta^\alpha(X)\,\{\eta^\alpha(Y)\,\xi_\beta - \eta^\beta(Y)\,\xi_\alpha\}$$

holds good for a S-structure (D. E. Blair [1]). Conversely, suppose a metric f-structure with complemented frames satisfies (1.21) and (1.17), then we see that

$$\begin{aligned} N_0(X,Y) &= (\nabla_{fX}f)\,Y - (\nabla_{fY}f)\,X - f(\nabla_X f)\,Y + f(\nabla_Y f)\,X \\ &\quad + 2\textstyle\sum d\eta^\alpha(X,Y)\,\xi_\alpha \\ &= 0 \end{aligned}$$

holds good. Thus we get the following:

THEOREM 1.2. *A metric f-structure with complemented frames is a S-structure if and only if it satisfies* (1.17) *and* (1.21).

It is easy to see that if a metric f-structure with complemented frames satisfies (1.20), then it satifies (1.17) and (1.18), and vice versa. Hence we get

COROLLARY 1.3. *A metric f-structure with complemented frames is a S-structure if and only if it satisfies* (1.20) *and* (1.21).

2. A definition of a locally f-symmetric space.

Let $M^{2n+2}(f, \xi_1, \xi_2, \eta^1, \eta^2, g)$ be a S-manifold. Since the f-structure is normal, we have $[\xi_1, \xi_2]=0$, and hence the distribution $\{\xi_1, \xi_2\}$, spaned by ξ_1 and ξ_2, is involutive. For an arbitrary point x of M^{2n+2}, we can find a flat coordinate neighborhood U of x with respect to $\{\xi_1, \xi_2\}$ (Palais [5]), and we have a local fibering

(2.1) $\pi: U \longrightarrow U/\{\xi_1, \xi_2\}$.

Since f and η^α are invariant by ξ_β, we have an induced Kählerian structure $(J, \bar{g})$ of $U/\{\xi_1, \xi_2\}$ defined by

(2.2) $$\begin{cases} J\bar{X}=\pi_* f\bar{X}^* \\ \pi^* \bar{g}+\Sigma\eta^\alpha \otimes\eta^\alpha=g, \end{cases}$$

where $\bar{X}$ is a vector field on $U/\{\xi_1, \xi_2\}$ and $\bar{X}^*$ is the horizontal life (with respect to the connection form $\gamma=(\eta^1, \eta^2)$) of $\bar{X}$ (cf. Blair-Ludden-Yano [2]). Since we have $g(\nabla_{\bar{X}^*}\bar{Y}^*, \xi_\alpha)=-g(\bar{Y}^*, \nabla_{\bar{X}^*}\xi_\alpha)=-g(\bar{Y}^*, f\bar{X}^*)=-d\eta^\alpha(\bar{X}^*, \bar{Y}^*)$, we have

(2.3) $$\nabla_{\bar{X}^*}\bar{Y}^*=(\bar{\nabla}_{\bar{X}}\bar{Y})^*-\Sigma\, d\eta^\alpha(\bar{X}^*, \bar{Y}^*)\xi_\alpha,$$

where $\bar{X}$ and $\bar{Y}$ are vector fields on $U/\{\xi_1, \xi_2\}$, $\bar{\nabla}$ is the Riemannian connection for $\bar{g}$, and $\bar{X}^*$, $\bar{Y}^*$ and $(\bar{\nabla}_{\bar{X}}\bar{Y})^*$ are the horizontal lifts of $\bar{X}$, $\bar{Y}$ and $\bar{\nabla}_{\bar{X}}\bar{Y}$, respectively. Since $2d\eta^\alpha(\bar{X}^*, \bar{Y}^*)=-\eta^\alpha([\bar{X}^*, \bar{Y}^*])$ holds good, (2.3) is the same as

(2.4) $$\nabla_{\bar{X}^*}\bar{Y}^*=(\bar{\nabla}_{\bar{X}}\bar{Y})^*+\frac{1}{2}\Sigma\eta^\alpha([\bar{X}^*, \bar{Y}^*])\,\xi_\alpha.$$

