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REAL SUBMANIFOLDS OF CONSTANT MEAN CURVATURE IN COMPLEX PROJECTIVE SPACE

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Introduction. Let S_N be complex projective N-space. The unitary group $U(N+1)$, or rather the projective unitary group $PSU(N+1)$, acts on S_N in the natural way. Suppose S_N be equipped with the *Fubini-Study metric*. Then the connected group of isometries of S_N coincides with $PSU(N+1)$, which we denote in brief by G below. We also write K for the isotropy subgroup of G at an appropriate point of S_N. Then S_N is isometric to G/K with an invariant metric. For any closed subgroup $H \not\subset K$ of G, the (not necessarily complex) submanifold $H/H \cap K$ has a constant mean curvature in G/K. The corresponding submanifold in S_N, of course, is of constant mean curvature.

The present authors have been interested in these submanifolds (especially real ones). But little has been known about them.

Suppose $N=n(n+2)$ where n runs through $1, 2 \ldots$. Let us consider the natural projection of $C^{N+1}-0$ to S_N. We denote by π_N the restriction of this projection to the unit sphere S^{2N+1} in C^{N+1}. Let m be the Mannoury imbedding of S_n (into $S^N \subset R^{N+1} \subset C^{N+1}$). We recall that, for $P \in S_n$, $m(P)$ is the point with the rectangular cartesian coordinates:

$$z_h \bar{z}_h, \quad \frac{1}{\sqrt{2}}(z_h \bar{z}_k + \bar{z}_h z_k), \quad \frac{\sqrt{-1}}{\sqrt{2}}(z_h \bar{z}_k - \bar{z}_h z_k)$$
$$(h,\ k=0, \ldots ,\ n;\ h<k)$$

where $z_0, \ldots, z_n$ are the homogeneous coordinates of P. Then $\pi_N \circ m(S_n)$ is a typical example of real submanifolds of the above kind. We will calculate its mean curvature in S_N, which eventually is $\dfrac{1}{\sqrt{2n}}$.

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1. Skew-Segre imbedding. The natural projection σ is the map that sends each point $(w_0, \ldots, w_N)$ of $C^{N+1}-0$ to the point complex projective N-space S_N

with homogeneous coordinates $w_0, \ldots, w_N$. Then the triple $(C^{N+1}-0, \sigma, S_N)$ is a fibre bundle. The unitary group $U(N+1)$ acts on both $C^{N+1}-0$ and S_N. This action satisfies the commutative relation: $g\circ\sigma=\sigma\circ g$ where $g\in U(N+1)$. Let π_N be the restriction of σ to the unit sphere S^{2N+1}. Then (S^{2N+1}, π_N, S_N) becomes a circle bundle each fibre of which has the constant length 2π.

Now let X_h, X_{hk} $(h, k=0, \ldots, n)$ be rectangular coordinates in $R^{(n+1)^2}$ and consider the map m of S_n into $R^{(n+1)^2}$ defined by the equations

$$X_h=z_h\bar{z}_h,\ X_{hk}=\frac{1}{\sqrt{2}}(z_h\bar{z}_k+\bar{z}_h z_k)$$

$$X_{kh}=\frac{\sqrt{-1}}{\sqrt{2}}(\bar{z}_k z_h-z_k\bar{z}_h)$$

where $h, k=0, \ldots, n$ and $h<k$. We may call m *Mannoury imbedding* of S_n. Up to a unitary transformation, this Mannoury imbedding is coincidental to the map defined by simpler equations

