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CLASSIFICATION OF LEFT INVARIANT COMPLEX STRUCTURES ON $GL(2, \boldsymbol{R})$ AND $U(2)$

Takeshi SASAKI

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1. Introduction

By a left invariant complex structure on a real Lie group G of even dimension we mean such a complex structure on G that the left multiplication of each elememt in G is holomorphic. When G is compact, H. C. Wang ([4]) already treated them systematically. For a non-compact group, a remarkable example is given by Professor A. Morimoto. For each element $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $GL(2, \boldsymbol{R})$, we associate a pair of complex numbers $(a+\sqrt{-1}\, b,\ c+\sqrt{-1}\, d)$. Then the group $GL(2, \boldsymbol{R})$ is a complex manifold as a subset of $\boldsymbol{C}^2$ under this identification. We can easily examine that this complex structure is left invariant. A. Morimoto ([2]) also proved that every reductive real Lie group of even dimension has at least one left invariant complex structure.

In this note we will treat those complex structures on the general linear group $GL(2, \boldsymbol{R})$. Namely, we classify left invariant complex structures on $GL^+(2, \boldsymbol{R})$. Using this classification, we will prove (1) Left invarinat complex structures on $GL^+(2, \boldsymbol{R})$ are parametrized by $\boldsymbol{R}$, (2) With each of these complex structures, the group $GL^+(2, \boldsymbol{R})$ is biholomorphic to $\mathscr{H} \times \boldsymbol{C}^*$ as a complex manifold, $\mathscr{H}$ being the upper half plane, and, hence, (3) It is Stein.

In the last section, we will give a remark on left invariant complex structures on the compact real form of $GL(2, \boldsymbol{C})$.

The discussions on some class of left invariant complex structures on a semi-simple Lie group and the classification of left invariant complex structures on another groups will be given in [3].

2. Preliminaries

To begin with let us fix notations and prepare definitions.

Let G denote a connected real Lie group of even dimension and $\mathfrak{g}$ be its Lie algebra which is identified with the tangent space of G at the unit element e. The complexification of $\mathfrak{g}$ is denoted by $\mathfrak{g}^C$ and σ is the complex conjugation with respect to $\mathfrak{g}$.

DEFINITION 1. A complex structure on G is said to be *left invariant* (*l. i.*) when the left multiplication of each element in G is holomorphic.

If we denote by J the structure tensor at e of a l. i. complex structure on G, then J is a linear transformation on $\mathfrak{g}$ satisfying

(1) $$J^2=-I$$

(2) $$[X,Y]+J[JX,Y]+J[X,JY]-[JX,JY]=0$$

for every X,Y in $\mathfrak{g}$. The condition (2) is the integrability condition. Conversely every J satisfying (1) and (2) determines a l. i. complex structure on G. So, we say the tensor J with above conditions a l. i. complex structure on G.

Given a l. i. complex structure J we associate a complex subspace $\mathfrak{m}$ of $\mathfrak{g}^C$ by the equation

$$\mathfrak{m}=\{X+\sqrt{-1}JX;\ X\in\mathfrak{g}\}.$$

Then it is seen that

PROPOSITION 1. *$\mathfrak{m}$ is a complex subalgebra of $\mathfrak{g}^C$ satisfying*

(3) $$\mathfrak{m}\cap\sigma\mathfrak{m}=0, \qquad \mathfrak{m}+\sigma\mathfrak{m}=\mathfrak{g}^C.$$

Conversely, each complex subalgebra $\mathfrak{m}$ satisfying (3) *arises from some l. i. complex structure.*

With this proposition we can call a complex subalgebra satisfying the condition (3) *a left invariant complex subalgebra* of $\mathfrak{g}^C$ with respect to $\mathfrak{g}$.

DEFINITION 2. We say l. i. complex structures J_1 and J_2 are *equivalent*, if there exists an automorphism x of $\mathfrak{g}$ such that $xJ_1=J_2x$.

Then the equivalence in terms of l. i. complex subalgebras are stated as follows.

