Kumamoto J. Sci. (Math.)
Vol. 13, 43-47 (1979)

REMARK ON THE UNCONDITIONAL STABILITY OF A FINITE ELEMENT SCHEME FOR NONLINEAR PLATE PROBLEMS

Tetsuhiko MIYOSHI and Makoto ARAKI

(Received Dec. 21, 1978)

1. Introduction

The nonlinear vibration of a thin elastic plate can be represented by a system of two nonlinear partial differential equations. For solving this system by the finite element method, one of the present authors derived a stability criterion with respect to the time increment Δt under a certain condition. The purpose of this note is to show that this condition is unnecessary to ensure the stability.

We consider the equations in the cylindrical domain $Q=(0, T)\times\Omega$, being Ω the shape of the plate. Let f be the Airy's stress function and w the deflection of the plate. Then the system of equations is

$$(1.1)\qquad \begin{cases} \Delta^2 f = -[w, w] \\ w_{tt} - \Delta w_{tt} + \Delta^2 w = [f+f_0, w] + p, \end{cases}$$

where Δ^2 and $[f, w]$ denote the biharmonic operator and $f_{xx}w_{yy}+f_{yy}w_{xx}-2f_{xy}w_{xy}$, respectively. The functions p and f_0 correspond to given lateral loard and a stress function derived from given plain-stress problem, respectively. We assume that the boundary $\partial\Omega$, p and f_0 are sufficiently smooth. The system (1.1) is solved under the boundary condition $f=w=\frac{df}{dn}=\frac{dw}{dn}=0$ on $\partial\Omega$ and the initial condition $w\Big|_{t=0}=w_0$, $\frac{\partial w}{\partial t}\Big|_{t=0}=w_1$, where n is the outward normal to $\partial\Omega$.

2. Approximating scheme.

We use (u,w) and $\|u\|$ to denote the inner product and the norm in $L_2(\Omega)$. The space H is the completion of the set of all C^∞-functions with support in Ω under the norm $\|w\|_H=\sqrt{(\Delta w, \Delta w)}$. By $\hat{H}$ we denote the finite element subspace of H spanned by a finite element basis $\{\phi_i\}(i=1,\dots,K)$.

The approximating scheme presented in [1] is as follows. The interval $[0,T]$ is divided into equal pieces of length $\Delta t=T/N$ by the points $t=n\Delta t$ $(n=0,1,\dots,N)$.

Then the approximate solution (f^n, w^n) at the time level $n\Delta t$ is determined by the following system of equations in $\hat{H}$.

$$(2.1)\qquad (\Delta f^n, \Delta\phi) = -\frac{1}{2}([w^{n+1}+w^{n-1}, w^n], \phi) \qquad \text{for all } \phi \in \hat{H},$$

$$(2.2)\qquad \begin{aligned}&(D_t D_{\bar{t}} w^n, \phi) - (D_t D_{\bar{t}}\, \Delta w^n, \phi) + \frac{1}{2}(\Delta(w^{n+1}+w^{n-1}), \Delta\phi)\\ &= ([f^n + f_0^n, w^n], \phi) + (p^n, \phi) \qquad \text{for all } \phi \in \hat{H},\end{aligned}$$

where D_t and $D_{\bar{t}}$ are the forward and backward difference operators, respectively. The initial conditions are approximated as

$$(2.3)\qquad w^0 = \dot{w}_0,\ w^1 = \dot{w}_0 + \dot{w}_1 \Delta t,$$

where $\dot{w}$ denotes the interpolate of w in $\hat{H}$. This scheme is well defined. Precisely, the coefficient matrix to determine w^{n+1} is symmetric and positive definite through all time steps (see [1]).

To analyze the approximating scheme it is conveniet to represent it by a system of operator equations in H.

