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ISOTOPY GROUPS OF 2-SPHERE WITH BOUNDARY HOLES

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1. Introduction.

In this paper we consider the orientation preserving homeomorphisms of the manifold obtained from the 2-sphere by removing the interiors of n disjoint subdisks, where the boundary curves will be denoted by $C_1, C_2, \ldots, C_n$. $H(M_n)$ will denote the group of homeomorphisms of M_n onto itself topologized by the compact open topology. The arc-component of the identity $H_o(M_n)$ is a normal subgroup of $H(M_n)$ and $H(M_n)/H_o(M_n)=\pi_0 [H(M_n)]$ is the group of the arc-components of $H(M_n)$, which is called the isotopy group of $H(M_n)$. The isotopy groups for the subspaces of $H(M_n)$ are similarly defined. The equivalence relation defined by $H_o(M_n)$ is called isotopy.

We denote by $\mathscr{A}[\pi_1(M_n, x_o)]$ the group of automorphisms of the homotopy group $\pi_1(M_n, x_o)$. A homeomorphism $h \varepsilon H(M_n, x_o)$, the isotropy group at the point $x_o \varepsilon M_n$, induces an automorphism $h_\#$ in $\mathscr{A}[\pi_1 (M_n, x_o)]$. Thus $H(M_n, x_o)$ has a representation α: $h \rightarrow h_\#$ as a group of automorphisms of $\pi_1(M_n, x_o)$. Furthermore, if $h \varepsilon H_o(M_n, x_o)$, then an arc from h to the identity in $H(M_n, x_o)$ provides a homotopy of $h(\gamma)$ with γ where $[\gamma] \varepsilon \pi_1(M_n, x_o)$, and hence $h_\#$ is the identity automorphism of $\pi_1(M_n, x_o)$. Thus α induces a representation

$$\alpha_\#:\ \pi_0[H(M_n, x_o)] \rightarrow \mathscr{A}[\pi_1(M_n, x_o)]$$

of the isotopy group $\pi_0[H(M_n, x_o)]$ as a subgroup of automorphisms of $\pi_1(M_n, x_o)$.

Thus we can define a homomorphism

$$\phi:\ \pi_0[H(M_n,x_o)] \rightarrow \mathscr{A}[\pi_1(M_n, x_o)]$$

by $\phi([h])=h_\# \varepsilon \mathscr{A}[\pi_1(M_n, x_o)]$, where $h^\# (\gamma)=[h(\gamma)] \varepsilon \pi_1(M_n, x_o)$ for any γ in a homotopy class $[\gamma] \varepsilon \pi_1(M_n, x_o)$. The domain of the homomorphism ϕ will be taken to be the isotopy groups of various subspaces of the orientation preserving homeomorphisms fixing the base point x_o. Throughout this paper we assume that the base point x_o is on the boundary curve C_n and $H^1_{n-1}(M_n)=\{h \varepsilon H^+_n(M_n) \mid h=e$ on $C_n\}$. By using the homomorphism ϕ, we study the isotopy groups of certain subspaces of $H(M_n)$.

2. Preliminaries

We state preliminary definitions and lemmas which will be needed in the next section.

M_n will denote a manifold obtained from the 2-sphere by removing the interiors of n disjoint subdisks. $H^+(M_n)=\{h \in H(M_n) \mid h$ is orientation preserving on $M_n\}$ $H^t_{n-t}(M_n)=\{h \in H(M_n) \mid h=e$ on certain t boundary curves and $h(C_i)=C_i$ for other $n-t$ curves$\}$. The notation "$\simeq$" will mean the homotopy relation and $\mathbf{Z}$ the group of integers.

DEFINITION 1.1. An isotopy of a space X is a collection $\{G_t\}$, $t \in I$ [0, 1], of homemorphisms of X onto itself such that the mapping G: $X \times I \rightarrow X$ defined by $G(x, t)=G_t(x)$ is continuous. An isotopy which moves no point on $Bd(X)$ is called a B-isotopy. $h \approx g$ will denote that h is isotopic to g. The imbeddings f_0, f_1: $X \rightarrow Y$ are ambient isotopic if there is a level preserving homeomorphism G: $Y \times I \rightarrow Y \times I$ such that $G(y, 0)=(y, 0)$ for all $y \in Y$ and $(f_1(x), 1)=G(f_0(x), 1)$ for all x $\in$ X.

