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ON EXISTENCE PROOF IN PLASTICITY THEORY

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Introduction

In [1], Duvaut-Lions presented an idea to formulate the plasticity problems mathematically. They also gave a proof of the existence and uniqueness of the solution to the elastic-perfectly plastic problems. The key of their formulation is in deleting, by introducing an inequality representing the orthogonality condition of the plastic strain increments, the distinction of two different states "elastic, plastic", which causes all mathematical difficulties in plasticity theory.

One of the important techniques of [1] in proving the existence of the solution is the penalty method, in which the elastic-plastic problem is regarded as a limit problem of a sequence of problems with viscosity. This method has great generality and, in a certain sense, is even natural for proving the existence and uniqueness of the solution.

Although the inequality as in [1] may be inevitable for the mathematical treatment of the plasticity problems, the use of the penalty method in constructing the solution is not necessarily convenient when we want to discuss about some problems. One example of such case arises in the derivation and analysis of the numerical methods, since the plasticity problems in engineering are treated usually in the classical form.

In a previous paper [3] we presented another approach to the plasticity problems, which is based on a simple "discretization". The purpose of this paper is to apply its basic idea to the general problems of plasticity. We again consider the dynamic elastic-plastic problem. We first introduce a finite element scheme, which is a system of ordinaly differential equations with hysteresis and called the semidiscrete system. This system can be regarded as a discrete model of the original problem. We show that the "classical" elastic-plastic problem can be well-posed for this system. Since the solution of this system is smooth enough to get some energy inequalities including the higher derivatives, we can attain, by passing to the limit, to the desired solution of the original elastic-plastic problem which is governed by the partial differential equations.

Our results can be used immediately as a theoretical background of some numerical methods to compute the solution. We refer to [4] for the numerical approach to the present problem, in particular for the stability analysis of the approximating schemes.

1. An incremental formulation of the dynamic elastic-plastic problem

The problem considered in this paper is formulated, originally, as follows. For simplicity, we consider the two-dimensional case. Let Ω be a domain in (x_1, x_2) plane and T the bounded time interval $(0, T)$. Let $u=(u_1, u_2)$, $\varepsilon=(\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{12})$, $\sigma=(\sigma_{11}, \sigma_{22}, \sigma_{12})$ $(\sigma_{21}=\sigma_{12})$ and $\alpha=(\alpha_{11}, \alpha_{22}, \alpha_{12})$ be the displacement, the strain, the stress and the parameter representing the center of the yield surface. Also, $u,_j$ and $\dot{u}$ are used to denote the derivative on x_j and on time t. The notation $*$ denotes the transpose of a vector. Hence a^*b denotes the inner product of vectors. The inner product and the norm of (vector) functions in $L^2(\Omega)$ are denoted by (α, β) and $\|\alpha\|$, respectively. We use these notations for both the single and vector functions, unless ambiguity occurs.

The dynamic elastic-plastic problem with Mises' yield condition and Ziegler's hardening rule is to solve the following equations under the given initial-boundary conditions (see [5], [6]).

(1.1) $$\rho \ddot{u}_i - \sum_j \sigma_{ij,j} = b_i \qquad \text{in } T\times\Omega,$$

(1.2) $$\begin{cases} \dot{\sigma} = D\dot{\varepsilon} \\ \dot{\alpha} = 0 \end{cases} \qquad \text{if } f(\sigma-\alpha) < \bar{\sigma},$$

(1.3) $$\begin{cases} \dot{\sigma} = (D-D')\dot{\varepsilon} \\ \dot{\alpha} = (\sigma-\alpha)\, \dfrac{\partial f^* \dot{\sigma}}{f} \end{cases} \qquad \text{if } f(\sigma-\alpha) = \bar{\sigma} \text{ and } \partial f^* \dot{\sigma} \geqslant 0,$$

where ρ is positive constant and $\{b_i\}$ are given smooth functions with piecewise analytic first derivative with respect to t. The function f is given by

(1.4) $$f^2(\sigma) = \sigma_{11}^2 + \sigma_{22}^2 - \sigma_{11}\sigma_{22} + 3\sigma_{12}^2.$$

The yield surface is thus given in the form $f^2(\sigma-\alpha) = \bar{\sigma}^2$ for a positive constant $\bar{\sigma}$. The 3×3 matrices D and D', and the vector $\partial f = \partial f(\sigma-\alpha)$ are given as follows.

(1.5) $$D = \frac{E}{1-\nu^2}\begin{vmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \nu' \end{vmatrix} \qquad \left(\nu' = \frac{1-\nu}{2}\right),$$

(1.6) $$D' = \frac{D\partial f \partial f^* D}{\eta + \partial f^* D \partial f},$$

(1.7) $$\partial f = \left(\frac{\partial f}{\partial \sigma_{11}}, \frac{\partial f}{\partial \sigma_{22}}, \frac{\partial f}{\partial \sigma_{12}} \right).$$

Here ν is the Poisson's ratio and η corresponds to the modulus of strain-hardening. We assume that η is a positive constant, for simplicity. The strain is assumed to be small, i. e., $\varepsilon = (\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{12}) = (u_{1,1}, u_{2,2}, u_{1,2} + u_{2,1})$. Also we assume that the initial displacement is 0 and that a certain part Γ_0 of the boundary (Γ_0: closed and length $(\Gamma_0) > 0$) is fixed ($u_i = 0$) and the remaining part is free ($\sum_j \sigma_{ij} \cos(n, x_j) = 0$). Our purpose is to show the existence of a (weak) solution to this problem.