PROPOSITION 2. *Let $\mathfrak{m}_i$ be complex subalgebras corresponding to l. i. compex structures J_i, $i=1, 2$. Then J_1 and J_2 are equivalent if and only if there exists an automorphism x of $\mathfrak{g}^C$ such that $x\sigma=\sigma x$ and $x\mathfrak{m}_1=\mathfrak{m}_2$.*

With these definitions and propositions, to classify l. i. complex structures on G, it is sufficient to give all equivalence classes of l. i. complex subalgebras of $\mathfrak{g}^C$.

3. Classification of l.i. complex subalgeras of $\mathfrak{gl}(2, C)$

Let $\mathfrak{g}=\mathfrak{gl}(2,\boldsymbol{R})$ denote the Lie algebra of $GL(2,R)$. It is decomposed as $\mathfrak{g}=\mathfrak{c}+\mathfrak{sl}(2,\boldsymbol{R})$, $\mathfrak{c}$ being the center. Let $\mathfrak{h}'$ be one of Cartan subalgebras of $\mathfrak{sl}(2,\boldsymbol{R})$. Then, denoting $\mathfrak{h}=\mathfrak{c}+\mathfrak{h}'$, $\mathfrak{g}^C=\mathfrak{gl}(2,C)$ has a root decomposition with respect to $\mathfrak{h}^C$ in the following manner. First, we fix a basis X_i of $\mathfrak{g}$:

$$(4) \qquad X_1=\frac{1}{2}\begin{pmatrix}1 & \\ & 1\end{pmatrix},\ X_2=\frac{1}{2}\begin{pmatrix} & 1\\ -1 & \end{pmatrix},\ X_3=\frac{1}{2}\begin{pmatrix} & 1\\ 1 & \end{pmatrix},\ X_4=\frac{1}{2}\begin{pmatrix}1 & \\ & -1\end{pmatrix}.$$

$\mathfrak{h}'=\boldsymbol{R}\{X_2\}$ is the fundamental Cartan subalgebra of $\mathfrak{g}$. Setting

$$(5) \qquad H=2\sqrt{-1}\,X_2, \qquad A=X_3+\sqrt{-1}\,X_4, \qquad \bar{A}=X_3-\sqrt{-1}\,X_4,$$

we have

$$(6) \qquad [H,A]=2A, \qquad [H,A]=-2\bar{A}, \qquad [A,\bar{A}]=H.$$

Denoting by C one non-zero element of $\mathfrak{c}$, $\mathfrak{g}^C$ has a root decomposition

$$\mathfrak{g}^C=\mathfrak{h}^C+\mathfrak{p}$$

where $\mathfrak{h}^C=\boldsymbol{C}\{C,H\}$, $\mathfrak{p}=\boldsymbol{C}\{A,\bar{A}\}$. We have obviously

$$(7) \qquad \sigma(C)=C,\ \sigma(H)=-H,\ \sigma(A)=\bar{A},\ \sigma(A)=\bar{A}.$$

Now let us begin to classify l. i. complex subalgebras $\mathfrak{m}$.

LEMMA 1. $\dim(\mathfrak{h}^C\cap\mathfrak{m})\leqq 1$ *and* $\dim(\mathfrak{p}\cap\mathfrak{m})\leqq 1$.

PROOF. If $\mathfrak{m}$ contains $\mathfrak{p}$, then $\mathfrak{m}\supset\mathfrak{sl}(2,\boldsymbol{C})$ contradicting (3). If $\mathfrak{m}\supset\mathfrak{h}^C$, then $\mathfrak{m}=\mathfrak{h}^C$ since $\mathfrak{m}$ is 2-dimensional. But $\mathfrak{h}^C+\sigma(\mathfrak{h}^C)=\mathfrak{h}^C$ by (7), $\mathfrak{m}$ cannot be a l. i. complex subalgebra by (3).