$$X'_h=z_h\bar{z}_h,\ X'_{hk}=z_h\bar{z}_k\ (h\neq k).$$

We, in fact, have

$$\begin{pmatrix} X_0 \\ \cdot \\ \cdot \\ \cdot \\ X_n \\ X_{01} \\ \cdot \\ \cdot \\ \cdot \\ X_{n-1,\,n} \\ X_{10} \\ \cdot \\ \cdot \\ \cdot \\ X_{n,\,n-1} \end{pmatrix} = \begin{pmatrix} 1_{n+1} & & \\ & \frac{1}{\sqrt{2}}1_{\frac{n(n+1)}{2}} & \frac{1}{\sqrt{2}}1_{\frac{n(n+1)}{2}} \\ & \frac{\sqrt{-1}}{\sqrt{2}}1_{\frac{n(n+1)}{2}} & \frac{-\sqrt{-1}}{\sqrt{2}}1_{\frac{n(n+1)}{2}} \end{pmatrix} \begin{pmatrix} X'_0 \\ \cdot \\ \cdot \\ \cdot \\ X'_n \\ X'_{01} \\ \cdot \\ \cdot \\ \cdot \\ X'_{n-1,\,n} \\ X'_{10} \\ \cdot \\ \cdot \\ \cdot \\ X'_{n,\,n-1} \end{pmatrix}$$

where 1_L denotes the unit matrix of degree L for $L=n+1$, $n(n+1)/2$. The matrix

$$\begin{pmatrix} 1_{n+1} & & \\ & \frac{1}{\sqrt{2}} 1_{\frac{n(n+1)}{2}} & \frac{1}{\sqrt{2}} 1_{\frac{n(n+1)}{2}} \\ & \frac{\sqrt{-1}}{\sqrt{2}} 1_{\frac{n(n+1)}{2}} & \frac{-\sqrt{-1}}{\sqrt{2}} 1_{\frac{n(n+1)}{2}} \end{pmatrix}$$

is clearly unitary. We note that the image by the Mannoury imbedding of S_n is contained in S^{n^2+2n}. From now on we assume

$$N = n(n+2).$$

We denote by m' the product $\pi_N \circ m$. Then m' is a diffeomorphism of S_n onto a real submanifold of S_N. Let P be the point with homogeneous coordinates $z_0, \dots, z_n$ and define

$s(P) =$ the image by π_N of the point with rectangular coordinates X'_h, X'_{hk} $(h, k=0, \dots, n;\ h \neq k)$.

Then s gives to a diffeomorphism of S_n onto a real submanifold of S_N, which can be transformed by the action of an element of $PSU(N+1)$ to $m'(S_n)$. We call s *skew-Segre imbedding* of S_n in S_N.

As is well known, we can introduce a hermitian metric into S_n by using the Mannoury imbedding (e. g., see [1]). It is called the *Fubini-Study* metric in the complex projective space. Explicitly it is given by

$$ds^2 = \frac{\sum_{h=0}^{n} dz_h\, d\bar{z}_h}{\sum_{h=0}^{n} z_h\, \bar{z}_h} - \frac{\sum_{h,k=0}^{n} (\bar{z}_h\, dz_h)(z_k\, d\bar{z}_k)}{\left(\sum_{h=0}^{n} z_h\, \bar{z}_h\right)^2}$$

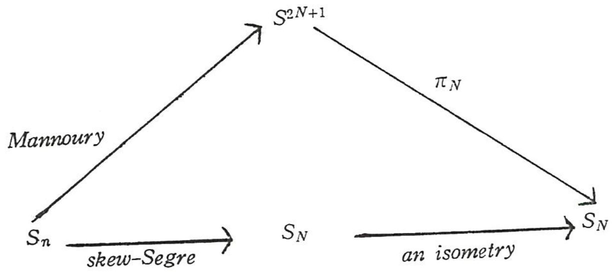
if we use the homogeneous coordinates $z_0, \dots, z_n$. Instead of the homogeneous coordinates, we can also use inhomogeneous coordinates

$$x_1 = \frac{z_1}{z_0}, \dots, x_n = \frac{z_n}{z_0}$$

on the coordinate neighbourhood $U = \{P \in S_n \mid z_0 \neq 0\}$. Then the metric ds can be expressed in the form:

$$ds^2 = \frac{\sum_{i=1}^{n} dx_i\, d\bar{x}_i}{1 + \sum_{i=1}^{n} x_i\, \bar{x}_i} - \frac{\left(\sum_{i=1}^{n} x_i\, d\bar{x}_i\right)\left(\sum_{j=1}^{n} \bar{x}_j\, dx_j\right)}{(1 + \sum_{i=1}^{n} x_i\, \bar{x}_i)^2}.$$

