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REMARK ON MEASURABILITY FOR FLOWS

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1. Introduction.

Let $(\Omega, \mathscr{F}, \mu)$ be a complete measure space with $\mu(\Omega)<\infty$. Consider a flow $\{T_t\}$ defined on $(\Omega, \mathscr{F}, \mu)$, i. e., a one-parameter group of measure preserving transformations T_t of Ω onto itself with the real parameter t-set R. A flow $\{T_t\}$ is said to be measurable if the (ω, t)-set defined by $\{(\omega, t) \in \Omega\times R;\ T_t\omega \in A\}$ is a $(\mu\times\lambda)$-measurable set for every $A \in \mathscr{F}$. Where $\mu\times\lambda$ is the product measure of μ with the ordinary Lebesgue measure λ on the real line R. We note that the σ-algebra of all $(\mu\times\lambda)$-measurable sets is the completion of the product σ-algebra $\mathscr{F}\times\mathscr{B}$ for the product measure $\mu\times\lambda$, where $\mathscr{B}$ is the σ-algebra of all Borel sets of the real line R.

The main assertion of this note is to remark the following property for a measurable flow. That is, for every measurable flow $\{T_t\}$ on a complete finite measure space $(\Omega, \mathscr{F}, \mu)$ there exists a σ-subalgebra $\mathscr{F}_0$ of $\mathscr{F}$ which satisfies that the mapping $(\omega, t)\longrightarrow T_t\omega$ is a measurable mapping of the measurable space $(\Omega\times R, \mathscr{F}_0\times\mathscr{B})$ onto the measurable space $(\Omega, \mathscr{F}_0)$ and that for any set $A\in\mathscr{F}$ there exists a set $A_0 \in \mathscr{F}_0$ such that $\mu(A\ominus A_0)=0$. Where $A\ominus B$ denotes the symmetric difference of A and B.

We shall deduce this property mainly from the representation theorem for a measurable proper flow in [1] and [2].

As an example of the consequence of this property, we shall show that the set $\bigcup_{t\in I} T_tA$ is universally measurable for every set $A \in \mathscr{F}_0$ and for every interval I of the real line R.

2. Definitions and notation.

DEFINITION 1. A *measurable space* $(X, \mathscr{A})$ is a system of a set X and a σ-algebra $\mathscr{A}$ of subsets of X. An *$\mathscr{A}$-measurable set* is a set in $\mathscr{A}$. A *measure* $\mu(A)$ on $(X, \mathscr{A})$ is a countably additive and non-negative set function defined for every set A of $\mathscr{A}$. $(X, \mathscr{A}, \mu)$ is called a *measure space*. A *finite measure space* is a space with $\mu(X)<\infty$. $(X, \mathscr{A}, \mu)$ is *σ-finite* if X is a countable union of $\mathscr{A}$-measurable sets of finite measure. $(X, \mathscr{A}, \mu)$ is *completed* or $\mathscr{A}$ is *completed for μ* if whenever A is in $\mathscr{A}$ and $\mu(A)=0$ then every subset of A is also in $\mathscr{A}$. We recall that if $(X, \mathscr{A}, \mu)$ is a finite or σ-finite measure space, the σ-algebra $\mathscr{A}^*$ of all μ-measurable subsets of X is completed for μ. We shall say that $\mathscr{A}^*$ is the *completion of $\mathscr{A}$ for μ.*

Throughout this note $(R, \mathscr{B}, \lambda)$ will denote the measure space of the real line R, the Borel σ-algebra $\mathscr{B}$ determined by all open intervals of R and the ordinary Lebesgue measure λ on R.

Let $\{(X_i, \mathscr{A}_i, \mu_i);\ i=1,2,\cdots,n\}$ be a family of finite or σ-finite measure spaces. $(X_1\times\cdots\times X_n, \mathscr{A}_1\times\cdots\times\mathscr{A}_n, \mu_1\times\cdots\times\mu_n)$ will denote the product measure space. $(\mathscr{A}_1\times\cdots\times\mathscr{A}_n)^*$ will denote the completion of $\mathscr{A}_1\times\cdots\times\mathscr{A}_n$ for $\mu_1\times\cdots\times\mu_n$.