DEFINITION 1.2. An isotopy $\{G_t\}$, $0 \leq t \leq 1$, is called invertible if the collection $\{G_t^{-1}\}$, $0 \leq t \leq 1$, of the inverse homeomorphisms is also an isotopy.

LEMMA 1.3. (Crowell [2]). Every isotopy G_t, $0 \leq t \geq 1$, of a locally compact Hausdorff space is invertible.

LEMMA 1.4. (Epstein [3]). Let M be a 2-manifold with boundary. Let α and β be two arcs in M such that

$$Bd(M) \cap \alpha = Bd(\alpha) = Bd(\beta) = Bd(M) \cap \beta,$$

and which are homotopic keeping the end points fixed. Then they are ambient isotopic by a B-isotopy.

Let $A=S^1 \times I$ and $H^2(A)=\{h \in H(A) \mid h=e$ on $Bd(A)\}$. H. Gluck [4] defined the winding number for a homeomorphism $h \in H^2(A)$ as follows. Let η be the isomorphism of $\pi_1(S^1, 0)$ with $\mathbf{Z}$ which takes the class of the path $f(t)=t$ onto 1. Let α be any path in $S^1 \times I$ from (0, 0) to (0, 1) and P_1: $S^1 \times I \rightarrow S^1$ the natural projection. Then $P_1(\alpha)$ is a closed path in S^1 based at 0. Hence $[P_1(\alpha)]$ is an element of $\pi_1(S^1, 0)$ and $\eta([P_1(\alpha)])=\omega(\alpha)$ is an integer. The integer $\omega(h\alpha)-\omega(\alpha)$ is independent of the path α for any $h \in H^2(A)$.

DEFINITION 1.5. Let h be a homeomorphism in $H^2(A)$ and α a path in A from (0, 0) to (0, 1). Then the integer $W[h;A]=\omega(h\alpha)-\omega(\alpha)$ is called the winding number of h on A.

We note that W defines a homomorphism W: $H^2(A)\rightarrow \mathbf{Z}$. But it is shown that the kernel of W is the arc-component of the identity $H_o^2(A)$ and thus W is in fact an isomorphism of H^2(A) onto **Z**[4].

DEFINITION 1.6. Let A_i be an annulus in $Int(M_n)$ around the boundary curve C_i. Then there is a homeomorphism h of the annulus A_i onto itself such that $W[h;\ A_i]=1$ and $h=e$ on $Bd(A_i)$. This homeomorphism can be extended to M_n by the identity on M_n-A_i. We call the extended homeomorphism an A_i-homeomorphism and denote it by h_{A_i}.

3. Isotopy groups and automorphisms of fundamental group

THEOREM 3.1. Let h be a homeomorphism in $H^n(M_n)$. Then h induces the identity automorphism $h_{\#}=e_{\#}$ of the homotopy group $\pi_1(M_n,\ x_o)$ if and only if h is B-isotopic to a product of the homeomorphisms h_{A_i} $(1\leq i\leq n-1)$.

PROOF: For the n$-$1 generators of $\pi_1(M_n,\ x_o)$, we take the closed paths γ_i $(1\leq i\leq n-1)$ which are obtained by tracing the arcs $\overline{x_o\, a_i}$, $\overline{a_i\, a_i'}$, $\overline{a_i'\, b_i'}$, $\overline{b_i'\, b_i}$ and $\overline{b_i\, x_o}$ with $\gamma_i\cap\gamma_j\cap\ [Int(M_n)]=\phi$ for $i\neq j$ as in **Figure I.**