2. Semidiscrete system

To get some a priori estimates of the solution, we examine the simplest finite element approximation to the present problem. We assume that Ω is a closed polygonal region. This is only for avoiding the technical complexity in passing to the limit. By $\hat{\Omega}$ we denote a triangulation of Ω. We require that the end points of Γ_0 is always the node of $\hat{\Omega}$ and the ratio of the maximum and the minimum side length of all triangles (=elements) is uniformly bounded as the former tends to zero. Let $\{\varphi_p\}$ be the piecewise linear finite element basis which takes 1 at the node p. The approximate value of u_i at time t is sought in the form

(2.1) $$u_i(t) = \sum_{p \in P} u_i^p(t) \varphi_p,$$

where P is the set of the nodes in $\hat{\Omega} - \Gamma_0$. The unkowns $\{u_i^p(t)\}$ are determined by solving the following Galerkin system.

(2.2) $$(\rho \ddot{u}_i, \varphi_p) + \sum_j (\sigma_{ij}, \varphi_{p,j}) = (b_i, \varphi_p) \qquad p \in P,$$

(2.3) $$\begin{cases} \dot{\sigma} = D\dot{\varepsilon} \\ \dot{\alpha} = 0 \end{cases} \qquad \text{for elastic element: } f(\sigma - \alpha) < \bar{\sigma},$$

(2.4) $$\begin{cases} \dot{\sigma} = (D - D')\dot{\varepsilon} \\ \dot{\alpha} = (\sigma - \alpha) \dfrac{\partial f^* \dot{\sigma}}{f} \end{cases} \qquad \text{for plastic element: } f(\sigma - \alpha) = \bar{\sigma} \text{ and } \partial f^* \dot{\sigma} \geqslant 0.$$

The strain-displacement relation and the initial condition are the same as in § 1. We assume that the given initial speed is smooth in x, vanishes on Γ_0 and is interpolated by $\{\varphi_p\}$.

We shall call (2.2) $\sim$ (2.4) the semidiscrete system. The first step is to seek a solution (u, σ, α) of this system. we require that σ and α are continuous with integrable first derivative with respect to t. We first show that this problem can be well posed as an initial-value problem for the unknown (u, σ, α). Then the existence of a unique solution follows at once.

Now, for any element, the elastic stress-strain relation (2.3) is taken as far as $f(\sigma)<\bar{\sigma}$ holds. Suppose that $f(\sigma)=\bar{\sigma}$ holds at $t=t_0$ for some elements. Then the plastic stress-strain relation (2.4) may or may not apply to these elements after t_0. The situation is very delicate. We can take both relations beyond $t=t_0$, if we only require a solution of a suitable initial value problem. However, what we want is a solution which is physically admissible. In other words, if we employ the plastic (resp. elastic) rule, then the obtained solution must behave like a plastic (resp. elastic) solution.

This means that, before posing the initial-value problem at $t=t_0$, we have to know the behavior of the solution in a certain neighborhood of $t=t_0$, or at least, at $t=t_0+0$. Is this however, possible? In what follows we shall show that this is in fact possible. The state of the element (i. e., elastic or plastic) at $t_0+\delta$ for small $\delta(>0)$ is determined by the information of the solution before t_0 and, of course, that of b.

Consider an element of which the stress σ just reached to the yield surface at $t=t_0$ Assume that σ is analytic in $t\,(\geqslant t_0)$. If $\partial f^*(\sigma)\dot{\sigma}|_{t_0+0}<0$ and $\alpha=\text{const.}=0$ for $t\geqslant t_0$, then there is a positive constant δ such that during the interval $I_\delta=(t_0, t_0+\delta)$, it holds that

$$f(\sigma)<\bar{\sigma}$$

If $\partial f^*(\sigma)\,\dot{\sigma}|_{t_0+0}>0$, and if α satisfies (2.4), then, for some $\delta>0$ the followings hold in I_δ.

$$\partial f^*(\sigma-\alpha)\dot{\sigma}>0, \qquad f^2(\sigma-\alpha)=\bar{\sigma}^2.$$

These will be evident, since

$$f^2(\sigma)-\bar{\sigma}^2=\int_{t_0}^{t}\frac{d}{d\tau}\,f^2(\sigma(\tau))\,d\tau=2\int_{t_0}^{t} f\,\partial f^*(\sigma)\dot{\sigma}\,d\tau$$

if $\alpha\equiv 0$, and $\partial f^*(\sigma-\alpha)(\dot{\sigma}-\dot{\alpha})=0$ otherwise. This observation suggests that for the yielded element satisfying $\partial f^*(\sigma)\dot{\sigma}|_{t_0+0}\neq 0$ we must choose its stress-strain relation for $t>t_0$ according to the sign of this quantity. (For the time being, we assume that this sign can be determined in advance.) However, if $\partial f^*(\sigma)\dot{\sigma}|_{t_0+0}=0$

this rule can not apply and the problem to pose an initial-value problem at $t=t_0$ is still pending. In this case it is needed to examine the higher derivatives of $\partial f^*(\sigma)\dot{\sigma}$. Generally, we have

LEMMA 1. *Assume that σ and α for an element satify $f(\sigma-\alpha)=\bar{\sigma}$ and $\partial f^*(\sigma-\alpha)\dot{\sigma}=0$ at $t=t_0+0$. Assume further that σ and α are analytic in $t(\geqslant t_0)$. Expand $\partial f^*(\sigma-\alpha)\dot{\sigma}$ in the Taylor series:*

$$g(t)\equiv\partial f^*(\sigma-\alpha)\dot{\sigma}=\sum_{k=1}^{\infty} g_0^{(k)}(t-t_0)^k.$$

Let $k_0(\neq\infty)$ be such integer (if it exists) that

$$g_0^{(k_0)}\neq 0, \qquad g_0^{(k)}=0 \qquad (k<k_0). \tag{2.5}$$

Then there is a positive constant δ such that in $I_\delta=(t_0, t_0+\delta)$ it holds that
(1) if $g_0^{(k_0)}<0$ and $\alpha\equiv\alpha_0=$ const., then

$$f(\sigma-\alpha_0)<\bar{\sigma}, \tag{2.6}$$

(2) if $g_0^{(k_0)}>0$ and α satisfies the second equality of (2.4), then, for some $\delta>0$, the followings hold in I_δ.

$$\partial f^*(\sigma-\alpha)\dot{\sigma}>0, \qquad f(\sigma-\alpha)=\bar{\sigma}. \tag{2.7}$$

PROOF. If the assumption of (1) is satisfied, then during a certain interval $\partial f^*(\sigma-\alpha_0)\dot{\sigma}<0$ holds, since σ is analytic. Hence the first assertion follows. (2) will be evident, since $\partial f^*(\sigma-\alpha)(\dot{\sigma}-\dot{\alpha})=0$.