DEFINITION 2. Let $(\Omega, \mathscr{F})$ be a measurable space and let A be any subset of Ω. A *trace σ-algebra of $\mathscr{F}$ on A, $\mathscr{F}\cap A$,* is the class $\{F\cap A;\ F\in\mathscr{F}\}$.

It is evident that if a set A is $\mathscr{F}$-measurable and if μ is a measure on $(\Omega, \mathscr{F})$, then the restriction of μ to $\mathscr{F}\cap A$ is also a measure on $(A, \mathscr{A}\cap A)$.

DEFINITION 3. Let $(X_1, \mathscr{A}_1)$ and $(X_2, \mathscr{A}_2)$ be two measurable spaces. A mapping, S, of X_1 into X_2 is said to be *$(\mathscr{A}_1/\mathscr{A}_2)$-measurable* if $S^{-1}\mathscr{A}_2\subset\mathscr{A}_1$.

DEFINITION 4. Let $(X_1, \mathscr{A}_1, \mu_1)$ and $(X_2, \mathscr{A}_2, \mu_2)$ be two measure spaces. A *measure preserving transformation*, S, is a 1:1 mapping of X_1 onto X_2 with the property that $S^{-1}\mathscr{A}_2\subset\mathscr{A}_1$, $S\mathscr{A}_1\subset\mathscr{A}_2$ and $\mu_1(A_1)=\mu_2(SA_1)$ for every $A_1\in\mathscr{A}_1$.

We shall define a flow according to [2] and [4].

DEFINITION 5. A *flow* is a one-parameter family, $\{T_t\}$, of measure preserving transformations of a complete finite measure space $(\Omega, \mathscr{F}, \mu)$ onto itself, which has the group property: $T_{t+s}=T_t\cdot T_s$ for all t and s in R.

In this note, when we say "a flow on $(\Omega, \mathscr{F}, \mu)$", we assume that $(\Omega, \mathscr{F}, \mu)$ is a complete finite measure space.

DEFINITION 6. Let $\{T_t\}$ be a flow on $(\Omega, \mathscr{F}, \mu)$. The set $A\subset\Omega$ is *invariant*, if $A\in\mathscr{F}$ and $T_tA\subset A$ for any $t\in R$.

If A is an invariant set of positive measure, the restriction of $\{T_t\}$ to $(A, \mathscr{F}\cap A, \mu)$ is also a flow.

DEFINITION 7. Let $\{T_t\}$ be a flow on $(\Omega_1, \mathscr{F}_1, \mu_1)$ and $\{S_t\}$ be a flow on $(\Omega_2, \mathscr{F}_2, \mu_2)$. $\{T_t\}$ is *isomrphic* to $\{S_t\}$ if, for $i=1, 2$, it is possible to split Ω_i into two disjoint subsets Ω_i' and Ω_i'' in such a way that:

1) Ω_i' and Ω_i'' are in $\mathscr{F}_i$ and $\Omega_i = \Omega_i' \cup \Omega_i''$, $i=1, 2$.
2) Ω_i' and Ω_i'' are invariant, $i=1, 2$.
3) $\mu_i(\Omega_i'')=0$, $i=1, 2$.
4) There exists a measure preserving transformation, V, of $(\Omega_1', \mathscr{F}_1 \cap \Omega_1', \mu_1)$ onto $(\Omega_2', \mathscr{F}_2 \cap \Omega_2', \mu_2)$ such that $VT_t = S_t V$ for all $t \in R$.

DEFINITION 8. Let $\{T_t\}$ be a flow on $(\Omega, \mathscr{F}, \mu)$ and S be a mapping of $\Omega \times R$ onto Ω defined by $S(\omega, t) = T_t \omega$ for every (ω, t) in $\Omega \times R$. The flow $\{T_t\}$ is *measurable* if S is $((\mathscr{F} \times \mathscr{B})^* / \mathscr{F})$-measurable.