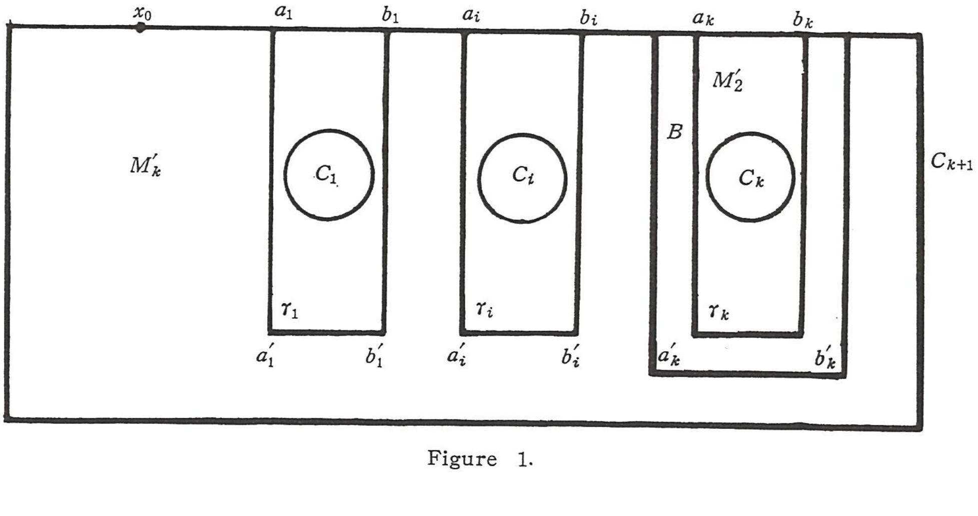

Figure 1.

Now assuming that h *is* B–isotopic to a product of $h_{A_i}(1 \leq i \leq n-1)$, we show that $h_\# = e_\# \in \mathscr{A}[\pi_1(M_n, x_o)]$. It is sufficient to consider the product of the homeomorphisms h_{A_i}. Define the annuli A_i around the corresponding boundary holes C_i for $1 \leq i \leq n-1$ so small that the annuli A_i do not meet any of the generators γ_i and $Bd(M_n)$. Then it is clear that $h_{A_i}(\gamma_j) = \gamma_j$ for $1 \leq i, j \leq n-1$ and thus the product induces the identity automorphism. Thus $h_\# = e_\#$ in $\mathscr{A}[\pi_1(M_n, x_o)]$.

Conversely, letting h be a homeomorphism in $H^n(M_n)$ such that $h_\# = e_\#$, we show by an induction on n that h is B–isotopic to a product of $h_{A_i}(1 \leq i \leq n-1)$. We first note that the theorem follows trivially for the case n=2. Now assuming that our theorem is true for $n=k$, we prove it for $n=k+1$ where the base point x_o is assumed to be on the boundary curve C_{k+1} as in Figure I. By our assumption we have $h(\gamma_i) \simeq \gamma_i$ keeping the parts $\overline{x_o a_i}$ and $\overline{b_i x_o}$ held fixed for $1 \leq i \leq k$. Denote $\gamma_k' = \overline{a_k a_k'} \cup \overline{a_k' b_k'} \cup \overline{b_k' b_k}$ and $\gamma_k'' = \gamma_k' \cup \overline{b_k a_k}$. Then $h(\gamma_k') \simeq \gamma_k'$ keeping the end points a_k and b_k held fixed and by Lemma 1.4 these two arcs are ambient isotopic by a B–isotopy, so there is an isotopy G_t: $M_{k+1} \rightarrow M_{k+1}$, $0 \leq t \leq 1$, such that $G_0 = e$ on M_{k+1} and $G_1^{-1} h = e$ on γ_k. Now let M_2' be the closed annulus defined by C_k and γ_k'', and $M_k' = (M_{k+1} - M_2') \cup \gamma_k'$. Then $G_1^{-1} h \mid M_k'$ is a homeomorphism of M_k' such that $G_1^{-1} h \mid M_k' \in H^k(M_k')$. Now observe that $(G_1^{-1} h)_\#$ is the identity automorphism of $\pi_1(M_k', x_o)$. We first note that $G_1^{-1} h(\gamma_i) \subset M_k'$ for $1 \leq i \leq k-1$, since each $\gamma_i \subset M_k'$ and $G_1^{-1} h \mid M_k' \in H^k(M_k')$. Let B be a narrow band around the arc γ_k' such that $B \cap [\gamma_i \cup G_1^{-1} h(\gamma_i)] = \emptyset$ for $1 \leq i \leq k-1$. Let g be a homeomorphism from $M_{k+1} \cup [Int\,(C_k)]$ onto M_k' such that $g(M_2' \cup B \cup [Int\,(C_k)]) = B$ and $g = e$ on $M_{k+1} - (M_2' \cup B) = M_k - B$. Then since $\gamma_i \simeq G_1^{-1} h(\gamma_i)$ on $M_{k+1} \cup [Int\,(C_k)]$ for $1 \leq i \leq k-1$, we have $g(\gamma_i) \simeq g G_1^{-1} h(\gamma_i)$ on $g(M_{k+1} \cup [Int\,(C_k)]) = M_k'$. But since $g = e$ on $M_k' - B$, $\gamma_i \simeq G_1^{-1} h(\gamma_i)$ on M_k' for $1 \leq i \leq k-1$ and hence $(G_1^{-1} h)_\#$is the identity automorphism of $\pi_1(M_k', x_o)$. Thus by our assumption $G_1^{-1} h \mid M_k'$ is B–isotopic to a product of h_{A_i} $(1 \leq i \leq k-1)$ on M_k'. On the other hand, $G_1^{-1} h \mid M_2'$ is also a homeomorphism of M_2' such that $G_1^{-1} h \mid M_2' \in H^2(M_2')$. Thus $G_1^{-1} h \mid M_2'$ is *B*–isotopic to a homeomorphism supported on an annulus A_k around the boundary hole C_k such that $A_k \subset Int\,(M_2')$. Hence $G_1^{-1} h$ is a homeomorphism of $M_{k+1} = M_k' \cup M_2'$ which is *B*–isotopic to a product of $h_{A_i}(1 \leq i \leq k)$ on M_{k+1}. But since $G_1^{-1} h$ is *B*–isotopic to h by Lemma 1.3, h is also *B*–isotopic to a product of $h_{A_i}(1 \leq i \leq k)$. Thus the theorem is proved for any integer $n \geq 1$.

