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A RELATION BETWEEN AN ASYMPTOTIC PROBABILITY AND THE MAXIMAL EGIENVALUE OF A RECURRENT POTENTIAL KERNEL

Yoichi ŌSHIMA

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1. Introduction

For certain transient Markov processes, a relation between the minimal eigenvalue of negative of the generator and an asymptotic behavior of the transition probability for large time is given by M. Kac [4] and generalized by Donsker-Varadhan [2]. Their results contain that, if L is the generator of a recurrent Markov process on a compact space E and $V(x)$ a nonnegative function on E then, under some additional hypothesis, the minimal eigenvalue σ_1 of $-L+V$ is given by

$$(1.1) \qquad \sigma_1 = -\lim_{t\to\infty} \frac{1}{t} \log E^x [exp\{-\textstyle\int_0^t V(X_s)ds\}]$$
$$= -\lim_{t\to\infty} \frac{1}{t} \log Q_t 1(x),$$

where Q_t is the transition probability of the Markov process corresponding to $L-V$.

If $V=0$ then the relation (1.1) is trivial. But if we restrict the domain of L to the space of null charges then a relation between the minimal eigenvalue of $-L$ and an asymptotic behavior of the transition probability P_t of the Markov process X corresponding to L holds. More explicitly, suppose that the resolvent $(G^p)_{p>0}$ of X is strong Feller and symmetric relative to the invariant measure μ. If we denote $\mathscr{H}$ the space of all square integrable functions relative to μ such that $<\mu, f> = 0$ then, under the additional hypothesis H2 in section 3, the minimal eigenvalue σ_1 of $-L$ restricted to $\mathscr{H}$ is given by

$$(1.2) \qquad \sigma_1 = -\lim_{t\to\infty} \frac{1}{t} \log < \mu, \|P_t - \mu\|(x) >$$

where $\|P_t-\mu\|(x)$ is the total variation of the signed measure $P_t(x, \cdot)-\mu(\cdot)$ (theorem 2). Under rather weaker hypothesis H1, the corresponding result for the resolvent holds (theorem 1).

The statements and proofs of theorems 1 and 2 are given by the maximal eigenvalue λ_1 of an operator G on $\mathscr{H}$ induced by a potential kernel of X. The eigenvalues σ_1 and λ_1 are related by $\sigma_1=1/\lambda_1$.

2. Preliminaries

Let E be a compact metric space and X a recurrent Hunt process on E with strong Feller resolvent. Denote $(P_t)_{t>0}$ and $(G^p)_{p>0}$ the transition function and resolvent of X, respectively. Let μ be the probability invariant measure of X then it is everywhere dense positive on E. We shall assume that our resolvent (G^p) is symmetric relative to μ, that is, for all $p>0$ and bounded measurable functions f and g

$$(2.1) \qquad (G^p f, g) = (f, G^p g),$$

where $(\cdot, \cdot)$ is the inner product relative to μ.

For all $p>0$ and $x \in E$, since the measures $G^p(x, \cdot)$ and μ are equivalent, there exists a jointly measurable function $g^p(x, y)$ ($p>0$, $x, y \in E$) such that $G^p(x, dy)= g^p(y, y)\, \mu(dy)$, $g^p(x, y)= g^p(y, x)$ and $g^p(\cdot, y)$ is p-excessive.

Let $G(x, dy)$ be a potential kernel of X defined by

$$(2.2) \qquad G(y, dy) = \sum_{n=1}^{\infty} \{(G^1)^n - \mu\}(x, dy).$$

Under our present hypothesis, (2. 2) is well defined. In our previous paper [4], it is shown that there exists unique jointly measurable function $g(x, y)$ ($x, y \in E$), called *the potential kernel function*, satisfying

(i) *$\int |g(x, y)|\, \mu(dy)$ is bounded.*

(ii) *Set $\Gamma=\{(x, y): |g(x, y)|<\infty\}$ then, for all $x \in E$, the complement Γ_x^c of x-section Γ_x of Γ is polar.*

(iii) *$g(x, y)=g(x, y)$ and $G(x, dy)=g(x, y)\, \mu(dy)$.*

(iv) *For all $(x, y) \in \Gamma$, $\int |g(x, z)|\, g^1(z, y)\, \mu(dz) < \infty$ and*

$$(2.3) \qquad \begin{aligned} g(x, y) &= g^1(x, y)+Gg^1(x, y)-1 \\ &= g^1(x, y)+g^1\hat{G}(x, y) -1, \end{aligned}$$

where $\hat{G}(dz, y)=G(y, dz)$.