LEMMA 2. *When* $\dim(\mathfrak{h}^C\cap\mathfrak{m})=1$, $\mathfrak{m}$ *is one of following algebras.*

(I) $\quad C\{H+dC, A\}$, $\quad$ Re $d\neq 0$,

($\bar{\text{I}}$) $\quad C\{H+dC, \bar{A}\}$. $\quad$ Re $d\neq 0$,

PROOF. Denote a non-zero element of $\mathfrak{h}^C\cap\mathfrak{m}$ by H_1 which is written as $cH+dC$. By conditions (3) and (7) we see $c\neq 0$, hence we can set $c=1$. Take X an element of $\mathfrak{m}$ linearly indepedent of H_1. It is written as $X=H_2+X'$; $H_2\in\mathfrak{h}^C\cap\mathfrak{m}$, $X'\in\mathfrak{p}\cap\mathfrak{m}$. X' is non-zero by Lemma 1. Setting $X'=aA+\bar{b}A$ we have $[H_1,X]=2(aA-\bar{b}A)$. Hence $[H_1,X]=kX$ for some constant $k\in\boldsymbol{C}$ and $k=\pm 2$ according as $b=0$ or $a=0$. In the case $b=0$, we have $a\neq 0$, $k=2$ and $H_1=H+dC$, $X=aA$, and, in the case $a=0$, we have $b\neq 0$, $k=-2$ and $H_1=H+dC$, $X=b\bar{A}$. In each case, $\mathfrak{m}$ satisfies (3) when Re $d\neq 0$. This proves Lemma 2.

LEMMA 3. *When* dim $(\mathfrak{m}\cap\mathfrak{h}^c)=0$, $\mathfrak{m}$ *is one of following algebras*

$$\text{(II)}\quad C\{H+aA-\frac{1}{a}\bar{A},\ bC+aA+\frac{1}{a}\bar{A}\};\qquad a\bar{a}\neq 1,\ \text{Re } b\neq 0.$$

PROOF. First consider the case dim $(\mathfrak{p}\cap\mathfrak{m})=1$. In this case we may choose $X=cH+dC+aA+b\bar{A}$ and $Y=pA+q\bar{A}$ as a basis of $\mathfrak{m}$. We may consider two cases (1) $p=1$ and (2) $q=1$. We can set $a=0$, $b\neq 0$ in the case (1) and $b=0$, $a\neq 0$ in the case (2). Then, for the case (1), $[X,Y]=2c(A-q\bar{A})-bH$. If $c=0$, bH must belong to $\mathfrak{m}$, which contradicts to dim $(\mathfrak{h}^c\cap\mathfrak{m})=0$. Therefore $c\neq 0$. Choosing X so as $c=1$, we get an element in $\mathfrak{m}$: $[X,Y]+bX-2Y=(b^2-4q)\bar{A}+bdC$. But this must be zero, since $\mathfrak{m}$ is 2-dimensional. Hence $d=0$ and $b^2=4q$, which means $\mathfrak{m}=C\{H+b\bar{A},\ A+(b^2/4)\bar{A}\}$ in the case (1). In the case (2), we see $\mathfrak{m}=C\{H+aA,\ (a^2/4)A+\bar{A}\}$. But, in both cases, $\mathfrak{m}$ does not satisfy (3) by (7).

Consider, next, the case dim $(\mathfrak{p}\cap\mathfrak{m})=0$. Choose $X=H+aA+b\bar{A}$ and $Y=C+cA+d\bar{A}$ as a basis of $\mathfrak{m}$. Then $[X,Y]=(ad-bc)H+2cA-2d\bar{A}$. Since, by the assumption, $ad-bc\neq 0$, we must have $[X,Y]=(ad-bc)X$. Solving this equation we see $b=-1/a$ and $c=a^2d$. Hence $\mathfrak{m}$ is one of subalgebras listed in (II). In order for $\mathfrak{m}$ to satisfy the condition (3) it is necessary and sufficient that Re $b\neq 0$ and $a\bar{a}\neq 1$.

Next let us compute the equivalence classes of (I), ($\bar{\text{I}}$) and (II).