This observation shows that the sign of $g_0^{(k_0)}$ is essential to determine the next state beyond $t=t_0$. We shall show below that the sign of this quantity can be in fact determined independently of the choice of the next state, for any element yielded at $t=t_0$. In what follows ∂f denotes $\partial f(\sigma-\alpha)$.
(1) The sign of $\partial f^*\dot{\sigma}|_{t_0+0}$ is independent of the choice of the next state beyond t_0. Because, if it is elastic, this quantity is equal to $\partial f^*D\dot{\varepsilon}|_{t_0+0}$ and, if plastic, to

$$\begin{aligned}\partial f^*\Big(D\dot{\varepsilon}-&\frac{D\partial f\partial f^*D}{\eta+\partial f^*D\partial f}\dot{\varepsilon}\Big)\Big|_{t_0+0}\\ &=\partial f^*D\dot{\varepsilon}|_{t_0+0}(1-\theta)\qquad\Big(\theta=\frac{\partial f^*D\partial f}{\eta+\partial f^*D\partial f}\Big).\end{aligned} \tag{2.8}$$

Hence the independence follows from the continuity of $\dot{\varepsilon}$.

For the element satisfying $\partial f^{*}\dot{\sigma}|_{t_0+0} \neq 0$, therefore, only one choice of the next state is admissible. We suppose that the next state is already chosen for this kind of elements and exclude them from the object of our consideration.

(2) Let $\mathscr{E}$ be the set of all elements of $\hat{\Omega}$ and $\mathscr{E}_0$ be the set of elements for which $\partial f^{*}\dot{\sigma}|_{t_0+0}=0$ holds in (1). It is evident that for the elements in $\mathscr{E}_0$ hold

(a) $\quad \partial f^{*}D\dot{\varepsilon}|_{t_0+0}=0,$

(b) $\quad \dot{\sigma}|_{t_0+0}$ is independent of the next state of the elements in $\mathscr{E}_0$.

For the elements of $\mathscr{E}_0$, the sign of $(\partial f^{*}\dot{\sigma})_t|_{t_0+0}$ is determined independently of the choice of the next state of the elements in $\mathscr{E}_0$. To see this, we first note that

$$(\partial f^{*}\dot{\sigma})_t|_{t_0+0} = \begin{cases} (\partial f^{*}D\dot{\varepsilon})_t|_{t_0+0} & \text{(if elastic)} \\ (\partial f^{*}D\dot{\varepsilon})_t|_{t_0+0}(1-\theta) & \text{(if plastic).} \end{cases}$$

By (b) and the continuity of $\dot{\varepsilon}$ and $\ddot{\varepsilon}$, $(\partial f^{*}D\dot{\varepsilon})_t|_{t_0+0}$ is determined independently of the next state, which proves our assertion. We thus suppose that the next state is already determined for the elements of $\mathscr{E}_0$, except those for which $(\partial f^{*}\dot{\sigma})_t|_{t_0+0}=0$ holds.

(3) To generalize this argument, let $\mathscr{E}_k$ be the set of the elements for which $f(\sigma-\alpha)=\bar{\sigma}$ holds and the sign of all derivatives $(\partial f^{*}\dot{\sigma})^{(i)}(i\leqslant k)$ are determined and vanish at $t=t_0+0$ independently of the next state of $\mathscr{E}_k$. The next state of $\mathscr{E}-\mathscr{E}_k$ is assumed to be already determined independently of that of $\mathscr{E}_k$.

Now, assume that for all elements of $\mathscr{E}_k$ hold

(A) $\quad (\partial f^{*}D\dot{\varepsilon})^{(i)}|_{t_0+0}=0 \qquad (i\leqslant k),$

(B) $\quad \sigma^{(i+1)}|_{t_0+0}(i\leqslant k)$ are determined independently of the next state of $\mathscr{E}_k$,

and another extra condition

(C) $\quad u^{(i+2)}|_{t_0+0}(i\leqslant k)$ are determined independently of the next state of $\mathscr{E}_k$.

Then, we have

(3_a) for all elements of $\mathscr{E}_k$, the sign of $(\partial f^{*}\dot{\sigma})^{(k+1)}|_{t_0+0}$ is independent of the next state of $\mathscr{E}_k$. (Hence the next state can be determined for such element of $\mathscr{E}_k$ that this quantity does not vanish.)

(3_b) for such element of $\mathscr{E}_k$ that $(\partial f^{*}\dot{\sigma})^{(k+1)}|_{t_0+0}=0$ holds (note that the totality of elements satisfying this condition is, by definition, equal to $\mathscr{E}_{k+1}$), (A), (B) and (C) hold, replacing k by $k+1$.

PROOF OF (3_a). There are two possibilities according to the choice of the next state.

$$(\partial f^*\dot{\sigma})^{(k+1)}|_{t_0+0}=\begin{cases}(\partial f^*D\dot{\varepsilon})^{(k+1)}|_{t_0+0}\\(\partial f^*D\dot{\varepsilon})^{(k+1)}|_{t_0+0}(1-\theta).\end{cases}$$

By (B) and (C), $(\partial f^*D\dot{\varepsilon})^{(k+1)}|_{t_0+0}$ is independent of the next state of $\mathscr{E}_k$.