COROLLARY **3.2.** The kernel of the homomorphism ϕ is $\mathbf{Z}^{n-1}$, if the domain of ϕ is taken to be $\pi_o[H^n(M_n)]$.

PROOF: By Theorem 3.1 we know that only the homeomorphisms h, which are B-isotopic to the products of the homeomorphisms $h_{A_i}(1 \leq i \leq n-1)$, induce the identity automorphism of $\pi_1(M_n, x_o)$. But each h_{A_i} generates the isotopy classes $\mathbf{Z}$ classified by the winding numbed $W[h; C_i]$ for $1 \leq i \leq n-1$, and thus the kernel of the homomorphism ϕ is $\mathbf{Z}^{n-1}$.

COROLLARY 3.3. Let h be a homeomorphism in $H^n(M_n)$. Then h is isotopic to the identity in $H^1_{n-1}(M_n)$ if and only if h is B-isotopic to a product of the homeomorphisms $h_{A_i}(1 \leq i \leq n-1)$.

PROOF: Assume that h is B-isotopic to a product of the homeomorphisms h_{A_i} $(1 \leq i \leq n-1)$. Each h_{A_i} in the product can be defor med to the identity by rotating the corresponding boundary curve C_i through 0 to $2(-m_i)\pi$ where $m_i = W[h; C_i]$ for $1 \leq i \leq n-1$. Thus h is isotopic to the identity in $H^1_{n-1}(M_n)$.

Conversely, since h is isotopic to the identity, an arc from h to the identity in $H^1_{n-1}(M_n)$ provides a homotopy of $h(\gamma)$ with γ for any loop γ in $[\gamma] \in \pi_1(M_n, x_o)$. Thus we have $h_{\#} = e_{\#}$ in $\mathscr{A}[\pi_1(M_n, x_o)]$ and the corollary is proved by Theorem 3.1.

LEMMA 3.4. The homomorphism $\emptyset$ is an isomorphism of $\pi_0[H^1_{n-1}(M_n)]$ into $\mathscr{A}[\pi_1(M_n, x_o)]$.

PROOF: We need to show that the kernel of ϕ is the isotopy class $[e]$ in $\pi_0[H^1_{n-1}(M_n)]$. By Theorem 3.1 it is clear that the kernel of ϕ is the collection of the classes $[h]$ of the homeomorphisms h which are isotopic to the products of the homeomorphisms $h_{A_i}(1 \leq i \leq n-1)$ by the istopy paths in $H^1_{n-1}(M_n)$. But by Corollary 3.3, any of such products can be deformed to the identity and hence every h such that $[h] \in \ker \phi$ is isotopic to the identity in $H^1_{n-1}(M_n)$. Thus ker $\phi = \{[e]\} \subset \pi_0[H^1_{n-1}(M_n)]$ and the homomorphism ϕ is a monomorphism, which implies that ϕ is an isomorphism onto $\phi(\pi_0[H^1_{n-1}(M_n)])$.