LEMMA 4. *Every automorphism* x *of* $\mathfrak{g}^c$ *which satisfies* $x\sigma=\sigma x$ *has the following form*: $x(C)=kC$, $x(H)=pH+qA-\overline{qA}$, $x(A)=rH+sA+t\bar{A}$, $x(A)=-\bar{r}H+\bar{t}A+\overline{sA}$, *where* k *is a real and* p, q, r, s, t *are complex numbers satisfying* $p=s\bar{s}-t\bar{t}$, $q=2(r\bar{t}+s\bar{r})$, $st=-r^2$, $s\bar{s}+t\bar{t}-2r\bar{r}=1$.

PROOF. Since C is a center element, $x(C)=kC$ for some $k\in C$. k is real by $\sigma(C)=C$. Since $C\{H,A,\bar{A}\}$ is the semi-simple part, x has the form: $x(H)=pH+qA+q'\bar{A}$, $x(A)=rH+sA+t\bar{A}$, $x(\bar{A})=r'H+s'A+t'\bar{A}$. We have, by (7), $q'=\bar{q}$, $p=\bar{p}$, $r'=\bar{r}$, $s'=\bar{t}$, $t'=\bar{s}$. The equalities in (6) give the relations in Lemma between these numbers.

PROPOSITION 3. *Any algebra in (II) is equivalent to some algebra in (I).*

PROOF. Let $\mathfrak{m}_1$ be an algebra in (II) with parameter (a, b). Define an automorphism x setting $k=-1$, $r=l$, $s=la$ and $t=-l/a$, where $l=|a|/(1-a\bar{a})$. Define, also, $m=((a\bar{a})^2-1)/(1-a\bar{a})^2$. Then we see that $x(A)=l(H+aA-\frac{1}{a}\bar{A})$ and $x(H+dC)=m(H+aA-\frac{1}{a}\bar{A})-(dC+aA+\frac{1}{a}\bar{A})$. This means $x\mathfrak{m}_2=\mathfrak{m}_1$ for $\mathfrak{m}_2$ the algebra in (I) with parameter $d=b$.

REMARK. Obviously any algebra in ($\bar{\text{I}}$) is transformed into one in (I) under the conjugation σ, i. e. the complex structure corresponding to an algebra in ($\bar{\text{I}}$) is conjugate to that corresponding to some algebra in (I). So, we can exclude the case ($\bar{\text{I}}$) from onsideration. Moreover note that, by our definition of equivalence, any algebra in ($\bar{\text{I}}$) is equivalent to some in (I).

THEOREM 1. *Every equivalence class of left invariant complex structures on* $\mathfrak{g}$ *is represented by one and only one l. i. complex subalgebra* $C\{A, H+dC\}$ *for* Re $(1/d)=-1$.

PROOF. It remains to examine the equivalence between algebras in (I). Let $\mathfrak{m}_1=C\{A, H+d_1C\}$, $\mathfrak{m}_2=C(A, H+d_2C\}$ be algebras in (I). If $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are equivalent under an automorphism x, then, using the form of x in Lemma 4, we see easily that $r=t=0$, hence $q=0$ and $p=1$. This means that $x(A)=sA$ and $x(H+D_1C)=H+kd_1C$, which gives an equivalence between $\mathfrak{m}_1$ and $\mathfrak{m}_2$ for $d_2=kd_1$. Hence, choosing a real number k appropriately, we may take d_2 as Re $(1/d_2)=-1$. It is obvious that algebras with different values of Im $(1/d)$ are not equivalent to each other.

REMARK. The example by Morimoto in Introduction corresponds to the complex subalgebra with parameter d when Im $(1/d)=0$.

4. Left invariant complex structures on $GL^+(2, \boldsymbol{R})$

In this section we will investigate some properties of left invariant complex structures on $GL^+(2, \boldsymbol{R})$.

