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ON FONG'S REDUCTIONS

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P. Fong gave many interresting results on modular representations of p-solvable groups, p a prime. In his work two reductions (theorems (2B) and (2D) in Fong [3]) play an important rôle. On the second reduction (Theorem (2D) in [3], W. Feit improves it in [2] ((1. 1) and (1. 2) of Chapter X). Our paper concerns the reductions. In § 1, we shall supply the gap of the proof of (1. 1) and give a remark to (1. 1) applying our argument. In § 2, we shall give a remark to Theorem (3C) in [3] by using the reductions.

§ 1

(1.1) in [2] is described as follows:

Let G be a finite group and let $H \lhd G$. Let ζ be an irreducible character of H. Assume that $G=T(\zeta)$, the inertia group of ζ. Let F be an algebraically closed field such that char $F \nmid |H|$. Let V be an irreducible $F[H]$-module which affords ζ. Then there exist a finite group $\tilde{G}$ and an exact sequence

$$(1) \qquad \langle 1 \rangle \longrightarrow Z \longrightarrow \tilde{G} \xrightarrow{f} G \longrightarrow \langle 1 \rangle$$

which satisfy the following conditions (i) and (ii).

(i) Z is a cyclic group in the center of $\tilde{G}$ and $|Z| \mid |H|^2$. Also $\tilde{G}$ contains a normal subgroup $\tilde{H}$ such that $Z\tilde{H}=Z\times\tilde{H}=f^{-1}(H)$. The group $\tilde{G}$ depends only on G and ζ, in particular it is independent of the choice of F.

(ii) Let F_1 be the subfield of F generated by a primitive $|H|^2$-*th* root of unity. There exists an $F_1[\tilde{G}]$-module $\tilde{V}_1$ such that if $\tilde{V}=\tilde{V}_1 \otimes_{F_1} F$ then $f(\tilde{V}_{\tilde{H}}) \cong V$.

Further the following holds for the group $\tilde{G}$. If W is an irreducible $F[G]$-module such that V is a constituent of W_H, then $W \cong \tilde{V} \otimes \tilde{W}$ for some absolutely irreducible $F[\tilde{G}/\tilde{H}]$-module $\tilde{W}$. Let $\Delta(F)$ be the set of all Brauer characters afforded by irreducible $F[G]$-modules W such that V is a constituent of W_H. Let $\tilde{\Delta}(F)$ be the set of all Brauer characters afforded by irreducible $F[\tilde{G}/\tilde{H}]$-modules U such that Z is in the kernel of $\tilde{V} \otimes U$. Then the map sending W to $\tilde{W}$ induces a one to one mapping from $\Delta(F)$ onto $\tilde{\Delta}(F)$.

We recall the part of the proof of (1. 1), which we discuss in this section. Let F' be a field with $F_1 \subseteq F' \subseteq F$. Let V' be an irreducible $F'[H]$-module which affords ζ and A be a representation with underlying module V'. Let $S=\{\det A(y) \mid y \in H\}$. For $x \in G$ let N_x be the set of all linear transformations z on V' such that $z^{-1}A(y)z=A(x^{-1}yx)$ for all $y \in H$ and such that $\det z \in S$. Then

$$(*) \qquad N_x \neq \phi$$

for each $x \in G$. The proof of $(*)$ is not complete. We have showed in [4] that $(*)$ is true when F_1 is the subfield of F generated by a primitive $\zeta(1)|H|$*-th* root of unity. Using the following Lemma, here, we show $(*)$ remains valid if F_1 is the subfield of F generated by a primitive $(d, 2\bar{n})h$*-th* root of unity, where $d=\zeta(1)$, $h=|H|$ and $\bar{n}$ is the exponent of G/H.

LEMMA *Let* G *be a group of finite order* g *and* F *be an algebraically closed field with* char $F \nmid g$. *Let* T *be an irreducible representation of* G *over* F *with degree* d *and* s *be the order of* $\det T$, *that is,* s *the smallest natural number such that* $(\det T(x))^s=1$ *for all* $x \in G$. *Then we have* $ds \mid 2g$.