Since we have $\eta^\alpha(\nabla_{\bar{X}^*}\bar{Y}^*)=g(\nabla_{\bar{X}^*}\bar{Y}^*, \xi_\alpha)=-g(\bar{Y}^*, f\bar{X}^*)=g(f\bar{Y}^*, \bar{X}^*)=-\eta^\alpha(\nabla_{\bar{Y}^*}\bar{X}^*)$, (2.4) implies

(2.5) $$\begin{aligned}(\bar{\nabla}_{\bar{X}}\bar{Y})^* &= \nabla_{\bar{X}^*}\bar{Y}^*-\Sigma\,\eta^\alpha(\nabla_{\bar{X}^*}\bar{Y}^*)\,\xi_\alpha \\ &= -f^2\,\nabla_{\bar{X}^*}\bar{Y}^*,\end{aligned}$$

and hence we get

(2.6) $$(\bar{\nabla}_{\bar{X}}\bar{\nabla}_{\bar{Y}}\bar{Z})^*=-f^2\,\nabla_{\bar{X}^*}\nabla_{\bar{Y}^*}\bar{Z}^*+2g(f\bar{Y}^*, \bar{Z}^*)\bar{X}^*.$$

Taking account of (1.21), (2.5), (2.6) and

(2.7) $$[\bar{X}^*, \bar{Y}^*]=[\bar{X}, \bar{Y}]^*+\Sigma\,\eta^\alpha[\bar{X}^*, \bar{Y}^*])\,\xi_\alpha,$$

we get

$$(2.8)\quad (\bar{R}(\bar{X},\bar{Y})\bar{Z})^* = R(\bar{X}^*,\bar{Y}^*)\bar{Z}^* + 2g(f\bar{Y}^*,\bar{Z}^*)f\bar{X}^* - 2g(f\bar{X}^*,\bar{Z}^*)f\bar{Y}^* - 4g(f\bar{X}^*,\bar{Y}^*)f\bar{Z}^*,$$

where we have used

$$(2.9)\quad R(\bar{X}^*,\ \bar{Y}^*)\xi_\alpha = 0,$$

which follows from (1.20) and (1.21). Since $[\xi_\alpha, \bar{Z}^*]=0$ holds, we get

$$(2.10)\quad R(\bar{X}^*,\xi_\alpha)\bar{Z}^* = -g(\bar{X}^*,\bar{Z}^*)\Sigma\xi_\beta,\qquad \alpha=1,2.$$

Making use of (1.21), (2.8), (2.9) and (2.10), we get

$$(2.11)\quad ((\bar{\nabla}_{\bar{V}}\bar{R})(\bar{X},\bar{Y})\bar{Z})^* = (\nabla_{\bar{V}^*}R)(\bar{X}^*,\bar{Y}^*)\bar{Z}^* + \{g(f\bar{V}^*,R(\bar{X}^*,\bar{Y}^*)\bar{Z}^*) - 2g(f\bar{V}^*,\bar{X}^*)g(\bar{Y}^*,\bar{Z}^*) + 2g(f\bar{V}^*,\bar{Y}^*)g(\bar{X}^*,\bar{Z}^*)\}\Sigma\xi_\alpha.$$

