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A NOTE ON ENVELOPES OF HOLOMORPHY

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Introduction

It is well known that if D_1, D_2 are domains of holomorphy in C^n, then the intersection $D_1 \cap D_2$ is a domain of holomorphy. Further, let $\{D_\lambda\}_{\lambda \in \Lambda}$ be a family of domains of holomorohy in C^n. Then the inner kernel of $\bigcap_{\lambda \in \Lambda} D_\lambda$ is a domain of holomorphy. It is also well known that the maximal domain of continuation $D_{\mathscr{F}}$ for a family $\mathscr{F}$ of holomorphic functions given in a domain D is not necessarily a domain in C^n even if D is a domain in C^n, and so the following statement is not necessarily true: the envelope of holomorphy of a domain D in C^n coincides with the inner kernel of the intersection of the family of domains of holomorphy in C^n that contains D.

In this note we define the *intersection* of a family of Riemann domains by the aid of the notion of the union of Riemann domains introduced in [1], see also [2], [4] and we ask if the above statement is true in the category of Riemann domains with base point. Our answer is as follows: *the envelope of a Riemann domain D with respect to a family $\mathscr{F}$ of holomorphic functions coincides with the intersection (in ous sense) of the family $\{H_f(D)\}_{f \in \mathscr{F}}$ of domains of maximal continuation $H_f(D)$ of f, $f \in F$.*

Notations and terminologies will be as in [1].

1. Union of Riemann domains

In this note, we consider only the Riemann domains over C^n with base point. As preparation, let us recall some results in [1]. There, Grauert and Fritzsche defined the envelope of holomorphy. Other constructions of envelope of holomorphy are found, for example, in [2] and [4].

Let $\mathfrak{z}_0$ be a fixed point in C^n, and let $\mathfrak{G} = (G, \pi, x_0)$ be a Riemann domain with base point x_0 such that $\pi(x_0) = \mathfrak{z}_0$.

DEFINITION 1. *Let $\mathfrak{G}_j = (G_j, \pi_j, x_j)$, $j=1, 2$ be Riemann domains over C^n with base point. We say that $\mathfrak{G}_1$ is contained in $\mathfrak{G}_2$ and denote it by $\mathfrak{G}_1 < \mathfrak{G}_2$, if there exists a continuous map ϕ: $G_1 \to G_2$ satisfying the following condition:*

i) $\pi_1 = \pi_2 \circ \phi$

ii) $\phi(x_1) = x_2$

With the relation $<$, $\{\mathfrak{G}_\iota\}\ \iota \in I$ is a partially orderd set, i. e., the following holds:

i) $\mathfrak{G} < \mathfrak{G}$,

ii) If $\mathfrak{G}_1 < \mathfrak{G}_2$, $\mathfrak{G}_2 < \mathfrak{G}_3$, then $\mathfrak{G}_1 < \mathfrak{G}_3$.

If $\mathfrak{G}_1 < \mathfrak{G}_2$ and $\mathfrak{G}_2 < \mathfrak{G}_1$, then $\mathfrak{G}_1$ is holomorphically equivalent to $\mathfrak{G}_2$ and we write $\mathfrak{G}_1 \simeq \mathfrak{G}_2$. In case $\mathfrak{G}_1$, $\mathfrak{G}_2$ are both schlicht, $\mathfrak{G}_1 < \mathfrak{G}_2$ if and only if $G_1 \subset G_2$.

Then, the union $\tilde{\mathfrak{G}}$ of $\{\mathfrak{G}_\iota\}\ \iota \in I$ is well defined and satisfies following fundamental properties, see [1]: let $\mathfrak{G}_1$, $\mathfrak{G}_2$, $\mathfrak{G}_3$ be Riemann domains with base point, and $\{\mathfrak{G}_\iota\}_{\iota \in I}$, $\{\mathfrak{G}_{\iota'}\}_{\iota' \in I'}$ are any families of Riemann domains, then

(1) if $\mathfrak{G}_\iota < \mathfrak{G}$, for all $\iota \in I$, then $\bigcup_{\iota \in I} \mathfrak{G} < \mathfrak{G}$,

(2) if $\mathfrak{G}_1 < \mathfrak{G}_2$, then $\mathfrak{G}_1 \cup \mathfrak{G}_2 \simeq \mathfrak{G}_2$,

(3) $\mathfrak{G}_1 \cup \mathfrak{G}_1 \simeq \mathfrak{G}_1$,

(4) $\mathfrak{G}_1 \cup \mathfrak{G}_2 \simeq \mathfrak{G}_2 \cup \mathfrak{G}_1$,