We define the following operators C and B by means of the Riesz representation theorem for bounded linear functional on H:

$$([u, v], \phi) = (C(u, v), \phi)_H \qquad \text{for all } \phi \in H,$$

$$(u, \phi) = (Bu, \phi)_H \qquad \text{for all } \phi \in H.$$

Let P be the projection on H onto $\hat{H}$. Then our scheme is represented on H as follows.

$$(2.4)\qquad \begin{aligned}&f^n = -\frac{1}{2} PC(w^{n+1}+w^{n-1}, w^n)\\ &PB(D_t D_{\bar{t}} w^n - D_t D_{\bar{t}} \Delta w^n) + \frac{1}{2}(w^{n+1}+w^{n-1})\\ &= PC(f^n + f_0^n, w^n) + PBp^n.\end{aligned}$$

As well known, the form $([u, v], w)$ is symmeric, that is,

$$(2.5)\qquad ([u, v], w) = [v, u], w) = ([w, u], v).$$

3. Stability in energy.

For proving the unconditional stability of our scheme, we provide the following lemma.

LEMMA 1. Let $\{x_n\}(n=0,1,\dots,N)$ be a sequence of nonnegative numbers satisfying the following inequality.

$$D_{\bar{t}}x_n \leqq C_1\, x_n^{\frac{1}{2}} + C_2 x_{n-1}^{\frac{1}{2}},$$

where the constants C_1, C_2 are positive, and x_0 is given. Then, holds the following inequality for all n.

$$x_n^{\frac{1}{2}} \leqq x_0^{\frac{1}{2}} + (C_1+C_2)T.$$

PROOF. Consider the positive solution of the equation,

$$D_{\bar{t}}\, x'_n = C_1 x_n'^{\frac{1}{2}} + C_2\, x_{n-1}'^{\frac{1}{2}}, \qquad x'_0 = x_0.$$

This equation has a unique nonnegative solution $\{x'_n\}$ satisfying

(3. 1) $$x_n \leqq x'_n \text{ and } x_0 = x'_0 \leqq x'_1 \leqq x'_2 \leqq \cdots \leqq x'_N.$$

Therefore, the following inequality holds for the sequence $\{x'_n\}$.

$$D_{\bar{t}}\, x'_n \leqq (C_1+C_2)\, x_n'^{\frac{1}{2}}.$$

We next consider the equation obtaind by equating the both sides of this inequality:

$$D_{\bar{t}}\, x''_n = C\, x_n''^{\frac{1}{2}} \qquad \text{where} \qquad C = C_1 + C_2, \quad x''_0 = x'_0.$$

This equation has a unique nonnegative solution $\{x''_n\}$ satisfying,

(3. 2) $$x'_n \leqq x''_n \text{ and } x'_0 = x''_0 \leqq x''_1 \leqq x''_2 \leqq \cdots \leqq x''_N.$$

It is easy to see that $x_n''^{\frac{1}{2}} \leqq x_{n-1}''^{\frac{1}{2}} + C\Delta t$. Hence we have

(3. 3) $$x_n''^{\frac{1}{2}} \leqq x_0''^{\frac{1}{2}} + CT.$$

The lemma follows from (3.1), (3.2) and (3.3).

Our conclution is then as follows.

THEOREM 1. Let E_n be defined by

$$E_n=\|D_t w^n\|^2+|D_t w^n|_1^2+\frac{1}{2}\|\Delta w^{n+1}\|^2+\frac{1}{2}\|\Delta w^n\|^2+\frac{1}{4}\|PC(w^n,w^{n+1})\|_H^2$$
$$+\frac{1}{4}\|PC(w^n,w^{n+1})-2f_0^n\|_H^2-\|f_0^n\|_H^2+M,$$

where $M=\underset{t}{Max}\|f_0\|_H^2$, and $|w|_1^2=\|w_x\|^2+\|w_y\|^2$. Then, holds the following energy inequality.

$$E_n^{\frac{1}{2}}\leqq E_0^{\frac{1}{2}}+CT \qquad (n=1,2,.., N),$$

where $C=2\underset{t}{Max}\,(\|D_{\bar{t}}f_0\|_H+\|p\|)$.