On the other hand, we get, for an arbitrary α, $\alpha=1,2$,

$$\begin{aligned} &g(f\bar{V}^*,\ R(\bar{X}^*,\bar{Y}^*)\bar{Z}^*)\\ &= g(\nabla_{\bar{V}^*}\xi_\alpha,\ R(\bar{X}^*,\bar{Y}^*)\bar{Z}^*)\\ &= -g(\xi_\alpha,(\nabla_{\bar{V}^*}R)(\bar{X}^*,\bar{Y}^*)\bar{Z}^*) - g(\xi_\alpha,R(\nabla_{\bar{V}^*}\bar{X}^*,\bar{Y}^*)\bar{Z}^*)\\ &\quad - g(\xi_\alpha,R(\bar{X}^*,\nabla_{\bar{V}^*}\bar{Y}^*)\bar{Z}^*) - g(\xi_\alpha,R(\bar{X}^*,\bar{Y}^*)\nabla_{\bar{V}^*}\bar{Z}^*)\\ &= -\eta^\alpha((\nabla_{\bar{V}^*}R)(\bar{X}^*,\bar{Y}^*)\bar{Z}^*) + g(R(\bar{Z}^*,\xi_\alpha)\bar{Y}^*,\nabla_{\bar{V}^*}\bar{X}^*)\\ &\quad - g(R(\bar{Z}^*,\xi_\alpha)\bar{X}^*,\nabla_{\bar{V}^*}\bar{Y}^*) + g(R(\bar{X}^*,\bar{Y}^*)\xi_\alpha,\nabla_{\bar{V}^*}\bar{Z}^*)\\ &= -\eta^\alpha((\nabla_{\bar{V}^*}R)(\bar{X}^*,\bar{Y}^*)\bar{Z}^*) + 2g(\bar{Z}^*,\bar{Y}^*)g(f\bar{V}^*,\bar{X}^*)\\ &\quad - 2g(\bar{Z}^*,\bar{X}^*)g(f\bar{V}^*,\bar{Y}^*). \end{aligned}$$

Hence we get

$$\begin{aligned} &\{g(f\bar{V}^*,R(\bar{X}^*,\bar{Y}^*)\bar{Z}^*) - 2g(f\bar{V}^*,\bar{X}^*)g(\bar{Y}^*,\bar{Z}^*)\\ &\quad + 2g(f\bar{V}^*,\bar{Y}^*)g(\bar{X}^*,\bar{Z}^*)\}\Sigma\xi_\alpha\\ &= -\eta^\beta((\nabla_{\bar{V}^*}R)(\bar{X}^*,\bar{Y}^*)\bar{Z}^*)\Sigma\xi_\alpha\\ &= -\Sigma\eta^\alpha((\nabla_{\bar{V}^*}R)(\bar{X}^*,\bar{Y}^*)\bar{Z}^*)\xi_\alpha, \end{aligned}$$

and hence we get

$$(2.12)\quad ((\bar{\nabla}_{\bar{V}}\bar{R})(\bar{X},\bar{Y})\bar{Z})^* = -f^2[(\nabla_{\bar{V}^*}R)(\bar{X}^*,\bar{Y}^*)\bar{Z}^*]$$

for any vector fields $\bar{X}$, $\bar{Y}$, $\bar{X}$ and $\bar{V}$ of $U/\{\xi_1,\xi_2\}$. Thus the following definition is reasonable:

DEFINITION. A S-manifold $M^{2n+2}(f,\xi_1,\xi_2,\eta^1,\eta^2,g)$ is said to be a locally f-

symmetric space if

(2.13) $f^2[(\nabla_V R)(X,Y)Z]=0$

holds for any horizontal vectors X, Y, Z and V.

From this definition and (2.12), we get the following:

THEOREM 2.1. *A S-manifold is a locally f-symmetric space if and only if each Kählerian manifold, which is a base space of the local fibering* (2.1), *is a Hermitian locally symmetric space.*

3. M-connection.

Let $M^{2n+2}(f,\xi_1,\xi_2,\eta^1,\eta^2,g)$ be a S-manifold, and let r be an arbitrary fixed real number. Let A be a tensor field defined by

(3.1) $A(X)Y=\Sigma\{d\eta^\alpha(X,Y)\xi_\alpha+r\eta^\alpha(X)fY-\eta^\alpha(Y)fX\}.$

The M-connection $\tilde{\nabla}$ is by definition

(3.2) $\tilde{\nabla}_X Y=\nabla_X Y+A(X)Y,$

where∇ is the Riemannian connection for g. We see that the structure tensors f, η^α, ξ_α and g are parallel with respect to $\tilde{\nabla}$, and hence A is also parallel.