From now on we suppose the complex projective space under consideration are always equipped with the Fubini-Study metrics. Then we have the following commutative diagram;



We can see that the image $m(S_n)$ by the Mannoury imbedding intersects each fibre of π_N orthogonally [2]. Since the image $s(S_n)$ by the skew-Segre imbedding can be transformed into $m'(S_n)$, $m'(S_n)$ has the same mean curvature as $s(S_n)$ in S_N. Hence we will calculate the mean curvature of $s(S_n)$ in S_N instead of that of $m'(S_n)$ (the calculation will be carried out in *Section* **3**).

2. The Christoffel symbols with respect to the Fubini-Study metric. Let us introduce again the inhomogeneous coordinates $x_1, \dots, x_n$ on the coordinate neighbourhood U_0. Then the fundamental tensor of the Fubini-Study metric is given by

$$(1) \qquad g_{h\bar h} = \frac{1+\sum_{i\neq h} x_i \bar x_i}{(1+\sum_{i=1}^{n} x_i \bar x_i)^2}, \quad g_{h\bar k} = \frac{-\bar x_h x_k}{(1+\sum_{i=1}^{n} x_i \bar x_i)^2}$$

where h, k range over $1, \dots, n$ and $h\neq k$. The conjugate tensor is given by

$$g^{h\bar h} = (1+x_h \bar x_h)(1+\sum_{i=1}^{n} x_i \bar x_i)$$

$$g^{h\bar k} = x_h \bar x_k (1+\sum_{i=1}^{n} x_i \bar x_i).$$

The calculation shows us that the Christoffel symbols are

$$\Gamma^{h}_{h\,h} = \frac{-2\bar x_h}{1+\sum_{i=1}^{n} x_i \bar x_i}, \quad \Gamma^{\bar h}_{\bar h\bar h} = \frac{-2x_h}{1+\sum_{i=1}^{n} x_i \bar x_i},$$

$$\Gamma^{h}_{h\,k} = \frac{-\bar x_k}{1+\sum_{i=1}^{n} x_i \bar x_i}, \quad \Gamma^{\bar h}_{\bar h\bar k} = \frac{-x_k}{1+\sum_{i=1}^{n} x_i \bar x_i}.$$

In fact all the other symbols vanish.

3. The mean curvature. The inhomogeneous coordinates on S_n, $x_1, \dots, x_n$, can also be considered as coordinates on $s(S_n)$. We can introduce inhomogeneous coordinates $y_1, \dots, y_N$ on S_N so that the skew-Segre imbedding is expressed by

$$y_h = \bar{x}_h, \quad y_{h(n+1)} = x_h, \quad y_{h(n+1)+k} = x_h \bar{x}_k$$

where h, k run through $1, \dots, n$. $x_1, \dots, x_n$ can be considered also as coordinates on $s(S_n)$. By multiplying (1) by 2, we can obtain the components of the metric tensor induced on $s(S_n)$ from the fundamental tensor of the Fubini-Study metric on S_N. The *Levi-Civita connection* on $s(S_n)$ (resp. S_N) will be denoted by ∇(resp. $^*\nabla$). Let X, Y be tangent vectors to $s(S_n)$ at $(x_1, \dots, x_n)$ and write the *Gauss's formula* (c. f. [3], p. 15) in the form:

$$^*\nabla_X Y = \nabla_X Y + \sum_{\alpha=1}^{2(N-n)} \Omega_\alpha(X, Y)\xi_\alpha,$$

where $\Omega_\alpha(X, Y)$ are the second fundamental forms of $s(S_n)$ and ξ_α are an orthogonal ennuple of unit vectors normal to $s(S_n)$. The components of Ω_α will be written as $\Omega_{\alpha|hk}$ etc. .