PROOF OF (3_b). By (3_a), $(\partial f^*\dot{\sigma})^{(k+1)}|_{t_0+0}=0$ implies $(\partial f^*D\dot{\varepsilon})^{(k+1)}|_{t_0+0}=0$, which proves (A) for $k+1$. To prove (B), we note that either

$$\sigma^{(k+2)}=D\varepsilon^{(k+2)},$$

or, since $\dot{\sigma}=D\dot{\varepsilon}-D/\eta\ \partial f\cdot\partial f^*\dot{\sigma}$ in plastic state,

$$\sigma^{(k+2)}=D\varepsilon^{(k+2)}-\frac{D}{\eta}[\sum_{r=0}^{k+1}{}_{k+1}C_r(\partial f)^{(k+1-r)}(\partial f^*\dot{\sigma})^{(r)}]$$

holds for $t\geqslant t_0$. The second term of the right side of the last equation vanishes at $t=t_0+0$. Hence by (C) of the assumption, (B) holds for $k+1$. Now we have

$$(\rho u_i^{(k+3)},\ \varphi_p)+\sum_j(\sigma_{ij}^{(k+1)},\ \varphi_{p,j})=(b_i^{(k+1)},\ \varphi_p). \tag{2.9}$$

The next state for the element of $\mathscr{E}-\mathscr{E}_{k+1}$ is already determined and thus $\sigma^{(i)}|_{t_0+0}$ ($i\leqslant k+1$) for such element is independent of the next state of $\mathscr{E}_{k+1}$. The quantity $\sigma^{(k+1)}|_{t_0+0}$ for the element of $\mathscr{E}_{k+1}$ is, of course, independent of the next state of $\mathscr{E}_{k+1}$, so that (2.9) implies that $u^{(k+3)}|_{t_0+0}$ is determined independently of the next state of $\mathscr{E}_{k+1}$, as we desired.

For the set $\mathscr{E}_0$, the conditions (A), (B) and (C) hold as we proved in (2) (condition (C) follows from the continuity of $u^{(2)}$). The set $\mathscr{E}_1$ is thus well defined and hold conditions (A), (B) and (C). Then the next state will be chosen for some elements of $\mathscr{E}_1$ as we desired. This procedure can be continued untill the next state is determined for all elements, except the special case that $(\partial f^*\dot{\sigma})^{(k)}|_{t_0+0}=0$ for all k. In such case we define that the element is plastic. The reason is as follows. Let $\mathscr{E}_\infty$ be the set of elements of such pathological character. We solve the initial value problem choosing the state as above for $\mathscr{E}-\mathscr{E}_\infty$ and plastic for $\mathscr{E}_\infty$. Near t_0, the solution is analytic and behaves as plastic (resp. elastic) one, if its state is chosen as plastic (resp. elastic). This is independent of the choice of the next state for $\mathscr{E}_\infty$. Also, by the analyticity of σ, the stress σ of $\mathscr{E}_\infty$ moves (or settles) on a fixed yield surface, which meets the requirements of the solution.

REMARK. In the above discussion, we considered only the case of yielding (the change of the state from elastic to plastic). The whole results, however, are valid to the case of unloading (i. e., from plastic to elastic). Hence, considering Theorem 3 below, our initial value problem is completely posed in T and has a unique C^2-class (w. r. t. time) solution u.

REMARK. Take an arbitrary $t_0 \in T$. Then there are three cases:
(1) No change of the state occurs in a certain neighborhood of t_0 for any element.
(2) There are some elements for which the change of the state occurs at $t=t_0$.
(3) t_0 is an accumulation point of the t_0 in the case (2).

It is important that, in the case (3), there is a positive δ such that there is no point of (2) during $(t_0, t_0+\delta)$. This follows from the (piecewise) analyticity of the solution.

Before deriving the energy inequalities we shall introduce another representation of the problem, which is essentially the same to that in [1] or [2].

THEOREM 2. *Let χ^e be the characteristic function of the element e and u, ε, σ and α be of the following form.*

$$u(t)=\sum_{p\in P} u^p(t)\varphi_p, \qquad (\varepsilon,\sigma,\alpha)(t)=\sum_{e\in\mathscr{E}}(\varepsilon^e,\sigma^e,\alpha^e)(t)\chi^e.$$

Then the initial value problem of the semidiscrete system is equivalent to the following problem: Seek (u,σ,α) which is continuous with respect to t and has integrable $(u^{(3)},\dot\sigma,\dot\alpha)$ such that

(2.10) $$(\rho\ddot u_i,\varphi_p)+\sum_j(\sigma_{ij},\varphi_{p,j})=(b_i,\varphi_p) \qquad \text{for any } p\in P,$$

(2.11) $$(\dot\varepsilon-C\dot\sigma,\tau-\sigma)\leqslant 0 \qquad \text{for any } \tau\in K,$$

(2.12) $$\dot\alpha=\eta S^{-1}(\dot\varepsilon-C\dot\sigma),$$

where $C=D^{-1}$, $\sigma\in K$ and

$$K=\{\tau=\sum_e \tau^e(t)\chi^e;\ \tau^e \text{ is continuous in } T \text{ and } f(\tau-\alpha)\leqslant\bar\sigma\}.$$

S is a symmetric, positive definite matrix which connects ∂f and $\sigma-\alpha$ by $f\partial f=S(\sigma-\alpha)$. The u-ε relation and the initial value for (u,σ,α) are the same as for the semidiscrete system.

PROOF. When the plastic state continues, ∂f is a vector which is pararell to the normal of the yield surface at the stress point. Therefore, the solution of the semidiscrete system satisfies (2.10)$\sim$(2.12). It is thus only necessary to prove the uniqueness of the solution.