In [9], if a flow is measurable then it is said to be *L-measurable*, and if the mapping S is $(\mathscr{F} \times \mathscr{B} / \mathscr{F})$-measurable then $\{T_t\}$ is said to be *B-measurable*. It is difficult to find examples, except trivial ones, of flows which are B-measurable, when we emphasize that $(\Omega, \mathscr{F}, \mu)$ is completed for μ.

However we shall show, in what follows, that for every measurable flow $\{T_t\}$ on $(\Omega, \mathscr{F}, \mu)$ there exists a σ-subalgebra $\mathscr{F}_0$ of $\mathscr{F}$ with the properties that the mapping S is $(\mathscr{F}_0 \times \mathscr{B} / \mathscr{F}_0)$-measurable and that for any set $A \in \mathscr{F}$ there exists a set $A_0 \in \mathscr{F}_0$ such that $\mu(A \ominus A_0) = 0$.

DEFINITION 9. ([2]) A flow $\{T_t\}$ on $(\Omega, \mathscr{F}, \mu)$ is *proper* if every $\mathscr{F}$-measurable set of positive measure contains an $\mathscr{F}$-measurble set A such that $\mu((\Omega \backslash A) \cap T_{t_0} A) > 0$ for some $t_0 \in R$; it is *completely improper* if $\mu(A \ominus T_t A) = 0$ for every set $A \in \mathscr{F}$ and for every $t \in R$.

DEFINITION 10. ([1]) Let $(M, \mathfrak{M}, m)$ be a complete finite measure space, T be a measure preserving transformation of M onto itself and $f(P)$ be a real valued m-integrable function with $f(P) > c > 0$ for some constant c and for all $P \in M$. Let Ω be the set of points (P, u) for which $0 \leqq u < f(P)$, and let $\mathscr{F}$ be the σ-algebra $(\mathfrak{M} \times \mathscr{B})^* \cap \Omega$ and $\mathscr{F}_0$ be the σ-algebra $(\mathfrak{M} \times \mathscr{B}) \cap \Omega$. Then, Ω is $(\mathfrak{M} \times \mathscr{B})$-measurable and $(\Omega, \mathscr{F}, m \times \lambda)$ is the completion of $(\Omega, \mathscr{F}_0, m \times \lambda)$.

Define a one-parameter familiy, $\{S_t\}$, of transformations of Ω onto itself by

$$(2.1)\quad \begin{cases} S_t(P,u)=(P,t+u) \text{ for } 0\leqq t+u<f(P), \\ S_t(P,u)=(T^nP, t+u-f(P)-\cdots-f(T^{n-1}P) \\ \qquad \text{for } 0\leqq t+u-f(P)-\cdots-f(T^{n-1}P)<f(T^nP) \text{ and } n>0, \\ S_t(P,u)=(T^{-n}P, t+u+f(T^{-1}P)+\cdots+f(T^{-n}P) \\ \qquad \text{for } 0\leqq t+u+f(T^{-1}P)+\cdots+f(T^{-n}P)<f(T^{-n}P) \text{ and } n>0; \\ \qquad \text{where } (P,u)\in\Omega \text{ and } t\in R. \end{cases}$$

Then $\{S_t\}$ is a flow on $(\Omega, \mathscr{F}, m\times\lambda)$. The proof of this fact is seen in [1], [5] and [6]. We call this flow, $\{S_t\}$, the *flow built on the measure preserving transformation T under the function $f(P)$* or briefly the *flow under the function $f(P)$*.

We denote this flow by $(f, T, M, \mathfrak{M}, m)$.