Let, generally, G be a real Lie group with a left invariant complex structure and H be a closed subgroup. Denote by $\mathfrak{g}$ (resp. $\mathfrak{h}$) the Lie algebra of G(resp. H). The complex structure tensor on $\mathfrak{g}$ is denoted, as before, by J which is integrable. Assume $\mathfrak{g}$ has a vector space decomposition $\mathfrak{g}=\mathfrak{h}+\mathfrak{p}$ such that

(8) $$J\mathfrak{h}=\mathfrak{h},$$

(9) $$[\mathrm{ad}h, J]|_{\mathfrak{p}}=0 \text{ for any } h\in\mathfrak{h},$$

then it is easily seen that the quotient space G/H becomes a complex manifold with the complex structure induced from J, and the right multiplication of an element of H on G/H is biholomorphic.

Let us turn to our case $G=GL^+(2, \boldsymbol{R})$. In the previous section we proved every l. i. complex subalgebras of $\mathfrak{g}^C=\mathfrak{gl}(2, C)$ is written as $C\{A, H+dC\}$ which turns out to be $C\{X_3+\sqrt{-1}\,X_4,\ dX_1+\sqrt{-1}\,X_2\}$ in terms of X_i defined by equations (4). Setting $\mathfrak{h}=\boldsymbol{R}\{X_1, X_2\}$ and $\mathfrak{p}=\boldsymbol{R}\{X_3, X_4\}$ we can see the decomposition $\mathfrak{g}=\mathfrak{h}+\mathfrak{p}$ satisfies (8) and (9). Here, we denote by $\mathscr{H}$ the upper half plane in $\boldsymbol{C}$. Every element of G acts on $\mathscr{H}$ by a linear fractional transformation and the quotient space G/H, where H is the closed Lie sugroup associated with $\mathfrak{h}$, is identified with $\mathscr{H}$. Since $\mathscr{H}$ has a unique complex structure, G/H is biholomorphic to $\mathscr{H}$. The fibre of the projection π: $G\longrightarrow G/H$ is a complex submanifold. Since H is homeomorphic to $\boldsymbol{C}^*$, H must be biholomorphic to $\boldsymbol{C}^*$. Therefore we have proved

THEROEM 2. *With any left invariant complex structure the complex manifold $GL^+(2, \boldsymbol{R})$ is a holomorphic fibre bundle over $\mathscr{H}$ with a general fibre $\boldsymbol{C}^*$, hence, it is biholomorphic to $\boldsymbol{C}^*\times\mathscr{H}$. Especially it is Stein.*

Now we will examine the above identification $GL^+(2, \boldsymbol{R})\cong\boldsymbol{C}^*\times\mathscr{H}$ more closely. For that purpose we fix a coordinate system of $GL^+(2, \boldsymbol{R})$ as follows. Every element $g=\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4\end{pmatrix}$ of $GL^+(2, \boldsymbol{R})$ can be written uniquely by the expression

(10) $$g=\begin{pmatrix} u & 0 \\ 0 & u\end{pmatrix}\begin{pmatrix} y & x \\ 0 & 1\end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{pmatrix},$$

when parameters take values in

(11) $$u>0,\quad y>0,\quad x\in\boldsymbol{R} \text{ and } 0\leq\theta<2\pi.$$

Hence, the quadruple (u, θ, x, y) gives a global coordinate system. (cf. S. Lang [1]). This system is related with the system (x_i) by equations

(12) $$\begin{cases} x=(x_2x_4+x_1x_3)/(x_3^2+x_4^2) \\ y=(x_1x_4-x_2x_3)/(x_3^2+x_4^2) \\ ue^{\sqrt{-1}\,\theta}=x_4-\sqrt{-1}\,x_3. \end{cases}$$

The projection π is given by $\pi(g)=x+\sqrt{-1}\,y$. For the sake of simplicity we will identify elements X_i in $\mathfrak{g}$ with left invariant vector fields on $GL^+(2, \boldsymbol{R})$. Then, using new coordinates, they are written as in

LEMMA 5.