Substituting (2.12) into (2.11), we have $(S\dot{\alpha}, \tau-\sigma)\leqslant 0$. Here τ is written as $\tau=\alpha+\bar{\sigma}\theta$ for a suitable θ satisfying $f^2(\theta)\leqslant 1$. If another solution (u', σ', α') exists, then we have two inequalities

$$(S\dot{\alpha}, \alpha+\bar{\sigma}\theta-\sigma)\leqslant 0, \qquad (S\dot{\alpha}', \alpha'+\bar{\sigma}\theta-\sigma')\leqslant 0.$$

Replacing θ by $(\sigma'-\alpha')/\bar{\sigma}$ and $(\sigma-\alpha)/\bar{\sigma}$ in the first and the second inequality, respectively, we have

$$(S(\dot{\alpha}-\dot{\alpha}'),\ \alpha-\alpha'-[\sigma-\sigma'])\leqslant 0.$$

Hence by (2.10)

$$\eta\|\dot{u}-\dot{u}'\|_\rho^2+\eta\|\sigma-\sigma'\|_C^2+\|\alpha-\alpha'\|_S^2\leqslant 0,$$

where $\|u\|_\rho^2$ and $\|\sigma\|_A^2$ (A; matrix) denote $\sum(\rho u_i, u_i)$ and $(A\sigma, \sigma)$, respectively. This proves the uniqueness.

3. Energy inequalities for the semidiscrete system

In the previous section we was able to pose an initial value problem for the semidiscrete system. Also, we know that, for any $t\in T$, there is a $\delta>0$ such that the solution is analytic in $(t, t+\delta)$. Therefore we can easily get a priori estimates of the solution and its higher derivatives, which are independent of the triangulation.

Let (u, σ, α) be the solution of the semidiscrete system. Introduce $E_0(t)$ by

$$E_0(t)=\|\dot{u}\|_\rho^2+\frac{1}{\eta}\|\alpha\|_S^2+\|\sigma\|_C^2.$$

THEOREM 3. *$E_0(t)$ is bounded uniformly on the triangulation of Ω.*

PROOF. By Theorem 2 we have

$$\begin{aligned}0&=-(\sigma, \dot{\varepsilon})+\frac{1}{\eta}(\sigma, S\dot{\alpha})+(\sigma, C\dot{\sigma})\\ &=\frac{1}{2}\frac{d}{dt}[\|\dot{u}\|_\rho^2+\frac{1}{\eta}\|\alpha\|_S^2+\|\sigma\|_C^2]-(b, \dot{u})+\frac{1}{\eta}(\sigma-\alpha, S\dot{\alpha}).\end{aligned}$$

The last term is non-negative, because by (2.11) and (2.12) we have

$$(\sigma-\tau,\ S\dot{\alpha})\geqslant 0,$$

and we can put $\tau=\alpha$ in this inequality. Since $(\dot{u},\sigma,\alpha)$ is continuous, the boundedness follows from the Gronwall's lemma.

To derive an estimate of the higher derivatives we prepare

LEMMA 4. *Assume that at $t=t_0$ the stress point σ leaves the yield surface $\{\tau\in E^3;\ f^2(\tau-\alpha)=\bar{\sigma}^2\}$. Then it holds that*

$$\partial f^*\dot{\sigma}=0 \qquad \text{at } t=t_0\pm 0.$$

PROOF. There are three cases.

(1). In a certain neighborhood of t_0, t_0 is the only t such that $f(\sigma(t)-\alpha(t))=\bar{\sigma}$ holds. In this case we have

$$0>\partial f^*|_{t_0}[\sigma(t_0\pm\delta)-\sigma(t_0)]$$

for small $\delta>0$. Therefore we have

$$\partial f^*\dot{\sigma}|_{t_0+0}=\partial f^*D\dot{\varepsilon}|_{t_0+0}\leqslant 0, \tag{3.1}$$

$$\partial f^*\dot{\sigma}|_{t_0-0}=\partial f^*D\dot{\varepsilon}|_{t_0-0}\geqslant 0. \tag{3.2}$$

Hence, by the contunuity of $\dot{\varepsilon}$, $\partial f^*\dot{\sigma}$ has to vanish at $t=t_0\pm 0$.

(2). The element is plastic for $t\in(t_0-\delta,t_0)$ and elastic for $t\in(t_0,t_0+\delta)$ $(\delta>0)$. In this case, we remember that $\partial f^*\dot{\sigma}\geqslant 0$ holds in plastic state. Assume that

$$\partial f^*\dot{\sigma}|_{t_0-0}>0. \tag{3.3}$$

The stress-strain relation in plastic state is given by

$$\dot{\sigma}=\left(D-\frac{D\partial f\partial f^*D}{\eta+\partial f^*D\partial f}\right)\dot{\varepsilon}.$$

Therefore, since

$$\partial f^*\dot{\sigma}=\partial f^*D\dot{\varepsilon}\left(1-\frac{\partial f^*D\partial f}{\eta+\partial f^*D\partial f}\right),$$

(3.3) implies $\partial f^*D\dot{\varepsilon}>0$ at $t=t_0-0$. On the other hand, by (3.1) $\partial f^*D\dot{\varepsilon}$ must be non-positive at $t=t_0+0$. This contradicts to the fact that $\dot{\varepsilon}$ is continuous. Hence $\partial f^*\dot{\sigma}|_{t_0-0}=0$ and thus $\partial f^*\dot{\sigma}|_{t_0+0}=0$.

(3). The case that t_0 is an accumulation point of t at which the state change occurs for some elements. We first note that the element is elastic for $t\in(t_0,t_0+\delta)$

for some $\delta>0$, since σ leaves the yield surface at $t=t_0$. Hence $\partial f^*\dot{\sigma}|_{t_0+0}=\partial f^*D\dot{\varepsilon}|_{t_0+0}$ holds. Now, by the assumption, the unloading points accumulate to t_0. Therefore $\partial f^*D\dot{\varepsilon}\to 0$ as $t\to t_0-0$. This implies $\partial f^*\dot{\sigma}\to 0$ as $t\to t_0-0$, i. e., $\partial f^*\dot{\sigma}|_{t_0-0}=\partial f^*D\dot{\varepsilon}|_{t_0-0}=0$, and thus $\partial f^*\dot{\sigma}|_{t_0+0}=0$ by the continuity of $\dot{\varepsilon}$. The lemma is thus proved.

