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AN INVARIANT FORM ON A COMPLEX MANIFOLD

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1. Introduction

Let D be a bounded domain in C^n. S. Bergman introduced the kernel function and Bergman metric of D which is invariant under the holomorphic transformations of D ([1], [2]).

As the generalization of these concepts, S. Kobayashi considered the kernel form and the Bergman metric of a complex manifold with certain conditions ([5]).

Let M be an n-dimensional complex manifold with certain conditions. In this paper, using the kernel form, we introduce a positive semidefinite quadratic form invariant under the holomorphic transformations of M. In particular, our form becomes the Bergman metric in the special case.

2. The kernel form and the Bergman metric.

Let M be a complex manifold of dimension n.

Let $F(M)$ be the set of holomorphic n-forms $f=udz_1\wedge\cdots\wedge dz_n$ on M such that

$$i^{n^2}\int_M f\wedge\bar{f}<\infty.$$

Then $F(M)$ is a separable complex Hilbert space with an inner product given by

$$(f, g)=i^{n^2}\int_M f\wedge\bar{g} \qquad (f, g\in F(M)).$$

Let $\{h_0, h_1, h_2, \cdots\}$ be an orthonormal basis for $F(M)$. We define a holomorphic $2n$-form on $M\times\overline{M}$, where $\overline{M}$ is the complex manifold conjugate to M, by

$$K(z,\bar{t})=i^{n^2}\sum_{j=0}^{\infty}h_j(z)\,\overline{h_j(t)},$$

where $h_j(z)=\phi_j(z)dz_1\wedge\cdots\wedge dz_n$, and $\phi_j(z)$ is a holomorphic function in a coordinate neighborhood.

It is known that $K(z,\bar{t})$ is independent of choice of orthonormal basis for $F(M)$.

$K(z,\bar{t})$ is called Bergman kernel form and written as

$$K(z,\bar{t})=i^{n^2}\, k(z,\bar{t})\; dz_1\wedge\cdots\wedge dz_n\wedge d\bar{z}_1\wedge\cdots\wedge d\bar{z}_n$$

with a locally defined Bergman kernel funtion

$$k(z,\bar{t}) = \sum_{j=0}^{\infty} \phi_j(z)\,\overline{\phi_j(t)}.$$

In particular, $K(z,\bar{z})$ can be considered as a $2n$-form on M.

PROPOSITION 1. *The form $K(z,\bar{z})$ is invariant under the group of holomorphic transformations of M* ([5]).

Let M be an n-dimensional complex manifold. We assume that M satisfies:

(A. 1) *For any $z\in M$, there is an $f\in F(M)$ such that $f(z)=0$. In other words, the kernel form $K(z,\bar{z})$ of M is different from zero at every point of M.*

Let $z_1, \cdots, z_n$ be a local coordinate system in M.

Let $K(z,\bar{z})=i^{n^2}\, k(z,\bar{z})\; dz_1\wedge\cdots\wedge dz_n\wedge d\bar{z}_1\wedge\cdots\wedge d\bar{z}_n$.

Then the kernel function $k(z,\bar{z})$ is positive. So, we define quadratic differential form ds^2 by

$$ds^2=\sum_{\alpha,\beta=1}^{n} \frac{\partial^2 \log k(z,\bar{z})}{\partial z_\alpha \partial \bar{z}_\beta}\, dz_\alpha d\bar{z}_\beta.$$

It is easy to see that ds^2 is independent of choice of local coordinate system.

Now, define $n\times n$ Hermitian matrix $T(z,\bar{z})$ by

$$T(z,\bar{z}) = \frac{\partial^2 \log k(z,\bar{z})}{\partial z^*\, \partial z}.$$

Then

$ds^2=dz^*\, T(z,\bar{z})\, dz$ holds, where $dz=(dz_1, \cdots, dz_n)'$

$$\frac{\partial}{\partial z}=\left(\frac{\partial}{\partial z_1}, \cdots, \frac{\partial}{\partial z_n}\right).$$

Here, a vector or a matrix marked with the symbol $'$ or $*$ is denoted the transposed and the transposed conjugate vector or matrix, respectively and $\frac{\partial w}{\partial z} = \frac{\partial}{\partial z} \times w$, where the sign $\times$ designates the kronecker product.

PROPOSITION 2. *The quadratic form ds^2 is positive semidefinite and invariant under the holomorphic transformations of M* ([5]).

It has been known that ds^2 is positive definite if and only if the following assumption is satisfied:

(A. 2) *For every holomorphic tangent vector ξ at $z \in M$, there exists an $f \in F(M)$ such that $f(z)=0$ and*

$$du \cdot \xi = \sum_{\mu=1}^{n} \frac{\partial u}{\partial z_\mu}(z)\xi_\mu \neq 0, \quad \textit{where } f=u dz_1 \wedge \cdots \wedge dz_n.$$

When ds^2 is positive definite, it is called the Bergman metric of M.