THEOREM 3.5. For the two different domains $\pi_0[H^n(M_n)]$ and $\pi_0[H^1_{n-1}(M_n)]$, the homomorphism ϕ induces an isomorphism; $\phi(\pi_0[H^n(M_n)]) \cong \phi(\pi_0[H^1_{n-1}(M_n)])$ in $\mathscr{A}[\pi_1(M_n, x_o)]$.

PROOF: We observe that

$$\pi_0[H^n(M_n)]/\pi_0[K] \cong \pi_0[H^1_{n-1}(M_n)] \qquad \text{(A)}$$

where K is the collection of the homeomorphisms in $H^n(M_n)$ which are B-isotopic to the products of the homeomorphisms $h_{A_i}(1 \leq i \leq n-1)$. Define a homomorphism

$$\psi: \pi_0[H^n(M_n)] \rightarrow \pi_0[H^1_{n-1}(M_n)]$$

by $\psi([h]^*)=[h]_*$, where $[h]^*$ and $[h]_*$ are the isotopy classes of the homeomorphism h in $\pi_0[H^n(M_n)]$ and $\pi_0[H^1_{n-1}(M_n)]$ respectively. Then ψ is an epimorphism and *ker* $\psi=\pi_0[K]$ by Corollary 3.3. Thus the relation (A) follows by the isomorphism induced by ψ. But by Theorem 3.1, only the homeomorphisms h in K induce the identity automorphism of the homotopy group $\pi_1(M_n, x_o)$, and thus

$$\pi_0[H^n(M_n)]/\pi_0[K] \cong \phi(\pi_0[H^n(M_n)]) \qquad \text{(B)}$$

On the other hand, by Lemma 3.4 we have

$$\pi_0[H^1_{n-1}(M_n)] \cong \phi(\pi_0[H^1_{n-1}(M_n)]) \qquad \text{(C)}$$

Combining the expressions (A), (B) and (C), the theorem is established.

LEMMA **3.6.** Let h be a homeomorphism in $H^n(M_n)$. Then h is isotopic to the identity in $H^+_n(M_n)$ if and only if h is B-isotopic to a product of the homeomorphisms $h_{A_i}(1 \leq i \leq n)$.

PROOF: It is easy to see that a product of the homeomorphisms $h_{A_i}(1 \leq i \leq n)$ is isotopic to the identity in $H^+_n(M_n)$, since each h_{A_i} can be deformed to the identity by rotating the corresponding boundary curve C_i through 0 to $2(-m_i)\pi$ where $m_i = W[h; C_i]$.

To prove the converse, let h be a product of A_i-homeomorphisms of the form

$$h=(h^{l_1}_{A_1} h^{l_2}_{A_2} \ldots h^{l_{n-1}}_{A_{n-1}} h^{l_n}_{A_n}).\ g,$$

where g is a homeomorphism which cannot be factored in such a way as to contain only the homeomorphisms h_{A_i} to some powers. Since each h_{A_i} can be deformed to the identity, we have

$$h=(h^{l_1}_{A_1} h^{l_2}_{A_2} \ldots h^{l_{n-1}}_{A_{n-1}} h^{l_n}_{A_n}).\ g \approx g$$

in $H^+_n(M_n)$. Thus it is enough to consider only the homeomorphism g. Now assume that $g \approx e$ in $H^+_n(M_n)$. Then the isotopy G_t, $0 \leq t \leq 1$, between g and e in $H^+_n(M_n)$ produces a rotation of the boundary curves, since G_t must keep each of the boundary curves held fixed setwise. Now we construct an isotopy H_t which agrees with G_t on C_n and is the identity outside the annulus A_n for $0 \leq t \leq 1$, where

$H_0=H_1=e$. Then $H_t^{-1}G_t$, $0\leq t\leq 1$, is an isotopy in $H^1_{n-1}(M_n)$ between $H_0^{-1}G_0=e$ and $H_1^{-1}G_1=g$. Thus the fact $g\approx e$ in $H^+_n(M_n)$ would imply that $g\approx e$ in $H^1_{n-1}(M_n)$. Corollary 3.3 then implies that g must be a product of the homeomorphism $h_{A_i}(1\leq i\leq n-1)$. This is a contradiction to our assumption and g (and thus h) is not isotopic to the identity in $H^+_n(M_n)$. Hence we can see that only the products of $h_{A_i}(1\leq i\leq n)$ are isotopic to the identity in $H^+_n(M_n)$, and the proof is complete.