THEOREM 5. *The quantities*

$$E_1(t\pm 0)=[\|\ddot{u}\|_\rho^2+\frac{1}{\eta}\|\dot{\alpha}\|_S^2+\|\dot{\sigma}\|_C^2](t\pm 0)$$

are uniformly bounded on the triangulation of Ω.

PROOF. Take an interval $T_i=(t_0, t_1)$ on which no change of the state occurs for any element. In T_i, we have

$$\frac{1}{2}\frac{d}{dt}[\|\ddot{u}\|_\rho^2+\frac{1}{\eta}\|\dot{\alpha}\|_S^2+\|\dot{\sigma}\|_C^2]-(\dot{b}, \ddot{u})+\frac{1}{\eta}(\dot{\sigma}-\dot{\alpha}, S\ddot{\alpha})=0.$$

The last term is non-negative. To see this, we note that for each element holds $\ddot{\alpha}=0$ or

$$S\ddot{\alpha}=\frac{d}{dt}\partial f\cdot\partial f^*\dot{\sigma}+\partial f\frac{d}{dt}(\partial f^*\dot{\sigma}).$$

In this case we have the identity

$$\frac{d}{dt}\partial f=\frac{1}{\bar{\sigma}}S(\dot{\sigma}-\dot{\alpha}).$$

Hence, taking into account $\partial f^*(\dot{\sigma}-\dot{\alpha})=0$ in plastic state, we get

$$(\dot{\sigma}-\dot{\alpha})^*S\ddot{\alpha}=\frac{1}{\bar{\sigma}}(\dot{\sigma}-\dot{\alpha})^*S(\dot{\sigma}-\dot{\alpha})\cdot\partial f^*\dot{\sigma}\geqslant 0.$$

Therefore the following inequality holds.

$$2\int_{t_0}^{t_1}(\dot{b}, \ddot{u})dt+E_1(t_0+0)\geqslant E_1(t_1-0). \tag{3.4}$$

In what follows we show $E_1(t_0-0)\geqslant E_1(t_0+0)$ for any $t_0\in T$.

By $E_1^e(t)$ we denote the restriction of $E_1(t)$ on the element e, that is, $E_1(t)=\sum_e E_1^e(t)$.

(1) Assume that t_0 is not the accumulation point of such t that the state change occurs at t for some elements. Let T_{i-1} be the preceeding interval on which no

change of the state occurs. t_0 is hence the right and the left coordinate of T_{i-1} and T_i, respectively.

(a) The case that e is elastic on T_{i-1} and plastic on T_i. On $T_i \times e$, it holds that

$$\|\dot\alpha\|_S^2 = (\partial f, S^{-1}\partial f)(\partial f^* \dot\sigma)^2 = (\partial f^* \dot\sigma)^2 \; area(e),$$

$$\begin{aligned}(C\dot\sigma, \dot\sigma) &= (\dot\varepsilon - \frac{1}{\eta} S\dot\alpha, \dot\sigma) = (\dot\varepsilon, \dot\sigma) - \frac{1}{\eta}(S\dot\alpha, \dot\sigma)\\ &= (\dot\varepsilon, (D-D')\dot\varepsilon) - \frac{1}{\eta}(\partial f^* \dot\sigma)^2 \; area(e),\end{aligned}$$

so that on the same region holds

$$\frac{1}{\eta}\|\dot\alpha\|_S^2 + \|\dot\sigma\|_C^2 = (\dot\varepsilon, (D-D')\dot\varepsilon).$$

Therefore, taking into account the fact $u \in C^2(T)$, we have

$$\begin{aligned}E_1^e(t_0+0) &= \|\ddot u\|_\rho^2 (t_0+0) + (\dot\varepsilon, \; (D-D')\dot\varepsilon)(t_0+0)\\ &= \|\ddot u\|_\rho^2 (t_0-0) + (\dot\varepsilon, \; (D-D')\dot\varepsilon)(t_0-0)\\ &\leqslant \|\ddot u\|_\rho^2 (t_0-0) + (\dot\varepsilon, \; D\dot\varepsilon)(t_0-0) = E_1^e(t_0-0).\end{aligned}$$

(b) The case that e is plastic on T_{i-1} and elastic on T_i. By Lemma 4 it holds that $\partial f^* \dot\sigma = 0$ at $t = t_0 \pm 0$. Therefore, on e $\|\dot\alpha\|(t_0-0) = 0$ and $\dot\sigma(t_0-0) = D\dot\varepsilon(t_0-0) = D\dot\varepsilon(t_0+0) = \dot\sigma(t_0+0)$, so that

$$\begin{aligned}E_1^e(t_0+0) &= [\|\ddot u\|_\rho^2 + \|\dot\sigma\|_C^2](t_0+0)\\ &= [\|\ddot u\|_\rho^2 + \frac{1}{\eta}\|\dot\alpha\|_S^2 + \|\dot\sigma\|_C^2](t_0-0)\\ &= E_1^e(t_0-0).\end{aligned}$$

(c) Assume finally that t_0 is an accumulation point. Samely as in the proof of Lemma 4, it is proved that $\partial f^* \dot\sigma = \partial f^* D\dot\varepsilon = 0$ at $t = t_0 \pm 0$. Hence we have

$$E_1^e(t_0-0) = [\|\ddot u\|_\rho^2 + (\dot\varepsilon, D\dot\varepsilon)](t_0-0).$$