(5) $\mathfrak{G}_1 \cup (\mathfrak{G}_2 \cup \mathfrak{G}_3) \simeq (\mathfrak{G}_1 \cup \mathfrak{G}_2) \cup \mathfrak{G}_3$, and

(6) if $I \subset I'$, then

$$\bigcup_{\iota \in I} \mathfrak{G}_\iota < \bigcup_{\iota' \in I'} \mathfrak{G}_{\iota'}$$

2. Intersection of Riemann domains

Let $\mathfrak{z}_0 \in C^n$ be a fixed point in C^n. We consider the Riemann domains $\{\mathfrak{G}_\iota\}\ \iota \in I$, $\mathfrak{G}_\iota = (G_\iota, \pi_\iota, x_\iota)$ over C^n with base point x_ι such that $\pi_\iota(x_\iota) = \mathfrak{z}_0$. Then, for $\{\mathfrak{G}_\iota\}\ \iota \in I$ we consider the set $\{\mathfrak{H}_\lambda\}_{\lambda \in \Lambda}$ such that $\mathfrak{H}_\lambda < \mathfrak{G}_\iota$ for all $\iota \in I$. We put

$$\mathfrak{H} = \bigcup_{\lambda \in \Lambda} \mathfrak{H}_\lambda.$$

It is obvious that $\pi'_\lambda(x'_\lambda) = \mathfrak{z}_0$ for every $\mathfrak{H}_\lambda := (H'_\lambda, \pi'_\lambda, x'_\lambda)$, $\lambda \in \Lambda$. After the method given in [1] we know that $\mathfrak{H}$ is a Riemann domain over C^n with base point. Further, we have the following:

i) Since $\mathfrak{H}_\lambda < \mathfrak{G}_\iota$ for all $\iota \in I$, from (1) in **1** we have

$$\bigcup_{\lambda \in \Lambda} \mathfrak{H}_\lambda < \mathfrak{G}_\iota \quad \text{for all } \iota \in I.$$

Hence

$$\mathfrak{H} < \mathfrak{G}_\iota \quad \text{for all } \iota \in I.$$

ii) If $\mathfrak{G}^* < \mathfrak{G}_\iota$ for all $\iota \in I$, then there exists some $\lambda_0 \in \Lambda$ such that $\mathfrak{G}^* \simeq \mathfrak{H}_{\lambda_0}$. Hence $\mathfrak{G}^* < \mathfrak{H}$ i. e., $\mathfrak{H}$ is the maximal Riemann domain which is contained in $\mathfrak{G}_\iota$ for every $\iota \in I$.

Considering the properties i) & ii), we call $\mathfrak{H}$ the *intersection* of $\{\mathfrak{G}_\iota, \iota \in I\}$ and denote it by $\bigcap_{\iota \in I} \mathfrak{G}_\iota$. Let $\{\mathfrak{G}_\iota\}_{\iota \in I}$ be a family of Riemann domains over C^n with base point. Since $\{\mathfrak{G}_\iota, \iota \in I\}$ is a lattice with operations $\cap$ and $\cup$, and since the properties (1)$\sim$(6) in **1** holds, it is easily verified that the following (1)$'\sim$(6)$'$, and, (7), (7)$'$ are also true. (1)$'\sim$(6)$'$ is dual to (1)$\sim$(6).

Let $\mathfrak{G}_1$, $\mathfrak{G}_2$, $\mathfrak{G}_3$ be Riemann domains and $\{\mathfrak{G}_\lambda\}_{\lambda \in \Lambda}$, $\{\mathfrak{G}_{\lambda'}\}_{\lambda' \in \Lambda'}$ are any famillies of Riemann domains, then

(1)$'$ if $\mathfrak{G}_\lambda > \mathfrak{G}_1$, for all $\lambda \in \Lambda$, then

$$\bigcap_{\lambda \in \Lambda} \mathfrak{G}_\lambda > \mathfrak{G}_1,$$

(2)$'$ if $\mathfrak{G}_1 < \mathfrak{G}_2$, then $\mathfrak{G}_1 \cap \mathfrak{G}_2 \simeq \mathfrak{G}_1$,

(3)$'$ $\mathfrak{G}_1 \cap \mathfrak{G}_1 \simeq \mathfrak{G}_1$,

(4)$'$ $\mathfrak{G}_1 \cap \mathfrak{G}_2 \simeq \mathfrak{G}_2 \cap \mathfrak{G}_1$,

(5)$'$ $\mathfrak{G}_1 \cap (\mathfrak{G}_2 \cap \mathfrak{G}_3) \simeq (\mathfrak{G}_1 \cap \mathfrak{G}_2) \cap \mathfrak{G}_3$.