3. Measurable flows.

LEMMA 1. (E. Hopf [4]) *Let $\{T_t\}$ be a measuarble flow on $(\Omega, \mathscr{F}, \mu)$, and let A be an $\mathscr{F}$-measurable set such that $\mu(A\ominus T_t A)=0$ for every $t\in R$. Then there exists an invariant set A_0 such that $\mu(A\ominus A_0)=0$.*

THEOREM 1. (W. Ambrose and S. Kakutani [2]) *Let $\{T_t\}$ be a measurable flow on $(\Omega, \mathscr{F}, \mu)$. Then there exist disjoint sets, Ω_1 and Ω_2, such that $\{T_t\}$ is completely improper on Ω_1 and proper on Ω_2 and that $\Omega=\Omega_1\cup\Omega_2$.*

4. Completely improper frows.

THEOREM 2. *Let $\{T_t\}$ be a measurable completely improper flow on $(\Omega, \mathscr{F}, \mu)$, and let $\mathscr{F}_0$ be the σ-algebra of all invariant subsets of Ω. Then the mapping $S(\omega,t)=T_t\omega$ is $(\mathscr{F}_0\times\mathscr{B}/\mathscr{F}_0)$-measurable and for any set $A\in\mathscr{F}$ there exists a set $A_0\in\mathscr{F}_0$ such that $\mu(A\ominus A_0)=0$.*

PROOF. That the mapping S is $(\mathscr{F}_0\times\mathscr{B}/\mathscr{F}_0)$-measurable follows from $S^{-1}A=A\times R$ for $A\in\mathscr{F}_0$.

The latter part is an immediate consequence of LEMMA 1.

REMARK. The completion for μ of the class $\mathscr{F}_0$ of all invariant subsets of Ω is the class of sets of the form $A_0\cup N$, where $A_0\in\mathscr{F}_0$ and N is a subset of an invariant null set. However it may happen that $\mathscr{F}$ contains null sets each of which is not contained in any invariant null set.

5. Proper flows.

THEOREM 3. (Representation theorem for proper flows [2]) *Let $\{T_t\}$ be a measurable proper flow on $(\Omega, \mathscr{F}, \mu)$. Then Ω is devided into at most countable invariant subsets $\Omega_n (n=1, 2, \cdots)$ of positive measure in such a way that each restriction of $\{T_t\}$ to $(\Omega_n, \mathscr{F} \cap \Omega_n, \mu)$ is isomorphic to a flow under a function respectively.*

THEOREM 4. *Let $\{S_t\}$ be a flow under a function $(f, T, M, \mathfrak{M}, m)$, and let $(\Omega, \mathscr{F}, m\times\lambda)$ and $(\Omega, \mathscr{F}_0, m\times\lambda)$ be the measure spaces in* DEFINITION 10. *Then the mapping $S(P, u, t) = S_t(P, u)$ of $\Omega\times R$ onto Ω is $(\mathscr{F}_0\times\mathscr{B}/\mathscr{F}_0)$-measurable and $\mathscr{F}$ is the completion of $\mathscr{F}_0$ for $m\times\lambda$.*

PROOF. This follows from that Ω is $(\mathfrak{M}\times\mathscr{B})$-measurable, the class, $\mathscr{F}_0 = (\mathfrak{M}\times\mathscr{B})\cap\Omega$, is determined by all $(\mathfrak{M}\times\mathscr{B})$-measurable rectangles contained in Ω and that (P, u, t)-functions

$$t+u+f(T^kP)+f(T^{k+1}P)+\cdots+f(T^{k+j}P),$$
$$(k=\cdots, -1, 0, +1, \cdots;\ j=1, 2, \cdots)$$

are $(\mathfrak{M}\times\mathscr{B}\times\mathscr{B})$-measurable.