If the element is elastic for $t > t_0$, clearly $E_1^e(t_0+0) = E_1^e(t_0-0)$. If it is plastic, then

$$\begin{aligned}E_1^e(t_0+0) &= [\|\ddot u\|_\rho^2 + (\dot\varepsilon, (D-D')\dot\varepsilon)](t_0+0)\\ &= [\|\ddot u\|_\rho^2 + (\dot\varepsilon, D\dot\varepsilon)](t_0+0)\\ &= E_1^e(t_0-0).\end{aligned}$$

By the above observation it is proved that in any situation the function $E_1^e(t)$ is non-increasing at $t = t_0$. Therefore we have by (3.4)

$$2\int_{t_0}^{t_1}(\dot{b},\ddot{u})dt+E_1(t_0-0)\geqslant E_1(t_1-0),$$

and finally

$$2\int_0^t(\dot{b},\ddot{u})dt+E_1(0)\geqslant E_1(t-0)\geqslant E_1(t+0)$$

for any $t\in T$, from which the boundedness of $E_1(t\pm 0)$ follows.

4. An existence theorem

The solution of the semidiscrete system converges to the solution of (a weak form) of the original system (1.1)～(1.3). Although the proof is standard, we shall sketch the outline. Let $\mathscr{D}_2^1(\Omega,\Gamma_0)$ be the $W_2^1(\Omega)$ completion of the set of functions belonging to $C^\infty(\bar{\Omega})$ and vanishing on Γ_0 (in what follows, $C^\infty(\bar{\Omega},\Gamma_0)$ denote this set of C^∞ functions). When $\alpha\in L^\infty(T;L^2(\Omega))$, we define $K=K_\alpha$ by

$$K=\{\tau\in L^\infty(T;\ L^2(\Omega));\ \text{a. e. on } T,\ f^2(\tau-\alpha)\leqslant\bar{\sigma}^2 \text{ a. e. on } \Omega\}.$$

THEOREM 6. *The following problem has a unique solution, which is a limit of the solution of the semidiscrete system: Seek* $(u,\sigma,\alpha)\in L^\infty(T;\ L^2(\Omega))$ *satisfying the following conditions a. e. on T.*

$$(4.1)\qquad (\rho\ddot{u}_i,\varphi)+\sum_j(\sigma_{ij},\varphi_{,j})=(b_i,\varphi)\qquad \textit{for all } \varphi\in\mathscr{D}_2^1(\Omega,\Gamma_0),$$

$$(4.2)\qquad (\dot{\varepsilon}-C\dot{\sigma},\tau-\sigma)\leqslant 0\qquad \textit{for all } \tau\in K,$$

$$(4.3)\qquad \dot{\alpha}=\eta S^{-1}(\dot{\varepsilon}-C\dot{\sigma}),$$

where $\sigma\in K$ *and* $(u,\dot{u})\in L^\infty(T;\ \mathscr{D}_2^1(\Omega,\Gamma_0))$, $(\ddot{u},\dot{\sigma},\dot{\alpha})\in L^\infty(T;\ L^2(\Omega))$. *The u-ε relation and the initial condition are the same as for the system* (1.1)～(1.3).

PROOF. Let $(\tilde{u},\tilde{\sigma},\tilde{\alpha})$ be the solution of the semidiscrete system. These functions thus satisfy the following conditions.

$$(4.4)\qquad (\rho\ddot{\tilde{u}}_i,\varphi_p)+\sum_j(\sigma_{ij},\varphi_{p,j})=(b_i,\varphi_p)\qquad \text{for all } p\in P,$$

$$(4.5)\qquad (\dot{\tilde{\varepsilon}}-C\dot{\tilde{\sigma}},\ \tilde{\tau}-\tilde{\sigma})\leqslant 0\qquad \text{for all } \tilde{\tau}\in\tilde{K},$$

$$(4.6)\qquad \dot{\tilde{\alpha}}=\eta S^{-1}(\dot{\tilde{\varepsilon}}-C\dot{\tilde{\sigma}}),$$

where

$$\tilde{K}=\{\tilde{\tau}=\sum_e \tau^e(t)\chi^e;\ \tau^e(t) \text{ is continuous in } T \text{ and } f^2(\tau^e-\tilde{\alpha}^e)\leqslant\bar{\sigma}^2\}.$$

Now, by the a priori estimates derived in the preceeding section, we have

$$\left.\begin{matrix}\|\dot{\tilde{u}}\|,\ \|\tilde{\sigma}\|,\ \|\tilde{\alpha}\| \\ \|\ddot{\tilde{u}}\|,\ \|\dot{\tilde{\sigma}}\|,\ \|\dot{\tilde{\alpha}}\|\end{matrix}\right\} \text{ uniformly bounded in } L^\infty(T;\ L^2(\Omega)),$$

as the size of the triangles tends to zero. Also the boundedness of $\|\tilde{\varepsilon}\|$ and $\|\dot{\tilde{\varepsilon}}\|$ implies the boundedness of $\|\tilde{u}_{i,j}\|$ and $\|\dot{\tilde{u}}_{i,j}\|$ by the Korn's inequality. Hence there are $(u,\dot{u})\in L^\infty(T;\ \mathscr{D}_2^1(\Omega,\Gamma_0))$ and $(\varepsilon,\sigma,\alpha)\in L^\infty(T;L^2(\Omega))$ such that, for a suitable subsequence,

$$\left.\begin{matrix}\ddot{\tilde{u}}\longrightarrow\ddot{u} \\ \tilde{\varepsilon},\tilde{\sigma},\tilde{\alpha}\longrightarrow\varepsilon,\sigma,\alpha \\ \dot{\tilde{\varepsilon}},\dot{\tilde{\sigma}},\dot{\tilde{\alpha}}\longrightarrow\dot{\varepsilon},\dot{\sigma},\dot{\alpha}\end{matrix}\right\} \text{ weakly* in } L^\infty(T;\ L^2(\Omega))$$