(6)$'$ if $\Lambda \supset \Lambda'$, then

$$\bigcap_{\lambda \in \Lambda} \mathfrak{G}_\lambda < \bigcap_{\lambda' \in \Lambda'} \mathfrak{G}_{\lambda'}.$$

Also we have

(7) $(\mathfrak{G}_1 \cup \mathfrak{G}_2) \cap \mathfrak{G}_3 > (\mathfrak{G}_1 \cap \mathfrak{G}_3) \cup (\mathfrak{G}_2 \cap \mathfrak{G}_3)$,

(7)$'$ $(\mathfrak{G}_1 \cap \mathfrak{G}_2) \cup \mathfrak{G}_3 < (\mathfrak{G}_1 \cup \mathfrak{G}_3) \cap (\mathfrak{G}_2 \cup \mathfrak{G}_3)$.

REMARK. In case every $\mathfrak{G}_\iota$ is schlicht, $\bigcup_{\iota \in I} \mathfrak{G}_\iota$ is the union in the usueal sense, but $\bigcap_{\iota \in I} \mathfrak{G}_\iota$ is not necessariliy the intersection in the usual sense. It should be pointed out that the intersection $\bigcap_{\iota \in I} \mathfrak{G}_\iota$ in our sense is connected and coincides with the connected component which contains base point such that $\pi_\iota(x_\iota) = \mathrm{id}(x_\iota) = \mathfrak{z}_0 \in C^n$. The intersection of Riemann domains over C^n with base point is always a domain, see **1**.

When $\mathfrak{G}_1$, $\mathfrak{G}_2$, $\mathfrak{G}_3$ are schlicht, in place of (7), (7)$'$ we have

(8) $$(\mathfrak{G}_1 \cup \mathfrak{G}_2) \cap \mathfrak{G}_3 \simeq (\mathfrak{G}_1 \cap \mathfrak{G}_3) \cup (\mathfrak{G}_2 \cap \mathfrak{G}_3),$$

(8)′ $$(\mathfrak{G}_1 \cap \mathfrak{G}_2) \cup \mathfrak{G}_3 \simeq (\mathfrak{G}_1 \cup \mathfrak{G}_3) \cap (\mathfrak{G}_2 \cup \mathfrak{G}_3).$$

3. Envelope of holomorphy

Using the notion of intersection of Riemann domains introduced in **2**, we are able to characterize a domain of holomorphy and the envelope of holomorphy.

DEFINITION 2. *Let* $\mathfrak{G}=(G, \pi, x)$ *be a Riemann domin over* C^n *with base point and let* $\mathscr{F}(\neq\phi)$ *be a family of holomorphic functions on* G. Let $\{\mathfrak{G}_\iota, \iota \in I\}$ *be the family of all Rieman domains which satisfies the following conditions*:
(1) $\mathfrak{G} < \mathfrak{G}_\iota$ *for all* $\iota \in I$,
(2) *if* $f \in \mathscr{F}$, *there exists some* $F_\iota \in A(\mathfrak{G}_\iota)$ *such that* $F_\iota | \mathfrak{G} = f$ *for every* $\iota \in I$.

We call the Riemann domain $H_{\mathscr{F}}(\mathfrak{G}) := \bigcup_{\iota \in I} \mathfrak{G}_\iota$ *the envelope of holomorphy of* $\mathfrak{G}$ *with respect to* $\mathscr{F}$. *In case* $\mathscr{F} = A(\mathfrak{G})$, *we call* $H_{A(\mathfrak{G})}(\mathfrak{G})$ *the envelope of holomorphy of* $\mathfrak{G}$ *and write* $H(\mathfrak{G})$. *If* $\mathscr{F} = \{f\}$, $H_f(\mathfrak{G}) := H_{\{f\}}(\mathfrak{G})$ *is the maximal domain of continuation of* f, *i. e., the maximal domain to which* f *can be analytically continued.*