THEOREM **3.7.** $\pi_0[H^t_{n-t}(M_n)]\cong\pi_0[H^+_n(M_n)]$ x $\mathbf{Z}^t$, where $0\leq t\leq n$.

PROOF: Define a homomorphism

$$\psi\colon\ \pi_0[H^t_{n-t}(M_n)]\rightarrow\pi_0[H^+_n(M_n)]$$

by $\psi([h]^*)=[h]_*$, where $[h]^*$ and $[h]_*$ are the isotopy classes of the homeomorphism h in $\pi_0[H^t_{n-t}(M_n)]$ and $\pi_0[H^+_n(M_n)]$ respectively. Then ψ is an epimorphism and the kernel of ψ is the collection of the classes $[h]$ of the homeomorphisms h isotopic to the products of h_{A_i} $(1\leq i\leq n)$ by the isotopy paths in $H^t_{n-t}(M_n)$, since $H^t_{n-t}(M_n)\subset H^+_n$ (M_n) and such products are isotopic to the identity in $H^+_n(M_n)\subset H(M_n)$ by Lemma 3.6, We note that the isotopy classes of the kernel of ψ in $H^t_{n-t}(M_n)$ are $\mathbf{Z}^t$ classified by the winding numbers $\{W[h;C_i]\mid[h]\ \varepsilon\ \ker\psi\}$ for each of the t boundary holes C_i. Thus $\pi_0[H^t_{n-t}(M_n)]/\mathbf{Z}^t\cong\pi_0[H^+_n(M_n)]$.

Now define a normal subgroup N of the isotopy group $\pi_0[H^t_{n-t}(M_n)]$, $N=\{[h]\ \varepsilon\ \pi_0[H^t_{n-t}(M_n)]\mid W[h;\ C_i]=0$ for the t holes $C_i\}$. We note that every isotopy class $[h]$ in $\pi_0[H^+_n(M_n)]$ contains a homeomorphism $h_o\ \varepsilon H^t_{n-t}(M_n)$ such that $W[h_o;C_i]=0$ for each of the t boundary holes C_i, since the isotopy in $H(M_n)$ is allowed to rotate each of the boundary curves. But $[h_o]$ belongs to the normal subgroup N and thus the restricted homomorphism $\psi'=\psi\mid N$ is an epimorphism. Further we can see that ψ' is a monomorphism. By the above arguments, the kernel of ψ' is the collection of the classes $[h]$ in N where the homeomorphism h are isotopic to the products of $h_{A_i}(1\leq i\leq n)$ by the isotopy paths in $H^t_{n-t}(M_n)$. But every h such that $[h]\ \varepsilon\ \ker\psi'$ does not contain any h_{A_i} in its isotopic product since $W[h;C_i]=0$, where the annuli A_i are determined around each of the t boundary holes, and thus it is isotopic to the identity in $H^t_{n-t}(M_n)$. Thus $\ker\psi'=[e]$ in N and ψ induces an isomorphism ψ' of the normal subgroup N onto $\pi_0[H^+_n(M_n)]$. Hence the proof is complete.

THEOREM **3.8.** $\pi_0[H^n(M_n)]\cong \mathit{ker}\ \phi\times\phi(\pi_0[H^1_{n-1}(M_n)])$, where the kernel of ϕ is defined on the domain $\pi_0[H^n(M_n)]$.

PROOF: $\pi_0[H^n(M_n)] \cong \mathbf{Z}^n \times \pi_0[H_n^+(M_n)]$ (by Theorem 3.7)

$\cong \mathbf{Z}^{n-1} \times \{\mathbf{Z} \times \pi_0[H_n^+(M_n)]\}$

$\cong \mathbf{Z}^{n-1} \times \{\pi_0[H_{n-1}^1(M_n)]\}$ (by Theorem 3.7)

$\cong \mathbf{Z}^{n-1} \times \phi(\pi_0[H_{n-1}^1(M_n)])$ (by Lemma 3.4)

$\cong ker\ \phi \times \phi(\pi_0[H_{n-1}^1(M_n)])$ (by Lemma (3.2))

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