Denote $L^2(\mu)$ the set of all square integrable functions relative to μ and

$$\mathscr{H}=\{f \in L^2(\mu): \langle \mu, f \rangle=0\}.$$

Then $\mathscr{H}$ is a closed linear subspace of the Hilbert space $L^2(\mu)$.

LEMMA 1. *The kernels G^p and G define the bounded linear operators on $\mathscr{H}$.*

PROOF. Let $f \in \mathscr{H} \cap b\mathscr{C}$ then, by Schwarz's inequality

$$\int (Gf)^2(x)\mu(dx) = \int \{\int g(x, y) f(y)\mu(dy)\}^2 \mu(dx)$$
$$\leqq \int \{\int |g(x, y)| \mu(dy) \int |g(x, y)| f(y)^2 \mu(dy)\} \mu(dx).$$

Since $\int |g(x, y)| \mu(dy)$ is bounded, $\sup_x \int |g(x, y)| \mu(dy) = \alpha < \infty$.
Hence

$$\int (Gf)^2(x)\mu(dx) \leqq \alpha \int\int |g(x, y)| f(y)^2 \mu(dy)\mu(dx)$$
$$\leqq \alpha^2 \int f(y)^2 \mu(dy),$$

so that G defines a bounded linear operator on $L^2(\mu)$. The equality $\langle \mu, Gf \rangle = 0$ is obvious. We shall use the same notation for this extended operator. Therefore $Gf \in \mathscr{H}$. Similarly $G^p f \in \mathscr{H}$ for all $p > 0$ and $f \in \mathscr{H}$.

As is well known, (G^p) is a strongly continuous resolvent on $L^2(\mu)$, and hence on $\mathscr{H}$, that is $\lim_{p \to \infty} pG^p h = h$ for all $h \in \mathscr{H}$. Hence, if $G^p f = 0$ for some $p > 0$ and $f \in \mathscr{H}$ then, by the resolvent equation, $G^p f = 0$ for all $p > 0$, so that, by the strong continuity of G^p, $f = 0$. Hence the generator

$$Lg = pg - (G^p)^{-1} g, \; g \in \mathscr{D}(L) = G^p(\mathscr{H}) \tag{2.4}$$

is well defined.

LEMMA 2. $-L = (G)^{-1}$

PROOF. It is enough to show that $\mathscr{D}(L) = G(\mathscr{H})$, $-LGf = f$ for all $f \in \mathscr{H}$ and $-GLg = g$ for all $g \in \mathscr{D}(L)$.

If $g \in \mathscr{D}(L)$ then $g = G^p f$ for some $f \in \mathscr{H}$. From (2.2),

$$g = G(I - pG^p) f \in G(\mathscr{H})$$

and

$$GLg = G(pg - f) = -G(f - pG^p f) = -g.$$

Similarly, since $(I - pG^p)Gf = G^p f$ for all $f \in \mathscr{H}$, $G(\mathscr{H}) \subseteq \mathscr{D}(L)$ and $-LGf = f$ for all $f \in \mathscr{H}$.

3. Main results

In this section, except the last part, we shall assume that E is compact and X is a recurrent Hunt process with strong Feller symmetric resolvent.

Denote $\Pi(x, dy)$ and $\pi(x, y)$ for $G^1(x, dy)$ and $g^1(x, y)$, respectively. We shall omit the trivial case that $\pi(x, y) = 1$ for all $x, y \in E$. Let g be the potential kernel function in section 2. We shall impose an additional hypothesis on π.

HYPOTHESIS H1. $\sup_{x\in E} \int \pi(x, y)^2 \mu(dy) < \infty$.

LEMMA 3. *Hypothesis H1 is equivalent to*

$$\sup_{x\in E} \int g(x, y)^2 \mu(dy) < \infty. \tag{3.1}$$

PROOF. Suppose that (3.1) holds. Then, since $\mu(\Gamma_x)=0$ for all $x \in E$ and