From the Gauss's formula it follows that

$$\sum_{\alpha=1}^{2(N-n)} \Omega_{\alpha|h\bar{k}}\, \xi_\alpha = {}^*\nabla_{\frac{\partial}{\partial x_h}} \frac{\partial}{\partial \bar{x}_k} \quad \text{for } h, k = 1, \dots, n.$$

Hence we have

$$\sum_{\alpha=1}^{2(N-n)} \Omega_{\alpha|h\bar{h}}\, \xi_\alpha = \frac{-1}{1+\sum_{i=1}^{n} x_i \bar{x}_i} \left[\bar{x}_h \frac{\partial}{\partial y_h} + x_h \frac{\partial}{\partial \bar{y}_h} + x_h \frac{\partial}{\partial y_{h(n+1)}} + \bar{x}_h \frac{\partial}{\partial \bar{y}_{h(n+1)}} \right.$$
$$+ x_h \sum_{i=1}^{n} \bar{x}_i \frac{\partial}{\partial y_{h(n+1)+i}} + \bar{x}_h \sum_{i=1}^{n} x_i \frac{\partial}{\partial \bar{y}_{h(n+1)+i}} + \bar{x}_h \sum_{i=1}^{n} x_i \frac{\partial}{\partial y_{i(n+1)+h}}$$
$$\left. + x_h \sum_{i=1}^{n} \bar{x}_i \frac{\partial}{\partial \bar{y}_{i(n+1)+h}} \right] + \frac{\partial}{\partial y_{h(n+2)}} + \frac{\partial}{\partial \bar{y}_{h(n+2)}}$$

and

$$\sum_{\alpha=1}^{2(N-n)} \Omega_{\alpha|h\bar{k}}\, \xi_\alpha = \frac{-1}{1+\sum_{i=1}^{n} x_i \bar{x}_i} \left[\bar{x}_h \frac{\partial}{\partial y_k} + x_k \frac{\partial}{\partial y_{h(n+1)}} + x_k \frac{\partial}{\partial \bar{y}_h} + \bar{x}_h \frac{\partial}{\partial \bar{y}_{k(n+1)}} \right.$$
$$+ x_k \sum_{i=1}^{n} \bar{x}_i \frac{\partial}{\partial y_{h(n+1)+i}} + \bar{x}_h \sum_{i=1}^{n} x_i \frac{\partial}{\partial \bar{y}_{i(n+1)+k}} + \bar{x}_h \sum_{i=1}^{n} x_i \frac{\partial}{\partial \bar{y}_{k(n+1)+i}}$$
$$\left. + x_k \sum_{i=1}^{n} \bar{x}_i \frac{\partial}{\partial \bar{y}_{i(n+1)+h}} \right] + \frac{\partial}{\partial y_{h(n+1)+k}} + \frac{\partial}{\partial \bar{y}_{k(n+1)+h}}$$

An easy calculation shows us

$$\frac{1}{2}\sum_{\alpha=1}^{2(N-n)}\left(\frac{1}{n}\sum_{h,k=1}^{n} g^{h\bar{k}}\ \Omega_{\alpha|h\bar{k}}\right)\xi_\alpha=\frac{1+\sum_{i=1}^{n}x_i\ \bar{x}_i}{2\,n}\Bigg[\sum_{h=1}^{n}\left(\frac{\partial}{\partial y_{h(n+2)}}+\frac{\partial}{\partial \bar{y}_{h(n+2)}}-x_h\frac{\partial}{\partial \bar{y}_h}\right.$$
$$\left.-\bar{x}_h\frac{\partial}{\partial y_h}-x_h\frac{\partial}{\partial y_{h(n+1)}}-\bar{x}_h\frac{\partial}{\partial \bar{y}_{h(n+1)}}\right)-\sum_{h,k=1}^{n}\bar{x}_h\ x_k\left(\frac{\partial}{\partial y_{k(n+1)+h}}+\frac{\partial}{\partial \bar{y}_{h(n+1)+k}}\right)\Bigg]$$

This is the mean curvature vector, denoted Ω, of $s(S_n)$ in S_N. The mean curvature of $s(S_n)$ in S_N by definition is the norm $|\Omega|$ of this vector.