Now, we consider the local fibering (2. 1). Let $\bar{X}$, $\bar{Y}$, $\bar{Z}$ and $\bar{V}$ be vector fields on $U/\{\xi_1,\xi_2\}$. Then we get

(3.3) $\tilde{\nabla}_{\bar{X}^*}\bar{Y}^*=(\bar{\nabla}_{\bar{X}}\bar{Y})^*,$

(3.4) $\tilde{\nabla}_{\xi_\alpha}\bar{Z}^*=(1+r)f\bar{Z}^*,$

and hence we get

(3.5) $\tilde{R}(\bar{X}^*,\bar{Y}^*)\bar{Z}^*=(\bar{R}(\bar{X},\bar{Y})\bar{Z})^*+4(1+r)g(f\bar{X}^*,\bar{Y}^*)f\bar{Z}^*.$

where $\tilde{R}$ is the curvature tenror of the M-connection $\tilde{\nabla}$. Making use of (3.3) and (3.5), we get

(3.6) $(\tilde{\nabla}_{\bar{V}^*}\tilde{R})(\bar{X}^*,\bar{Y}^*)\bar{Z}^*=((\bar{\nabla}_{\bar{V}}\bar{R})(\bar{X},\bar{Y})\bar{Z})^*.$

On the other hand, since we have $\tilde{\nabla}f=0$, we get

(3.7) $\tilde{R}(X,Y)fZ=f\tilde{R}(X,Y)Z,$

and hence we get

(3.8) $\tilde{R}(fX,Y)Z=-\tilde{R}(X,fY)Z$

for any tangent vectors X, Y and Z of M^{2n+2}. Since we have $\tilde{\nabla}\xi_\alpha=0$, we get

(3.9) $\tilde{R}(X,Y)\xi_\alpha=0$,

and hence

(3.10) $\tilde{R}(\xi_\alpha,Y)Z=0$

for any tangent vectors X, Y and Z of M^{2n+2}. Making use of (3.7)$\sim$(3.10) and (3.6), we get that $\bar{\nabla}\bar{R}=0$ holds good if and onl if $\tilde{\nabla}\tilde{R}=0$ holds good. Thus we get

THEOREM 3.1. *In order that a S-manifold is a locally f-symmetric space it is necessary and sufficient that the curvature tensor of the M-connection is parallel with respect to the M-connection in consideration.*

REMARK. From (3.8) and (3.10), we get

(3.11) $\tilde{R}(fX,fY)=\tilde{R}(X,Y)$.

4. f-geodesic symmetry.

A geodesic $\gamma=\gamma(s)$ in a S-manifold $M^{2n+2}(f,\xi_1,\xi_2,\eta^1,\eta^2,g)$ is said to be *a f-*geodesic if $\eta_\alpha(\gamma'(s))=0$, $\alpha=1,2$, holds at each point of the geodesic. A local diffeomorphism σ_x of M^{2n+2}, $x\in M^{2n+2}$, is said to be a f-geodesic symmetry at x if, for each f-geodesic $\gamma=\gamma(s)$ such that $\gamma(0)$ lies in the leaf of the distribution $\{\xi_1, \xi_2\}$ passing through x,

(4.1) $\sigma_x\gamma(s)=\gamma(-s)$

holds for s. In this section, we shall prove the following theorem:

THEOREM 4.1. *A necessary and sufficient condition for a S-manifold to be a locally f-symmetric space is that each f-geodesic symmetry is a local automorphism of the metric f-structure.*

By using the local fibering (2.1), it is easy to see that the sufficiency holds good. To prove the necessity, we use the M-connection and Theorem 3.1.

Suppose a S-manifold $M^{2n+2}(f,\xi_1,\xi_2,\eta^1,\eta^2\cdot g)$ be a locally f-symmetric space. We consider the M-connection $\tilde{\nabla}$ on M^{2n+2}. The torsion tensor $\tilde{T}$ of $\tilde{\nabla}$ is by definition

(4.2) $\tilde{T}(X,Y)=A(X)Y-A(Y)X$
$$=\Sigma\,[2d\eta^{\alpha}(X,Y)\,\xi_{\alpha}+(r+1)\,\{\eta^{\alpha}(X)\,fY-\eta^{\alpha}(Y)\,fX\}],$$

and it is parallel with respect to $\tilde{\nabla}$. By Theorem 3.1, the curvature tensor $\tilde{R}$ of $\tilde{\nabla}$ is parallel. Let x be an arbitrary point of M^{2n+2}. Let $\{e_1, e_2, \ldots, e_{2n}, \xi_{1x}, \xi_{2x}\}$ be an orthonormal basis of $T_x(M^{2n+2})$. We consider a linear isomorphism