$$g(x, y) = \pi(x, y) + \int \pi(x, z) g(z, y) \mu(dz) - 1$$

for all $(x, y) \in \Gamma$ by (2.3),

$$\int \pi(x, y)^2 \mu(dy) = \int \{g(x, y) - \Pi g(x, y) + 1\}^2 \mu(dy).$$

Hence, for the proof of Hl, it ie enough to show that

$$\sup_{x\in E} \int \{\Pi g(x, y)\}^2 \mu(dy) < \infty. \tag{3.2}$$

This follows easily from Schwarz's inequality, in fact,

$$\begin{aligned}\sup_{x\in E} \int \{\Pi g(x, y)\}^2 \mu(dy) &\leq \sup_{x\in E} \int \{\pi(x, z) \mu(dz) \int \pi(x, z) g(z, y)^2 \mu(dz)\} \mu(dy) \\ &= \sup_{x\in E} \int\int \pi(x, z) g(z, y)^2 \mu(dz) \mu(dy) \leqq \sup_{z\in E} \int g(z, y)^2 \mu(dy) < \infty.\end{aligned}$$

Conversely, if H1 holds then the proof of (3.1) is similar by noting that, for all $(x, y) \in \Gamma$

$$g(x, y) = \pi(x, y) + \int g(x, z) \pi(z, y) \mu(dz) - 1$$

and $\sup_{x\in E} \int |g(x, z)| \mu(dz) < \infty$.

LEMMA 4. *Under the hypothesis H1, the operator* G *and* G^p *on* $\mathscr{H}$ *are symmetric compact operators. Moreover, they are strictly positive definite on* $\mathscr{H}$.

PROOF. Since G and G^1 define the Hilbert-Schmidt type operators, the compactness of G and G^1 are obvious. By the resolvent equation, G^p is also compact.

Symmetry is contained in the hypothesis on X. From [7, lemma 6.3], G is strictly positive difinite. Similarly G^p is strictly positive difinite.

From lemma 5, as is well known, the operator G on $\mathscr{H}$ has at most countable positive eigenvalues with no accumulation points except zero. Let $\lambda_1 \geqq \lambda_2 \geqq \cdots > 0$ be the eigenvalues of G and $\varphi_1, \varphi_2, \cdots$ be the corresponding normalized eigenfunctions in $\mathscr{H}$. Then $\{\varphi_n\}_{n\geq 1}$is a complete orthonormal system on $\mathscr{H}$.

REMARK. If we consider G as an operator on $L^2(\mu)$ into $\mathscr{H}$ then it has zero as an eigenvalue and the only corresponding eigenfunction is constant.

LEMMA 5. *For all* $n \geqq 1$, ψ_n *has a bounded continuous version.*

PROOF. Since $\psi_n \in \mathscr{H}$ and $G\psi_n = \lambda_n \psi_n$,

$$\lambda_n \, ess\ sup \, |\psi_n(x)| = ess\ sup \, |G\psi_n|(x)$$
$$= ess\ sup \, |\int g(x, y)\, \psi_n(y)\, \mu(dy)| \leqq sup\, \{\int g(x, y)^2\, \mu(dy)\}^{1/2} < \infty.$$

Hence ψ_n is essentially bounded and hence $G\psi_n$ is bounded. Thus, from the strong Feller property of G_p, $G\psi_n = G^p(pG\psi_n + \psi_n)$ is continuons.

THEOREM 1. *Under the hypothesis* $H1$,

$$log\,(\lambda_1/1+\lambda_1) = \lim_{n\to\infty} log < \mu, \, \|\Pi^n - \mu\|\,(x)>. \tag{3.3}$$

where $\|\Pi^n - \mu\|\,(x)$ *is the total variation of the signed measure* $\Pi^n(x, \cdot) - \mu(\cdot)$.

PROOF. Since $G\psi_1 = \lambda_1 \psi_1$ and $(I - \Pi)\, G\psi_1 = \Pi\psi_1$,

$$\Pi^n \psi_1 = (\lambda_1/1+\lambda_1)^n \psi_1 \quad \text{for all } n \geqq 1. \tag{3.4}$$

Therefore, integrating by μ,

$$(\lambda_1/1+\lambda_1)^n < \mu, \, |\psi_1| > = < \mu, \, |\Pi^n \psi_1| >.$$

Since $\psi_1 \neq 0$, $< \mu, \, |\psi_1| > \, > 0$ and

$$n\, log(\lambda_1/1+\lambda_1) + log < \mu, \, |\psi_1| > = log < \mu, \, |\Pi^n \psi_1| >$$
$$= log < \mu, \, |\int \{\Pi^n(x, dy) - \mu(dy)\}\psi_1(y)| >$$
$$\leqq log(ess\ sup |\psi_1|) + log < \mu, \| \, \Pi^n - \mu \|(x) >.$$