We can see easily that Bergman metric is Kähler metric. Namely, any complex manifold with properties (A. 1) and (A. 2) is entitled to an invariant Kähler metric ds^2 of Bergman. Further we have the following

PROPOSITION 3. *Let M be an n-dimensional complex manifold with properties* (A. 1) *and* (A. 2). *Then the matrix $(n+1)(g_{\bar{\alpha}\beta})-(R_{\bar{\alpha}\beta})$ $(\alpha, \beta=1, 2, \cdots, n)$ is a positive semidefinite, where $g_{\bar{\alpha}\beta}$ and $R_{\bar{\alpha}\beta}$ are covariant metric tensor and Ricci curvature tensor for M, respectively. In particular, if M is a bounded domain in C^n, then the matrix $(n+1)(g_{\bar{\alpha}\beta})-(R_{\bar{\alpha}\beta})$ is positive definite* ([4], [5]).

3. Invariant form

Let M be an n-dimensional complex manifold with properties (A. 1) and (A. 2).

Let $z_1, \cdots, z_n$ be a local coordinate system at $z \in M$, and let $K(z, \bar{z})$ be the Bergman kernel form of M.

Let $K(z,\bar{z})=i^{n^2} k(z,\bar{z})\, dz_1\wedge\cdots\wedge dz_n \wedge d\bar{z}_1 \wedge\cdots\wedge d\bar{z}_n$, where $k(z, \bar{z})$ is a locally defined kernel function.

We set

$$T(z,\bar{z}) = \frac{\partial^2 \log k(z,\bar{z})}{\partial z^* \partial z}.$$

Now, we shall define $k_m(z,\bar{z})$, $T_m(z,\bar{z})$ $(m\geqq 1)$ as follows, respectively:

$$T_m(z,\bar{z}) = \frac{\partial^2}{\partial z^* \partial z} \log k_m(z,\bar{z}), \qquad (m\geqq 1)$$

$$k_m(z,\bar{z}) = \det\,(k^m(z,\bar{z})\; T(z,\bar{z})) \qquad (m\geqq 2)$$

$$k_1(z,\bar{z}) = k(z,\bar{z}).$$

Then we obtain the following

THEOREM. *Let M be an n-dimensional complex manifold with properties (A. 1) and (A. 2). Then under the above notations, the quadratic form $ds_m^2 = dz^*\, T_m(z,\bar{z})dz$ is positive semidefinite and invariant under the holomorphic transformations of M.*

PROOF. The $n\times n$ Hermitian matrix $T(z,\bar{z})$ $(\equiv T_1(z,\bar{z}))$ and $T_m(z,\bar{z})$ may be calculated as follows, respectively:

$$(1)\quad T_m(z,\bar{z}) = k_m^{-2}(z,\bar{z})\{k_m(z,\bar{z}) \frac{\partial^2 k_m(z,\bar{z})}{\partial z^* \partial z} - \frac{\partial}{\partial z^*} k_m(z,\bar{z}) \frac{\partial}{\partial z} k_m(z,\bar{z})\} \qquad (m\geqq 1).$$

$$(2)\quad T_m(z,\bar{z}) = mn\; T(z,\bar{z}) + \frac{\partial^2}{\partial z^* \partial z} \log \det T(z,\bar{z}) \qquad (m\geqq 2).$$

Since $T(z,\bar{z})$ is invariant from Proposition 2, $T_m(z,\bar{z})$ is also invariant from (2).

From Proposition 3,

$$(3)\quad (n+1)\,(g_{\bar{\alpha}\beta}) - (R_{\bar{\alpha}\beta}) \qquad (\alpha,\beta = 1, 2, \cdots, n) \text{ is positive semidefinite.}$$

In our case, we can take $T(z,\bar{z})$ as $(g_{\bar{\alpha}\beta})$ and $-\dfrac{\partial^2 \log\det T(z,\bar{z})}{\partial z^* \partial z}$ as $(R_{\bar{\alpha}\beta})$.

Then (3) becomes as follows:

$$(4)\quad v^*[(n+1)\; T(z,\bar{z}) + \frac{\partial^2}{\partial z^* \partial z} \log\det T(z,\bar{z})]\; v \geqq 0, \text{ where } v \text{ is}$$

an arbitrary n-tuple nonzero constant vector.

Making use of the relation (4), from (2), we have

$$v^* T_m(z,\bar{z})\ v = v^*[mnT(z,\bar{z}) + \frac{\partial^2}{\partial z^* \partial z} \log \det T(z,\bar{z})]\ v$$

$$\geqq v^*[(n+1)\ T(z,\bar{z}) + \frac{\partial^2}{\partial z^* \partial z} \log \det T(z,\bar{z})]\ v \geqq 0.$$

Namely, $T_m(z,\bar{z})$ is a positive semidefinite Hermitian matrix.

Q. E. D.

When $ds_m^2(\equiv dz^*\ T_m(z,\bar{z})\ dz)$ is positive definite, we call ds_m^2 m-th Bergman metric.

REMARK. If M is a bounded domain in C^n, then ds_m^2 is positive definite invariant metric under the holomorphic transformations of M. Therefore, if M is a bounded domain in C^n, ds_m^2 is m-th Bergman metric. In particular, for $m=1$ we have the Bergman metric.

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