Now we will calculate $|\Omega|$ in what follows. $^*g$ stands for the fundamental tensor of the Fubini-Study metric of S_N.

$$\left(\frac{2n\,|\Omega|}{1+\sum_{i=1}^{n}x_i\ \bar{x}_i}\right)^2=2\Bigg[\sum_{h,l=1}^{n}{}^*g_{h(n+2),\,\overline{l(n+2)}}-\sum_{h,l=1}^{n}\bar{x}_l\ {}^*g_{h(n+2),\,\overline{l(n+1)}}$$
$$-\sum_{h,l=1}^{n}x_l\ {}^*g_{h(n+2),\bar{l}}-\sum_{h,l,m=1}^{n}x_l\ \bar{x}_m\ {}^*g_{h(n+2),\overline{m(n+1)+l}}-\sum_{h,l=1}^{n}x_h\ {}^*g_{h(n+1),\,\overline{l(n+2)}}$$
$$+\sum_{h,l=1}^{n}x_h\ \bar{x}_l\ {}^*g_{h(n+1),\,\overline{l(n+1)}}+\sum_{h,l=1}^{n}x_h\ x_l\ {}^*g_{h(n+1),\bar{l}}+\sum_{h,l,m=1}^{n}x_h\ x_l\ \bar{x}_m\ {}^*g_{h(n+1),\overline{m(n+1)+l}}$$
$$-\sum_{h,l=1}^{n}\bar{x}_h\ {}^*g_{h,\overline{l(n+2)}}+\sum_{h,l=1}^{n}\bar{x}_h\ \bar{x}_l\ {}^*g_{h,\overline{l(n+1)}}+\sum_{h,l=1}^{n}x_l\ \bar{x}_h\ {}^*g_{h\bar{l}}+\sum_{h,l,m=1}^{n}\bar{x}_h\ x_l\ \bar{x}_m\ {}^*g_{h,\overline{m(n+1)+l}}$$
$$-\sum_{h,k,l=1}^{n}\bar{x}_h\ x_k\ {}^*g_{k(n+1)+h,\overline{l(n+2)}}+\sum_{h,k,=1}^{n}\bar{x}_h\ x_k\ \bar{x}_l\ {}^*g_{k(n+1)+h,\ \overline{l(n+1)}}$$
$$+\sum_{h,k,l=1}^{n}\bar{x}_h\ x_k\ x_l\ {}^*g_{k(n+1)+h,\bar{l}}+\sum_{h,k,l,m=1}^{n}\bar{x}_h\ x_k\ x_l\ \bar{x}_m\ {}^*g_{k(n+1)h,\overline{m(n+1)+l}}\Bigg]$$
$$=\frac{2}{(1+\sum_{I=1}^{N}y_I\ \bar{y}_I)^2}\Bigg[\sum_{h=1}^{n}(1+\sum_{I\neq h(n+2)}y_I\ \bar{y}_I)-\sum_{1\leq h\neq l\leq n}\bar{y}_{h(n+2)}y_{l(n+2)}+\sum_{h,l=1}^{n}\bar{x}_l\ x_h\ \bar{x}_h\ x_l$$
$$+\sum_{h,l=1}^{n}x_l\ \bar{x}_l\ x_h\ \bar{x}_h-\sum_{h=1}^{n}x_h\ \bar{x}_h\ (1+\sum_{I\neq h(n+2)}y_I\ \bar{y}_I)+\sum_{h,l,m=1}^{n}x_l\ \bar{x}_m x_h\ \bar{x}_h\ x_m\ \bar{x}_l$$
$$-\sum_{h=1}^{n}x_h\ \bar{x}_h\ y_{h(n+2)}\ \bar{y}_{h(n+2)}+\sum_{h,l=1}^{n}x_h\ \bar{x}_h\ x_l\ \bar{x}_l+\sum_{h=1}^{n}x_h\ \bar{x}_h\ (1+\sum_{I\neq h(n+1)}y_I\ \bar{y}_I)$$
$$-\sum_{h,l=1}^{n}x_h\ \bar{x}_l\ \bar{x}_h\ x_l+\sum_{h=1}^{n}x_h\ \bar{x}_h\ y_{h(n+1)}\ \bar{y}_{h(n+1)}-\sum_{h,l=1}^{n}x_h\ \bar{x}_h\ x_l\ \bar{x}_l$$
$$-\sum_{h,l,m=1}^{n}x_hx_l\bar{x}_m\bar{x}_h\bar{x}_lx_m+\sum_{h,l=1}^{n}x_h\bar{x}_hx_l\bar{x}_l-\sum_{h,l=1}^{n}x_h\bar{x}_hx_l\bar{x}_l+\sum_{h=1}^{n}x_h\bar{x}_h\ (1+\sum_{I\neq h}y_I\bar{y}_I)$$
$$-\sum_{h,l=1}^{n}x_h\bar{x}_hx_l\bar{x}_l+\sum_{h=1}^{n}x_h\bar{x}_hy_h\bar{y}_h-\sum_{h,l,m=1}^{u}x_h\bar{x}_hx_l\bar{x}_lx_m\bar{x}_m-\sum_{h=1}^{n}x_h\bar{x}_h\ (1+\sum_{I\neq h(n+2)}y_I\bar{y}_I)$$
$$+\sum_{h,k,l=1}^{n}x_h\bar{x}_hx_k\bar{x}_kx_l\bar{x}_l-\sum_{h=1}^{n}x_h\bar{x}_hy_{h(n+2)}\bar{y}_{h(n+2)}-\sum_{h,k,l=1}^{n}x_h\bar{x}_hx_k\bar{x}_kx_l\bar{x}_l$$
$$-\sum_{h,k,l=1}^{n}x_h\bar{x}_hx_k\bar{x}_kx_l\bar{x}_l+\sum_{h,k=1}^{n}x_h\bar{x}_hx_k\bar{x}_k(1+\sum_{I\neq k(n+1)+h}y_I\bar{y}_I)$$