Dividing by n and letting $n \to \infty$ we have

$$log(\lambda_1/1+\lambda_1) \leqq \lim_{n\to\infty} \inf \frac{1}{n} \log<\mu, \|\Pi^n-\mu\|(y)>.$$

For the proof of the converse statement, we shall define $\pi^n(x, y)$ inductively by

$$\pi^1(x, y)=\pi(x, y), \ \pi^n(x, y)=\Pi^{n-1} \pi(x, y),$$

then, by Schwarz's inequality

$$\begin{aligned} <\mu, \|\Pi^n-\mu\|(x)>^2 &= \{\int\mu(dx) \int|\pi^n(x, y)-1|\mu(dy)\}^2 \\ &\leqq \int\int|\pi^n(x, y)-1|^2\mu(dx)\, \mu(dy) \\ &= \int\int|\Pi^{n-1}(\pi-1)(x, y)|^2\, \mu(dx)\, \mu(dy). \end{aligned}$$

Set $f_y(z)=\pi(z, y)-1$ then

$$\begin{aligned} <\mu, \|\Pi^n-\mu\|(x)>^2 &\leqq \int\int|\Pi^{n-1}f_y(x)|^2\, \mu(dx)\, \mu(dy) \\ &= \int(\Pi^{n-1} f_y, \Pi^{n-1}f_y)\, \mu(dy)=\int(f_y, \Pi^{2n-2}f_y)\, \mu(dy) \\ &= \int\{(f_y, \Pi^{2n-2}f_y)/\|f_y\|_2^2\}\|f_y\|_2^2\, \mu(dy), \end{aligned}$$

where $\|\cdot\|_2$ is the norm in $L^2(\mu)$. Since $(\lambda_1/1+\lambda_1)^{2n-2}$ is the maximal eigenvalue of the operator Π^{2n-2} on $\mathscr{H}$, from a classical variational formla,

$$(3.5) \qquad (\lambda_1/1+\lambda_1)^{2n-2}=\sup_{f\in\mathscr{H}} (f, \Pi^{2n-2}f)/\|f\|_2^2.$$

From the definition of f_y, $f_y\neq 0$,

$$\|f_y\|_2^2=\int\{\pi(z, y)-1\}^2\, \mu(dz)\leqq\int\pi(z, y)^2\, \mu(dz)-1<\infty$$

and $<\mu, f_y>=0$, that is, $f_y\in\mathscr{H}$. Moreover, from the hypothesis H1,

$$(3.6) \qquad \int\|f_y\|_2^2\, \mu(dy)<\infty.$$

Hence,

$$\begin{aligned} <\mu, \|\Pi^n-\mu\|>^2 &\leqq \int\{\sup_{f\in\mathscr{H}}(f, \Pi^{2n-2}f)/\|f\|_2^2\}\|f_y\|_2^2\, \mu(dy) \\ &\leqq (\lambda_1/1+\lambda_1)^{2n-2}\int\|f_y\|_2^2\, \mu(dy). \end{aligned}$$

Hence we have

$$2\log<\mu,\|\Pi^n-\mu\|(x)>\leqq(2n-2)\log(\lambda_1/1+\lambda_1)+\log\{\int\|f_y\|_2^2\, \mu(dy)\}.$$

Therefore

$$\lim_{n\to\infty} \sup \frac{1}{n} \log<\mu, \|\Pi^n-\mu\|(x)>\leqq \log(\lambda_1/1+\lambda_1).$$

Thus we have the result.

To give a connection between λ_1 and the asymptotic behavior of P_t as $t\to\infty$, we must impose stronger hypothesis than H1.

HYPOTHESIS H2. *For all $x \in E$ and $t>0$ there exists a jointly measurable density $p_t(x, y)$ of $P_t(x, dy)$ relative to $\mu(dy)$ satisfying $p_t(x, y)=p_t(y, x)$ and*

$$\sup_{x\in E} p_t(x, x) < \infty. \tag{3.7}$$

Clearly, (3.7) is equivalent to

$$\sup_{x\in E} \int p_{t/2}(x, y)^2\, \mu(dy) < \infty. \tag{3.8}$$

Under the hypothesis H2

$$g^p(x, y) = \int_0^\infty e^{-pt}\, p_t(x, y)dt$$

satisfies the hypothesis H1. Let λ_n and $\psi_n(n=1,2,\cdots)$ be the eigenvalues and the corresponding normalized eigenfunctions of G, as before.