Clearly u, σ and α satisfy the initial conditions. To prove (4.1)$\sim$(4.3) and $\sigma\in K$, we start from (4.1). Take an arbitrary $\varphi\in C^\infty(\overline{T\times\Omega})$ such that $\varphi|_{\Gamma_0}=\varphi|_{t=0}=\varphi|_{t=T}=0$. Since φ can be approximated by $\{\varphi_p\}$ in the norm of $W_2^1(T\times\Omega)$, we have

$$\int_0^T[\rho\ddot{u}_i,\varphi)+\sum_j(\sigma_{ij},\varphi_{,j})]dt=\int_0^T(b_i,\varphi)dt. \tag{4.7}$$

For $t\in T$ define $T_\delta=[t-\delta/2,t+\delta/2]$ $(\delta>0)$. Take $\bar{\varphi}\in C^\infty(\bar{\Omega},\Gamma_0)$ and put

$$\psi=\begin{cases}\bar{\varphi} & \text{in } T_\delta,\\ 0 & \text{otherwise.}\end{cases}$$

Then it is easy to see that, for small δ, the equality (4.7) holds for ψ. Hence, deviding the both sides by δ and using Lebesgue's theorem we have, a. e. on T,

$$(\rho\ddot{u}_i,\bar{\varphi})+\sum_j(\sigma_{ij},\bar{\varphi}_{,j})=(b_i,\bar{\varphi})\qquad\text{for all } \bar{\varphi}\in C^\infty(\bar{\Omega},\Gamma_0).$$

(4.1) is thus proved by passing to the limit in this equality.

The equality (4.3) is proved by passing to the limit in (4.6). To prove (4.5), we first note that $\tau\in K$ can be written in the form

$$\tau=\alpha+\bar{\sigma}\theta$$

for a suitable $\theta\in L^\infty(T;\ L^2(\Omega))$, where θ satisfies a. e. on T $f^2(\theta)\leq\bar{\sigma}^2$ a. e. on Ω. For this θ we can find $\tilde{\theta}=\sum_e\theta^e(t)\chi^e\in\tilde{K}$ such that

$$\|\theta-\tilde{\theta}\|_{L^2(T\times\Omega)}\to 0.$$

To constructuct such $\tilde{\theta}$, mollify θ and interpolate the resulting function by using $\{\chi^e\}$. Substituting $\tilde{\tau}=\tilde{\alpha}+\bar{\sigma}\tilde{\theta}$ into (4.5) and integrating with respect to t, we have

$$0\geqslant\frac{1}{2\eta}\|\tilde{\alpha}\|_S^2(T)+\frac{\bar{\sigma}}{\eta}\int_0^T(S\dot{\tilde{\alpha}},\tilde{\theta})\,dt-\int_0^T(\dot{\tilde{\varepsilon}}-C\dot{\tilde{\sigma}},\tilde{\sigma})\,dt.$$

The last term of the right side is written as follows by (4.1).

$$-\int_0^T(\dot{\tilde{\varepsilon}}-C\dot{\tilde{\sigma}},\tilde{\sigma})dt=\frac{1}{2}\|\dot{\tilde{u}}\|^2(T)+\frac{1}{2}\|\tilde{\sigma}\|_C^2(T)-\int_0^T(b,\dot{\tilde{u}})dt.$$

Hence, if necessary taking a subsequence in passing to the $\underline{lim}$, we have

$$\begin{aligned}0\geqslant\frac{1}{2\eta}\|\alpha\|_S^2(T)+\frac{\bar{\sigma}}{\eta}\int_0^T(S\dot{\alpha},\theta)\,dt+\frac{1}{2}\|\dot{u}\|_\rho^2(T)+\frac{1}{2}\|\sigma\|_C^2(T)\\-\int_0^T(b,\dot{u})\,dt\\=\frac{1}{\eta}\int_0^T(S\dot{\alpha},\tau-\sigma)\,dt=\int_0^T(\dot{\varepsilon}-C\dot{\sigma},\tau-\sigma)\,dt.\end{aligned}$$

(4.2) is thus proved in the same way as before. Finally the proof of $\sigma\in K$. Let $f(\tau,\varphi)$ be defined by

$$f(\tau,\varphi)=\tau_{11}\varphi_{11}+\tau_{22}\varphi_{22}-\frac{1}{2}(\tau_{11}\varphi_{22}+\tau_{22}\varphi_{11})+3\tau_{12}\varphi_{12}.$$

Then $f^2(\tau)=f(\tau,\tau)$, and $(\int_\Omega f^2(\tau)dx)^{1/2}$ and $\|\tau\|$ are equivalent as the norm in $L^2(\Omega)$. Let χ be the characteristic function of an arbitrary measurable set in $T\times\Omega$. If $\tau\in L^\infty(T;\,L^2(\Omega))$, then clearly $\chi\tau$ is so too. Since $\chi(\tilde{\sigma}-\tilde{\alpha})$ is regarded also as a weakly convergent sequence in $L^2(T\times\Omega)$, we have

$$0\geqslant\underline{lim}\int_0^T\int_\Omega\chi\,[f^2(\tilde{\sigma}-\tilde{\alpha})-\bar{\sigma}^2]\,dxdt\geqslant\int_0^T\int_\Omega\chi\,[f^2(\sigma-\alpha)-\bar{\sigma}^2]\,dxdt$$

and thus

$$f^2(\sigma-\alpha)\leqslant\bar{\sigma}^2\qquad\text{a. e. on } T\times\Omega,$$

from which $\sigma\in K$ follows. Therefore (u,σ,α) is a solution of the present problem. The uniqueness of the solution is proved by just the same way as for the semi-discrete system. Hence the whole sequence $(\tilde{u},\tilde{\sigma},\tilde{\alpha})$ converges to the solution. This completes the proof.

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