$$- \sum_{h,k,l,m=1}^{n} x_h \bar{x}_h x_k \bar{x}_k x_l \bar{x}_l x_m \bar{x}_m + \sum_{h,k=1}^{n} x_h \bar{x}_h x_k \bar{x}_k y_{k(n+1)+h} \bar{y}_{k(n+1)+h} \Bigg]$$

$$= 2 \Bigg[\frac{n}{(1+\sum_{i=1}^{n} x_i \bar{x}_i)^2} - \sum_{h,l=1}^{n} \frac{x_h \bar{x}_h x_l \bar{x}_l}{(1+\sum_{i=1}^{n} x_i \bar{x}_i)^4} + \sum_{h,l=1}^{n} \frac{x_h \bar{x}_h x_l \bar{x}_l}{(1+\sum_{i=1}^{n} x_i \bar{x}_i)^4}$$

$$+ \sum_{h,l=1}^{n} \frac{x_h \bar{x}_h x_l \bar{x}_l}{(1+\sum_{i=1}^{n} x_i \bar{x}_i)^4} - \sum_{h=1}^{n} \frac{x_h \bar{x}_h}{(1+\sum_{i=1}^{n} x_i \bar{x}_i)^2} + \sum_{h,l,m=1}^{n} \frac{x_h \bar{x}_h x_l \bar{x}_l x_m \bar{x}_m}{(1+\sum_{i=1}^{n} x_i \bar{x}_i)^4}$$

$$+ \sum_{h,l=1}^{n} \frac{x_h \bar{x}_h x_l \bar{x}_l}{(1+\sum_{i=1}^{n} x_i \bar{x}_i)^4} + \sum_{h=1}^{n} \frac{x_h \bar{x}_h}{(1+\sum_{i=1}^{n} x_i \bar{x}_i)^2} - \sum_{h,l=1}^{n} \frac{x_h \bar{x}_h x_l \bar{x}_l}{(1+\sum_{i=1}^{n} x_i \bar{x}_i)^4}$$