$$(13)\quad \begin{cases} X_1=(u/2)\dfrac{\partial}{\partial u}, \qquad X_2=(1/2)\dfrac{\partial}{\partial \theta} \\ X_3=(-u/2)\sin 2\theta\dfrac{\partial}{\partial u}-(1/2)\cos 2\theta\dfrac{\partial}{\partial \theta}+y\left(\cos 2\theta\dfrac{\partial}{\partial x}+\sin 2\theta\dfrac{\partial}{\partial y}\right) \\ X_4=(-u/2)\cos 2\theta\dfrac{\partial}{\partial u}+(1/2)\sin 2\theta\dfrac{\partial}{\partial \theta}+y\left(-\sin 2\theta\dfrac{\partial}{\partial x}+\cos 2\theta\dfrac{\partial}{\partial y}\right). \end{cases}$$

PROOF. see [1].

By this Lemma we have

$$X_3+\sqrt{-1}\,X_4=e^{-2\sqrt{-1}\,\theta}\left\{-(1/2)(\sqrt{-1}\,u\frac{\partial}{\partial u}+\frac{\partial}{\partial \theta})+y\left(\frac{\partial}{\partial x}+\sqrt{-1}\frac{\partial}{\partial y}\right)\right\}$$

$$dX_1+\sqrt{-1}\,X_2=(1/2)\left(du\frac{\partial}{\partial u}+\sqrt{-1}\frac{\partial}{\partial \theta}\right).$$

Let us recall a function f on $GL^+(2, \boldsymbol{R})$ is holomorphic if and only if $(X_3+\sqrt{-1}\,X_4)\,f=(dX_1+\sqrt{-1}\,X_2)f=0$. Hence, the projection $\pi=x+\sqrt{-1}\,y$ is holomorphic. Now set $1/d=-1+\sqrt{-1}\,\nu$. Then we see

LEMMA 6. *The function w defined by*

$$w=u^{-1+\sqrt{1-\nu}}\,y^{\sqrt{-1}\nu/2}\,e^{\sqrt{-1}\,\theta}$$

is holomorphic.

Also we see

PROPOSITION 4. *The mapping $\psi=(\pi, w)$ from $GL^+(2, \boldsymbol{R})$ to $\mathscr{H}\times \boldsymbol{C}^*$ is biholomorphic.*

PROOF. It is enough to note that the jacobian of ψ is nowhere zero, which is verified by computation.

Due to the identification of $GL^+(2, \boldsymbol{R})$ and $\mathscr{H}\times \boldsymbol{C}^*$ by ψ, the left multiplication of an element $g=\begin{pmatrix} a & b \\ c & d\end{pmatrix}$ is an automorphism of $\mathscr{H}\times \boldsymbol{C}^*$, which we write by $\Phi_\nu(g)$, ν being the parameter defined above. Denote by (z, w) the coordinate of $\mathscr{H}\times \boldsymbol{C}^*$. Then we have

PROPOSITION 5. (1) *The action of $\Phi_\nu(g)$ on $\mathscr{H}\times \boldsymbol{C}^*$ is $\Phi_\nu(g)(z, w)=((az+b)/$*

$(cz+d)$, $(\det g)^{\sqrt{-1}\nu/2}(cz+d)^{-1}w)$.

(2) *If* $\nu\neq\nu'$, *then* Φ_ν *and* $\Phi_{\nu'}$ *are not conjugate in the automorphism group of* $\mathscr{H}\times C^*$.

PROOF. First note that the mapping ψ is given by the following equations in terms of the coordinate system $x=(x_i)$ of G:

$$z=(x_1-\sqrt{-1}\,x_2)/(x_3-\sqrt{-1}\,{}_4),$$

$$w=(\det x)^{\sqrt{-1}\nu/2}/(x_4+\sqrt{-1}\,x_3).$$

Then, using these equalities, the above expression of $\Phi_\nu(g)$ is easily seen. Next, assume that there exists an automorphism ϕ of $\mathscr{H}\times C^*$ satisfying $\phi\Phi_\nu(g)=\Phi_{\nu'}(g)\phi$ for every g. Writing this equality explicitly using (1), we see $\nu=\nu'$.