PROOF We prove by induction on g. Let G' be the commutator subgroup of G. If $G'=G$ or $\langle 1 \rangle$, then the lemma is trivial. We assume that G' is a proper subgroup of G, then there exists a normal subgroup N of G such that the index $|G:N|$ of N in G is a prime, say l. Let T_0 be an irreeducible constituent of T_N, the restriction of T to N, and s_0 be the order of $\det T_0$.

If T_N is irreducible, then $T_N=T_0$ and

$$(\det T(x))^l=\det T_0(x^l) \qquad \text{(for all } x \in G).$$

Hence

$$(\det T(x))^{ls_0}=(\det T_0(x^l))^{s_0}=1 \qquad \text{(for all } x \in G).$$

Thus, we have $s \mid ls_0$. Therefore, by the induction hypothesis, we see $ds|2g$.

If T_N is not irreducible then, by Clifford's theorem, we see that $T=T_0{}^G$. Therefore, for $x \in G$, there exists $y_i \in N$ $(i=1,2,\dots,l)$ such that

$$\det T(x)=\pm\det T_0(y_1)\det T_0(y_2)\cdots\det T_0(y_l).$$

Hence we have

$$(2) \qquad (\det T(x))^{s_0}=(\pm 1)^{s_0}.$$

If s_0 is even, then $(\det T(x))^{s_0}=1$ for all $x\in G$, that is, $s \mid s_0$. Hence we see $ds \mid 2g$ by the induction hyothesis. If s_0 is odd, then we can see that $\deg T_0 \cdot s_0=(d/l)s_0 \mid |N|$ by the induction hypothesis. On the other hand, from (2), we see $(\det T(x))^{2s_0}=1$ for all $x\in G$, hence $s \mid 2s_0$. Therefore, we see $ds \mid 2g$. This completes the proof.

REMARK *Since N_x for $x=1$ can be taken as Z of exact sequence (1), by Lemma, we may replace $|Z| \mid h^2$ in (i) of (1. 1) by $|Z| \mid (d, 2)h$. In particular, if d is odd the $|Z| \mid h$.*

PROOF OF (*) We may assume that char $F=0$ and A is a matrix representation over F_0, where F_0 is the subfield of F generated by a primitive *h-th* root of unity. Let x be a fixed element of G. Since A and $A^{(x)}$ are equivalent in F_0, there exists $Z_0 \in GL(d, F_0)$ such that

(3) $$Z_0^{-1}A(y)Z_0=A^{(x)}(y)=A(x^{-1}yx) \qquad (\text{for all } y\in H).$$

We set $z=\lambda Z_0$, $\lambda^d=(\det Z_0)^{-1}$. We have

(4) $$\det z=1,$$

(5) $$z^{-1}A(y)\,z=A(x^{-1}yx) \qquad (\text{for all } y\in H),$$

(6) $$z^d=(\det Z_0)^{-1}Z_0{}^d \in GL(d, F_0).$$

If we denote by $\bar{x}$ the element of the residue class group G/H represented by x and by $o(\bar{x})$ the order of $\bar{x}$, then

$$z^{-o(\bar{x})}A(y)\,z^{o(\bar{x})}=A(x^{-o(\bar{x})}yx^{o(\bar{x})})=A(x^{o(\bar{x})})^{-1}A(y)A(x^{o(\bar{x})})$$

for all $y\in H$. Hence by Schur's lemma we may write

$$z^{o(\bar{x})}=\eta A(x^{o(\bar{x})}) \qquad (\eta\in F).$$

From (4), $\eta^{d|S'}=1$. Since, by Lemma, η is a $(d, 2)h$*-th* root of unity, we have $\eta\in F_1$,

(7) $$z^{o(\bar{x})}\in GL(d, F_1),$$

(8) $$z^{2o(\bar{x})h}=I \quad (\text{identity matrix}).$$

From (6) and (7), we see that if $o(\bar{x})$ and d are relatively prime, then $z\in GL(d, F_1)$ and so N_x is not empty.