(4.3) $\sigma_0\colon T_x(M^{2n+2})\longrightarrow T_x(M^{2n+2})$

defined by

(4.4) $$\begin{cases}\sigma_0(e_i)=-e_i, & 1\leqq i\leqq 2n,\\ \sigma_0(\xi_{\alpha x})=\xi_{\alpha x}, & 1\leqq\alpha\leqq 2.\end{cases}$$

Then, by (4.2), we see that $\tilde{T}(\sigma_0 e_i, \sigma_0 e_j)=2g(fe_i, e_j)(\Sigma\xi_{\alpha})=\sigma_0\tilde{T}(e_i, e_j)$ and $\tilde{T}(\sigma_0 e_i, \sigma_0\xi_{\alpha x})=-(r+1)fe_i=\sigma_0\tilde{T}(e_i, \xi_{\alpha x})$ holds good for $1\leqq i, j\leqq 2n$ and $1\leqq\alpha\leqq 2$. Hence we get

(4.5) $\sigma_0\tilde{T}_x=\tilde{T}_x$.

On the other hand, (3.9) implies that $\tilde{R}(e_i, e_j)e_k$ is orthogonal to ξ_{α}, $\alpha=1, 2$. Hence we get $\tilde{R}(\sigma_0 e_i, \sigma_0 e_j)\sigma_0 e_k=\sigma_0\tilde{R}(e_i, e_j)e_k$. (3.9) and (3.10) imply $\tilde{R}(\sigma_0 e_i, \sigma_0 e_j)\sigma_0\xi_{\alpha x}=\sigma_0\tilde{R}(e_i, e_j)\xi_{\alpha x}$, $\tilde{R}(\sigma_0 e_i, \sigma_0\xi_{\alpha x})\sigma_0 e_k=\sigma_0\tilde{R}(e_i, \xi_{\alpha x})e_k$ and $\tilde{R}(\sigma_0 e_i, \sigma_0\xi_{\alpha x})\sigma_0\xi_{\beta x}=\sigma_0\tilde{R}(e_i, \xi_{\alpha x})\xi_{\beta x}$ for $1\leqq i, j, k\leqq 2n$ and $1\leqq\alpha, \beta\leqq 2$. Thus we get

(4.6) $\sigma_0\tilde{R}_x=\tilde{R}_x$.

Hence, according to Theorem 7.4 of Chapter VI in Kobayashi-Nomizu [3], for example, we see that the local diffeomorphism σ_x, defined by

(4.7) $\sigma_x(x_1, x_2, \ldots, x_{2n}, z_1, z_2)$
$$=(-x_1, -x_2, \ldots, -x_{2n}, z_1, z_2)$$

on a normal coordinate neighborhood U with a normal coordinate system $(x_1, x_2, \ldots, x_{2n}, z_1, z_2)$ determined by $\{e_1, e_2, \ldots, e_{2n}, \xi_{1x}, \xi_{2x}\}$, is a local affine transformation with respect to $\tilde{\nabla}$. Since $(\sigma_{x}*)_x=\sigma_0$, we get $(\sigma_x*)_x\circ f_x=f_x\circ(\sigma_x*)_x$, $(\sigma_x{}^*)_x\eta^{\alpha}{}_x=\eta^{\alpha}{}_x$, $(\sigma_x*)_x\xi_{\alpha x}=\xi_{\alpha x}$ and $(\sigma_x*)_x\,g_x=g_x$ for $1\leqq\alpha\leqq 2$. Hence, since σ_x is a local affine transformation and since f, ξ_{α}, η^{α} and g are parallel with respect to $\tilde{\nabla}$, σ_x is a local automorphism of the metric f-structure in consideration, and hence σ_x is a f-geodesic symmetry.

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