PROOF. Replacing ϕ by $(w^{n+1}-w^{n-1})/\Delta t$ in the both sides of the equation (2. 2), we have the next equality.

$$\begin{aligned}&(D_tD_{\bar{t}}w^n, w^{n+1}-w^{n-1})/\Delta t-(D_tD_{\bar{t}}\Delta w^n, w^{n+1}-w^{n-1})/\Delta t\\ &\quad+\frac{1}{2}(\Delta(w^{n+1}+w^{n-1}),\ \Delta(w^{n+1}-w^{n-1}))/\Delta t\\ &=([f^n+f_0^n,w^n],w^{n+1}-w^{n-1})/\Delta t+(p^n,w^{n+1}-w^{n-1})/\Delta t.\end{aligned}\tag{3. 4}$$

Each term of the left side is written as follows.

$$(D_tD_{\bar{t}}w^n,w^{n+1}-w^{n-1})/\Delta t=D_{\bar{t}}\|D_tw^n\|^2,$$
$$-(D_tD_{\bar{t}}\Delta w^n,w^{n+1}-w^{n-1})/\Delta t=D_{\bar{t}}|D_tw|_1^2,$$
$$\frac{1}{2}(\Delta(w^{n+1}+w^{n-1}),\ \Delta(w^{n+1}-w^{n-1}))/\Delta t=\frac{1}{2}D_{\bar{t}}(\|\Delta w^{n+1}\|^2+\|\Delta w^n\|^2).$$

On the other hand, by using the first equality of (2. 4) and the relation (2. 5), we have

$$\begin{aligned}&([f^n+f_0^n,w^n],w^{n+1}-w^{n-1})/\Delta t\\ &=-\frac{1}{2}D_{\bar{t}}\|PC(w^n,w^{n+1})\|_H^2+([f_0^n,w^n],w^{n+1}-w^{n-1})/\Delta t\\ &=-\frac{1}{2}D_{\bar{t}}\|PC(w^n,w^{n+1})\|_H^2+D_{\bar{t}}(PC(w^n,w^{n+1}),f_0^n)_H\\ &\qquad-(PC(w^n,w^{n-1}),D_{\bar{t}}f_0^n)_H\\ &=-D_{\bar{t}}\left[\frac{1}{4}\|PC(w^n,w^{n+1})\|_H^2+\frac{1}{4}\|PC(w^n,w^{n+1})-2f_0^n\|_H^2-\|f_0^n\|_H^2\right]\\ &\qquad-(PC(w^n,w^{n-1}),D_{\bar{t}}f_0^n)_H.\end{aligned}$$

Since $(p^n, w^{n+1}-w^{n-1})/\Delta t=(p^n, D_tw^n+D_tw^{n-1})$, the equality (3. 4) is written as follows.

$$D_{\bar{t}}\,[\|D_t w^n\|^2 + |D_t w^n|_1^2 + \frac{1}{2}\|w^{n+1}\|^2 + \frac{1}{2}\|\Delta w^n\|^2 + \frac{1}{4}\|PC(w^n, w^{n+1})\|_H^2$$
$$+ \frac{1}{4}\|PC(w^n, w^{n+1}) - 2f_0^n\|_H^2 - \|f_0^n\|_H^2]$$
$$= -(PC(w^n, w^{n-1}), D_{\bar{t}} f_0^n)_H + (p^n, D_t w^n + D_t w^{n-1}).$$

Since M is constant, we have

$$D_{\bar{t}} E_n \leqq \|PC(w^n, w^{n-1})\|_H \|D_{\bar{t}} f_0^n\|_H + \|p^n\|(\|D_t w^n\| + \|D_t w^{n-1}\|)$$
$$\leqq (2\|D_{\bar{t}} f_0^n\|_H + \|p^n\|) E_{n-1}^{\frac{1}{2}} + p^n \|E_n\|^{\frac{1}{2}}.$$

Hence, by the lemma, we have the following estimation.

$$E_n^{\frac{1}{2}} \leqq E_0^{\frac{1}{2}} + CT,$$

where $C = 2\, \underset{t}{Max}\, (\|D_{\bar{t}} f_0\|_H + \|p\|)$. The theorem is thus proved.

REMARK: In [1], the stability is proved under the condition

$$\Delta t < 2/(3\sqrt{3}) \underset{|\alpha|=2}{Max} |D^\alpha f_0^n|.$$

Reference

[1] T. Miyoshi: An unconditionally stable implicit finite element scheme for solving nonlinear dynamical problems of elastic plates, FUNCTIONAL ANALYSIS AND NUMERICAL ANALYSIS. Japan-France Seminar, Tokyo and Kyoto, 1976; H. Fujita (Ed.), Japan Society for the Promotion of Science, 1978.

Department of Mathematics
Faculty of Science
Kumamoto University

Ariake Technical College