THEOREM 2. *Under the hypothesis H2,*

$$-1/\lambda_1 = \lim_{t\to\infty} (1/t) \log<\mu, \|P_t-\mu\|(x)>. \tag{3.9}$$

PROOF. The proof is similar to the proof of theorem 1, so that we shall only outline it. Since $P_t\psi_1=\exp(-t/\lambda_1)\psi_1$,

$$-(t/\lambda_1)+\log<\mu, |\psi_1|> = \log<\mu, |P_t\psi_1|>$$
$$\leqq \log(\text{ess sup}|\psi_1|)+\log<\mu, \|P_t-\mu\|(x)>.$$

Hence

$$-(1/\lambda_1) \leqq \lim_{t\to\infty}\inf(1/t) \log<\mu, \|P_t-\mu\|(x)>.$$

Conversely, if we set $f_y(z)=p_s(z, y)-1\in \mathscr{H}$ for fixed $s(s<t)$ then

$$<\mu, \|P_t-\mu\|(x)>^2 \leqq \int\|P_{t-s}f\|_2^2\, \mu(dy)$$
$$\leqq \sup_{f\in\mathscr{H}}(\|P_{t-s}f\|_2^2/\|f\|_2^2) \int \|f_y\|_2^2\, \mu(dy)$$
$$\leqq \exp\{-2(t-s)/\lambda_1\} \int \|f_y\|_2^2\, \mu(dy).$$

Since $\int \|f_y\|_2^2 \mu(dy) < \infty$, we have

$$\lim_{t\to\infty} sup(1/t) \log <\mu, \|P_t - \mu\|(x)> \leqq -1/\lambda_1.$$

REMARK 1. From lemma 2, $1/\lambda_1$ is the smallest eigenvalue of $-L$.

REMARK 2. If μ is not a probability measure then the results of theorems 1 and 2 hold by replacing $\mu/\mu(E)$ for μ.

Finally, we shal remark some easy consequences for the case with non-compact state space. Let E be a separable metric space and X be a recurrent Hunt process with symmetric strong Feller resolvent (G^p). Then there exists a potential kernel function $g(x, y)$ (see [4], section 4). Let A be an arbitrary fixed non-negative finite continuous additive functional of X and (K^p) be the resolvent of the time changed process Y of X by A. Then the restriction of $g(x, y)$ to $supp\ (A) \times supp\ (A)$ is a potential kernel function of Y. Denote ν_A the measure associated with A then it vanishes outside of $supp(A)$.

HYPOTHESIS H1$'$. *$\nu_A(E) < \infty$ and $\sup_{x \in supp(A)} \int g(x,y)^2 \nu_A(dy) < \infty$.*

Note that, under the hypothesis H1$'$, $\int |g(x, y)| \nu_A(dy)$ is bounded on $supp\ (A)$. Since the potential kernel function is unique up to difference of a locally bounded function of the form $f_1(x) + f_2(y)$ ([4], theorem 4. 1), hypothesis H1$'$ is independent of the choice of the potential kernel function. Suppose, for simplicity, that $\nu_A(E) = 1$ and

$$\int G(x, y)\, \nu_A(dy) = 0.$$

This is possible by replacing μ and g by $\mu/\nu_A(E)$ and

$$g(x, y)\, \nu_A(E) - \int g(x, z)\, \nu_A(dz) - \int g(y, z)\, \nu_A(dz) + 1/\nu_A(E) \cdot \iint g(z, u)\, \nu_A(dz)\, \nu_A(du),$$

respectively. After this modification, we have

THEOREM 1.$'$ *Under the hypothesis H1$'$, the maximal eigenvalue λ_1 of the operator K_A defined by*

$$K_A f(x) = \int g(x, y) f(y)\, \nu_A(dy)$$

on $\mathscr{H}_A = \{f \in L^2(\nu_A): <\nu_A, f> = 0\}$ is given by the formula

$$(3.3)' \qquad log(\lambda_1/1+\lambda_1) = \lim_{n\to\infty} log\langle \nu_A, \|(K_A^1)^n - \nu_A\| (x)\rangle,$$

where, $K_A^1(s, dy)$ is the kernel defined by

$$K_A^1 f(x) = E^x \left[\int_0^\infty exp(-A_t)\, f(X_t)\, dA_t\right].$$

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Department of Mathematics,
Faculty of engineering,
Kumamoto University