$$- \sum_{h,l=1}^{n} \frac{x_h \bar{x}_h x_l \bar{x}_l}{(1+\sum_{i=1}^{n} x_i \bar{x}_i)^4} - \sum_{h,l,m=1}^{n} \frac{x_h \bar{x}_h x_l \bar{x}_l x_m \bar{x}_m}{(1+\sum_{i=1}^{n} x_i \bar{x}_i)^4} + \sum_{h,l=1}^{n} \frac{x_h \bar{x}_h x_l \bar{x}_l}{(1+\sum_{i=1}^{n} x_i \bar{x}_i)^4}$$

$$- \sum_{h,l=1}^{n} \frac{x_h \bar{x}_h x_l \bar{x}_l}{(1+\sum_{i=1}^{n} x_i \bar{x}_i)^4} + \sum_{h=1}^{n} \frac{x_h \bar{x}_h}{(1+\sum_{i=1}^{n} x_i \bar{x}_i)^2} - \sum_{h,l=1}^{n} \frac{x_h \bar{x}_h x_l \bar{x}_l}{(1+\sum_{i=1}^{n} x_i \bar{x}_i)^4}$$

$$- \sum_{h,l,m=1}^{n} \frac{x_h \bar{x}_h x_l \bar{x}_l x_m \bar{x}_m}{(1+\sum_{i=1}^{n} x_i \bar{x}_i)^4} - \sum_{h=1}^{n} \frac{x_h \bar{x}_h}{(1+\sum_{i=1}^{n} x_i \bar{x}_i)^2} + \sum_{h,k,l=1}^{n} \frac{x_h \bar{x}_h x_k \bar{x}_k x_l \bar{x}_l}{(1+\sum_{i=1}^{n} x_i \bar{x}_i)^4}$$

$$- \sum_{h,k,l=1}^{n} \frac{x_h \bar{x}_h x_k \bar{x}_k x_l \bar{x}_l}{(1+\sum_{i=1}^{n} x_i \bar{x}_i)^4} - \sum_{h,k,l=1}^{n} \frac{x_h \bar{x}_h x_k \bar{x}_l x_l \bar{x}_k}{(1+\sum_{i=1}^{n} x_i \bar{x}_i)^4}$$

$$+ \sum_{h,k=1}^{n} \frac{x_h \bar{x}_h x_k \bar{x}_k}{(1+\sum_{i=1}^{n} x_i \bar{x}_i)^2} - \sum_{h,k,l,m=1}^{n} \frac{x_h \bar{x}_h x_k \bar{x}_k x_l \bar{x}_l x_m \bar{x}_m}{(1+\sum_{i=1}^{n} x_i \bar{x}_i)^4} \Bigg]$$

$$= 2 \Bigg[\frac{n}{(1+\sum_{i=1}^{n} x_i \bar{x}_i)^2} - \sum_{h,l=1}^{n} \frac{x_h \bar{x}_h x_l \bar{x}_l}{(1+\sum_{i=2}^{n} x_i \bar{x}_i)^3} - \sum_{h,l,m=1}^{n} \frac{x_h \bar{x}_h x_l \bar{x}_l x_m \bar{x}_m}{(1+\sum_{i=1}^{n} x_i \bar{x}_i)^3}$$

$$+ \sum_{h,k=1}^{n} \frac{x_h \bar{x}_h x_k \bar{x}_k}{(1+\sum_{i=1}^{n} x_i \bar{x}_i)^2} \Bigg]$$

$$= 2 \Bigg[\frac{n}{(1+\sum_{i=1}^{n} x_i \bar{x}_i)^2} - \sum_{h,l=1}^{n} \frac{x_h \bar{x}_h x_l \bar{x}_l}{(1+\sum_{i=1}^{n} x_i \bar{x}_i)^2} + \sum_{h,k=1}^{n} \frac{x_h \bar{x}_h x_k \bar{x}_k}{(1+\sum_{i=1}^{n} x_i \bar{x}_i)^2} \Bigg]$$

$$= \frac{2n}{(1+\sum_{i=1}^{n} x_i \bar{x}_i)^2}$$

We can therefore find

$$|\Omega| = \frac{1}{\sqrt{2n}}.$$

4. Let d be the degree of the submanifold $s(S_n)$. The following inequality follows from Theorem 5.35 [4].

$$2^n > d.$$

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