LEMMA 2. *Let $\{T_t\}$ be a measurable flow on $(\Omega_1, \mathscr{F}_1, \mu_1)$, and suppose that $\{T_t\}$ is isomorphic to a flow $\{S_t\}$ on $(\Omega_2, \mathscr{F}_2, \mu_2)$. If $\mathscr{F}_2$ contains a σ-subalgebra $\mathscr{A}_2$ with the properties that $\mathscr{F}_2$ is the completion of $\mathscr{A}_2$ for μ_2 and that the mapping $S(\omega_2, t) = S_t\omega_2$ is $(\mathscr{A}_2\times\mathscr{B}/\mathscr{A}_2)$-measurable, then there exists a σ-subalgebra $\mathscr{A}_1$ of $\mathscr{F}_1$ with the properties that $\mathscr{F}_1$ is the completion of $\mathscr{A}_1$ for μ_1 and that the mapping $T(\omega_1, t) = T_t\omega_1$ is $(\mathscr{A}_1\times\mathscr{B}/\mathscr{A}_1)$-measurable.*

PROOF. Recalling DEFINITION 7, we take the class $(V^{-1}(\mathscr{A}_2\cap\Omega_2'))\cup(\mathscr{F}_1\cap\Omega_1'')$ for $\mathscr{F}_1$. It is clear that $\mathscr{A}_1$ is the required one.

From THEOREM 3, 4 and LEMMA 2 we can deduce the following theorem.

THEOREM 5. *Let $\{T_t\}$ be a measurable proper flow on $(\Omega, \mathscr{F}, \mu)$. Then there exists a σ-subalgebra $\mathscr{F}_0$* of *$\mathscr{F}$ with the properties that the mapping $S(\omega, t) = T_t\omega$ is $(\mathscr{F}_0\times\mathscr{B}/\mathscr{F}_0)$-measurable and that $\mathscr{F}$ is the completion of $\mathscr{F}_0$ for μ.*

Moreover, from THEOREM 1, 2 and 5 it follows:

THEOREM 6. *Let $\{T_t\}$ be a measurable flow on $(\Omega, \mathscr{F}, \mu)$. Then there exists a σ-subalgebra $\mathscr{F}_0$ of $\mathscr{F}$ with the properties that the mapping $S(\omega, t) = T_t\omega$ is $(\mathscr{F}_0 \times \mathscr{B}/\mathscr{F}_0)$-measurable and that for any $A \in \mathscr{F}$ there exists a set $A_0 \in \mathscr{F}_0$ such that $\mu(A \ominus A_0) = 0$.*

More precisely, if Ω_2 is the part of maximal measure on which $\{T_t\}$ is proper, then $\mathscr{F} \cap \Omega_2$ is the completion of $\mathscr{F}_0 \cap \Omega_2$ for μ.

6. An application.

Let $\{T_t\}$ be a measurable flow on $(\Omega, \mathscr{F}, \mu)$ and let $\mathscr{F}_0$ be the σ-algebra which emerges in THEOREM 6. For any set $A \in \mathscr{F}_0$ and any interval I of R, the set $A(I) = \bigcup_{t \in I} T_t A$ is the projection of the (ω, t)-set $A^*(I) = \{(T_t\omega, t);\ \omega \in A,\ t \in I\}$ on Ω.

Because the mapping $(\omega, t) \longrightarrow (T_t\omega, t)$ is $(\mathscr{F}_0 \times \mathscr{B}/\mathscr{F}_0 \times \mathscr{B})$-measurable the set $A^*(I)$ is a set in $\mathscr{F}_0 \times \mathscr{B}$.

On one hand, the σ-algebra $\mathscr{B}$ is determined by all compact subsets of R and consequently the projection of $(\mathscr{F}_0 \times \mathscr{B})$-measurable set on Ω belongs to the Souslin (or analytic) class $(\mathscr{F}_0)_S$ generated by $\mathscr{F}_0$. From the capacity theorem it follows that $(\mathscr{F}_0)_S$ is contained in the completion $(\mathscr{F}_0)^*$ of $\mathscr{F}_0$ for μ. Clearly a set B which is measurable with respect to every finite measure on $(\Omega, \mathscr{F}_0)$ belongs to $(\mathscr{F}_0)^*$ and $(\mathscr{F}_0)^*$ is contained in $\mathscr{F}$. It is an immediate consequence from these that the set $A(I) = \bigcup_{t \in I} T_t A$ for $A \in \mathscr{F}_0$ is universally measurable, given $(\Omega, \mathscr{F}_0)$.

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