We consider the group $<z, A(y) \mid y \in H>$ generated by z and $A(y)$, $y \in H$. By (5) and (8), the order of the group is finite and the exponent of it divides $2o(\bar{x})h$, for $\eta^{2h}=1$. Hence, by Brauer's theorem ([1, Theorem 1]) there exists a non-singular matrix P over F such that $P^{-1}zP$, $P^{-1}A(y)P$ $(y \in H)$ are matrices over F_2, where F_2 is the subfield of F generated by a primitive $2o(\bar{x})h$-th root $\sqrt[2o(\bar{x})h]{1}$ of unity. Hence $P^{-1}AP$ is a representation of H over F_2. Hence there exists $Q \in GL(d, F_2)$ such that

$$Q^{-1}(P^{-1}A(y)P)\,Q=A(y) \qquad \text{(for all } y \in H).$$

By Schur's lemma, PQ is a scalar matrix and

$$z=(PQ)^{-1}z(PQ)=Q^{-1}(P^{-1}zP)\,Q \in GL(d, F_2).$$

Hence $\lambda \in F_2$.

Case I ($2\nmid d$ *or* $4|h$) To show that $N_x \neq \phi$, we may assume $o(\bar{x})=q^a$, q a prime. Further we may assume $q \mid d$, since $N_x \neq \phi$ when $(o(\bar{x}), d)=1$. In this case $F_2 =F_0(\sqrt[2o(\bar{x})h]{1})$ is a cyclic extension over F_0 with degree $(h, 2)o(\bar{x})$. Hence, if we set $(F_0(\lambda): F_0)=m$, then

$$m \mid (h, 2)\,o(\bar{x}), \qquad F_0(\lambda)=F_0(\sqrt[mh]{1}).$$

On the other hand, $m \mid d$, since λ^d, $\sqrt[d]{1} \in F_0$. Hence, $mh|(d, 2\bar{n})\,h$ and $F_0(\lambda)\subseteq F_1$. Therefore, $z \in GL(d, F_1)$ and $z \in N_x$. This completes the proof in this case.

Case II ($2|d$ *and* $4\nmid h$) Let y_0 be an involution of H. In this case, $<y_0>$ is a 2-Sylow subgroup of H. By Burnside's theorem, H has a normal 2-complement H_0. Since $2|d$ and H_0 is a $2'$-group, by Clifford's theorem, there exists an irreducible character ζ_0 of H_0 such that $\zeta=\zeta_0{}^H$ and $\zeta_{H_0}=\zeta_0+\zeta_0{}^{(y_0)}$. Let A_0 be a matrix representation of H_0 over F_0 which affords ζ_0. We may assume that

$$(9) \qquad A(y)=\begin{pmatrix} A_0(y) & 0 \\ 0 & A_0(y_0{}^{-1}yy_0)\end{pmatrix} \quad (y \in H_0), \quad A(y_0)=\begin{pmatrix} 0 & I_0 \\ I_0 & 0\end{pmatrix},$$

where I_0 is the identity matrix of degree $d/2$.

Since $<y_0>$ is a 2-Sylow subgroup of H, $G=H\cdot N_G(<y_0>)=H\cdot C_G(y_0)$. Hence we may assume $x \in C_G(y_0)$, because $N_x \neq \phi$ is evident for $x \in H$. Since the inertia group $T(\zeta)$ is G, $\zeta_0{}^{(x)}=\zeta_0$ or $\zeta_0{}^{(y_0)}$. Hence we may also assume that $\zeta_0{}^{(x)}=\zeta_0$, replacing x by xy_0 if $\zeta_0{}^{(x)}=\zeta_0{}^{(y_0)}$.