5. Left invariant complex structures on $U(2)$

It is well known that $\mathfrak{gl}(2,C)$ has two real forms. One is $\mathfrak{gl}(2,\boldsymbol{R})$ considered above and the other is $\mathfrak{u}(2)$, the Lie algebra of the group $U(2)$ of 2×2 unitary matrices. As a topological manifold $U(2)$ is homeomorphic to $S^1\times S^3$. The manifold $S^1\times S^3$, on the other hand, has a complex structure realized as a usual Hopf surface: the quotient space of $C^2-\{0\}$ by the group generated by a transformation $z\longrightarrow \alpha z$, for $z\in C^2-\{0\}$ and a complex number α whose absolute value is not equal to one. And this structure is invariant under the natural action as a group manifold. But, we can see that these are all l. i. complex structures on $U(2)$ in the following way.

We take, as a basis of $\mathfrak{u}(2)$, four elements:

$$(4)'\quad Y_1=\frac{1}{2}\begin{pmatrix}\sqrt{-1} & \\ & \sqrt{-1}\end{pmatrix},\ Y_2=\frac{1}{2}\begin{pmatrix}\sqrt{-1} & \\ & -\sqrt{-1}\end{pmatrix},\ Y_3=\frac{1}{2}\begin{pmatrix} & 1\\ -1 & \end{pmatrix},\ Y_4=\frac{1}{2}\begin{pmatrix} & \sqrt{-1}\\ \sqrt{-1} & \end{pmatrix}$$

and set

$$(5)'\qquad h=2\sqrt{-1}\,Y_2,\ B=\sqrt{-1}\,(Y_3+\sqrt{-1}\,Y_4),\ \bar{B}=\sqrt{-1}\,(Y_3-\sqrt{-1}\,Y_4).$$

Then we have

$$(6)'\qquad [h,B]=2B,\ [h,\bar{B}]=-2\bar{B},\ [B,\bar{B}]=h.$$

Denoting the element Y_1 by C, the conjugation τ of $\mathfrak{gl}(2,C)$ with respect to $\mathfrak{u}(2)$ is given by

$$(7)'\qquad \tau(C)=C,\ \tau(h)=-h,\ \tau(B)=-\bar{B},\ \tau(\bar{B})=-B.$$

With these identities, Lemma 1 and Lemma 2 hold also for $\mathfrak{u}(2)$ replacing H, A,

$\bar{A}$ by h, B, $\bar{B}$ respectively. Lemma 3 holds without the condition $a\bar{a}\neq 1$. Lemma 4 needes some modification. But Proposition 3 is also true. And, hence, we can prove Theorem 1 for $\mathfrak{u}(2)$ which says that the l. i. complex subalgebras for $\mathfrak{u}(2)$ are $C\{B, h+dC\}$.

Now let us set $\mathfrak{h}=\boldsymbol{R}\{Y_1, Y_2\}$ and $\mathfrak{p}=\boldsymbol{R}\{Y_3, Y_4\}$. We see that the decomposition $\mathfrak{u}(2)=\mathfrak{h}+\mathfrak{p}$ satisfies (8) and (9). The subgroup T of $U(2)$ corresponding to $\mathfrak{h}$ is a complex torus. And the quotient space $U(2)/T$, which is simply connected and compact, must be the complex projective space. Therefore, the space $U(2)$, as a complex manifold, is the holomorphic torus bundle over the projective space which is a Hopf surface in the above sense.

REMARK. The reasonings in **4** and in **5** show that there is a complete duality between l. i. complex structures on $\mathfrak{u}(2)$ and those on $\mathfrak{gl}(2, \boldsymbol{R})$. This duality is due to the low dimensionality of Lie algebras. We will discuss on the matter of this kind in [3] for higher dimensional case.

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Department of Mathematics
Faculty of Science
Kumamoto University