PROPOSITION 3. *For* $\mathfrak{G}=(G, \pi, x)$ *and* $\mathscr{F} = \{f_k | k \in K, f_k \in A(G)\}$, *we have*

$$H_{\mathscr{F}}(\mathfrak{G}) = \bigcap_{k \in K} H_{f_k}(\mathfrak{G}).$$

PROOF. We put $\tilde{\mathfrak{H}} = \bigcap_{k \in K} H_{f_k}(\mathfrak{G})$. Then $\tilde{\mathfrak{H}} = \bigcup_{\lambda \in \Lambda} \tilde{\mathfrak{H}}_\lambda$, where Λ is the set of all λ for which $\tilde{\mathfrak{H}}_\lambda < H_{f_k}(\mathfrak{G})$ for all $k \in K$. Since $\tilde{\mathfrak{H}}_\lambda := (\tilde{H}_\lambda, \tilde{\pi}_\lambda, \tilde{x}_\lambda)$ $\lambda \in \Lambda$ is the Riemann domain such that some $F_{k,\lambda} \in A(\tilde{\mathfrak{H}}_\lambda)$ exists for every $k \in K$ with $F_{k,\lambda} | G = f_k$, we have

$$\tilde{\mathfrak{H}}_\lambda < H_{\mathscr{F}}(\mathfrak{G}) \qquad \text{for all } \lambda \in \Lambda.$$

Hence,

$$\bigcup_{\lambda \in \Lambda} \tilde{\mathfrak{H}}_\lambda < H_{\mathscr{F}}(\mathfrak{G}) \qquad \text{(see (1) in } \mathbf{1}\text{)},$$

which implies

$$\tilde{\mathfrak{H}} = \bigcap_{k \in K} H_{f_k}(\mathfrak{G}) < H_{\mathscr{F}}(\mathfrak{G}).$$

Next, we prove that $H_{\mathscr{F}}(\mathfrak{G}) < H_{f_k}(\mathfrak{G})$ for all $k \in K$. Put $H_{\mathscr{F}}(\mathfrak{G}) = \bigcup_{\lambda' \in \Lambda'} \hat{\mathfrak{H}}_{\lambda'} = \hat{\mathfrak{H}}$, then for any $\lambda' \in \Lambda'$ all $f_k \in \mathscr{F}$ can be continued to $\hat{\mathfrak{H}}_\lambda$. Furthermore, putting

$H_{f_k}(\mathfrak{G})=\bigcup_{\mu\in M}\bar{\mathfrak{H}}_{k,\mu}$, we see that for any $\hat{\mathfrak{H}}_{\lambda'}=(\hat{H}_{\lambda'}, \hat{\pi}_{\lambda'}, \hat{x}_{\lambda'})$, $\lambda'\in\Lambda'$, there exists an $\hat{F}_{k,\lambda'}\in A(\hat{\mathfrak{H}}_{\lambda'})$ such that $\hat{F}_{k,\lambda'}|G=f_k$. Hence, there exists some $\mu_0\in M$ such that

$$\hat{\mathfrak{H}}_{\lambda'}<\bar{\mathfrak{H}}_{\mu_0}.$$

Consequently, for any $k\in K$ we have

$$H_{\mathscr{F}}(\mathfrak{G})<\bigcup_{\lambda'\in\Lambda'}\bar{\mathfrak{H}}_{\lambda'}<\bigcup_{\mu\in M}\bar{\mathfrak{H}}_{\mu}=H_{f_k}(\mathfrak{G}),$$

which implies

$$H_{\mathscr{F}}(\mathfrak{G})<\bigcap_{k\in K}H_{f_k}(\mathfrak{G}).$$

Thus, we proved

$$H_{\mathscr{F}}(\mathfrak{G})\simeq\bigcap_{k\in K}H_{f_k}(\mathfrak{G}).$$

COROLLARY 4. *The intersection of domains of holomorphy is a domain of holomorphy.*

PROOF. From Proposition 3 the proof is obvious.

THEOREM 5. *If* $\mathscr{F}=A(\mathfrak{G})$, *then*

$$H(\mathfrak{G})\simeq\bigcap_{f\in A(\mathfrak{G})}H_f(\mathfrak{G}).$$

PROOF. It is enough to put $\mathscr{F}=A(\mathfrak{G})$ in Proposition 3.

REMARK. A class of Riemann domains over C^n with base point is a category whose objects are $\{(G, \pi, x)\}$, where $\mathfrak{G}:=(G, \pi, x)$ is a Riemann domain, π is a projection and $x\in G$. A morphism from (G, π, x) to (G', π', x') is a locally topological map ϕ from G to G' such that $\phi(x)=x'$. The intersection $\mathfrak{G}\cap\mathfrak{G}'$ then coincides with the *pull back* for π and π'.

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