We set $\tilde{x}=xH_0(\in<x, H_0>/H_0)$ and $|H_0|=h_0$. Let F_1' be the subfield of F generated by a primitive $(d/2, 2o(\tilde{x}))h_0$-th root of unity. From Case I, in which

$\langle x, H_0 \rangle$, H_0 and ζ_0 are taken as G, H and ζ, there exists a non-singular matrix Z' over F_1' such that

$$(10) \qquad (Z')^{-1} A_0(y) Z' = A_0(x^{-1}yx) \qquad (\text{for all } y \in H_0), \qquad \det Z' = 1.$$

Since $d/2$ is odd and $o(\bar{x}) \mid o(\tilde{x}) \mid 2o(\bar{x})$, we have

$$(d/2,\ 2o(\bar{x}))h_0 = (d/2,\ o(\tilde{x}))h_0 \mid (d,\ 2\bar{n})h,$$

hence $F_1' \subseteq F_1$. If we set

$$Z_1 = \begin{pmatrix} Z' & 0 \\ 0 & Z' \end{pmatrix}$$

then $Z_1 \in GL(d, F_1)$. From (9), (10) and the hypothesis $x \in C_G(y_0)$, we see $\det Z_1 = 1$ and

$$Z_1^{-1} A(y) Z_1 = A(x^{-1}yx) \qquad (\text{for all } y \in H).$$

Hence $N_x \neq \phi$, as required. This completes the proof.

§ 2

Let p be a fixed prime and G be a p-solvable group of finite order. Let B be a p-block with defect group D and p^c be the index of the center $Z(D)$ in D. For an ordinary irreducible character $\chi \in B$, denote by $h(\chi)$ the height of χ. P. Fong showed that

$$h(\chi) \leqq c$$

holds for every $\chi \in B$ ([3, Theorem (3C)]). To the Fong's result we add the following.

PROPOSITION *Let p be a prime and G be a p-solvable group of finite order g. Let B be a p-block with defect group D and p^c be the index of $Z(D)$ in D. If*

$$h(\chi) = c \qquad (\textit{for some } \chi \in B),$$

then D is abelian.

PROOF We prove by induction on g. The proposition is trivial for p'-groups. By reduction theorems (2B) and (2D) in [3], we may assume that B is a block with maximum defect, that is, D is a p-Sylow subgroup of G. Let H be a maximal normal subgroup of G and b be a block of H covered by B. b is of maximum defect and has a defect group $D_0=D\cap H$. If $p\nmid |G:H|$ then, by the induction hypothesis, we can easily see that D is abelian. So we may assume $|G:H|=p$. Let ζ be an irreducible constituent of χ_H belonging to b.

First we show $Z(D)\not\subseteq H$. Assume $Z(D)\subseteq H$. Then $Z(D)\subseteq Z(D_0)$. Hence $\nu(|D:Z(D)|>\nu(|D_0:Z(D_0)|)$, where if n is a non-zero integer then $p^{\nu(n)}$ is the heighest power of p which divides n. On the other hand, if χ_H is irreducible, then $\chi_H=\zeta$ and we have, by [3, Theorem (3C)],

$$\nu(|D_0:Z(D_0)|)\geq h(\zeta)=\nu(\zeta(1))=\nu(\chi(1))=h(\chi),$$

therefore

$$\nu(|D:Z(D)|)>h(\chi).$$

This contradicts the assumption of Proposition. If χ_H is not irreducible, then $\chi=\zeta^G$ and

$$\nu(|D_0:Z(D_0)|)\geq h(\zeta)=\nu(\zeta(1))=\nu(|D_0:Z(D)|)\geq \nu(|D_0:Z(D_0)|).$$

Hence, we have

$$\nu(|D_0:Z(D_0)|)=h(\zeta), \quad Z(D_0)=Z(D).$$

By the induction hypothesis, D_0 is abelian and $Z(D_0)=D_0$. Therefore D is abelian, which yields a contradiction. Hence $Z(D)\not\subseteq H$. Therefore by [3, Lemma (3B)], χ_H is irreducible and

$$\begin{aligned}\nu(|D_0:Z(D_0)|)\geq h(\zeta)=h(\chi)=\nu(|D:Z(D)|)\\ =\nu(|D_0:Z(D)\cap D_0|)\geq \nu(|D_0:Z(D_0)|).\end{aligned}$$

Hence the equalities

$$h(\zeta)=\nu(|D_0:Z(D_0)|), \quad Z(D)\cap D_0=Z(D_0)$$

hold. By the induction hypothesis, D_0 is abelian and $D_0=Z(D_0)\subseteq Z(D)$. Hence D is abelian. This completes the proof.

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