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JOINT ACTION AND TYPES OF TENSOR PRODUCTS OF VON NEUMANN ALGEBRAS*

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1. Introduction.

In the theory of operator algebras it is interesting and important to study the tensor products of von Neumann algebras. It is not only interesting for the study of the tensor product von Neumann algebras themselves but also indispensable for the elucidation of the structure of von Neumann algebras which is one of the important problems in the theory of operator algebras. There are some methods to construct von Neumann algebras from given von Neumann algebras. One of those is the method by virtue of the tensor products. The study of more complicated von Neumann algebras constructed by these methods helps us to elucidate the structure of general von Neumann algebras. Now, the recent developments in the theory of operator algebras show the importance of the study of the group actions on von Neumann algebras ([3], [4], [12], [15], [16], etc.). A group action is a continuous action of a locally compact abelian group on a von Neumann algebra. The Tomita's modular automorphism group is a group action which is a one-parameter automorphism group induced by the modular operator and plays important roles in the study of von Neumann algebras of type III. For example, Connes gave the classification of factors of type III to those of type III_λ $(0 \leq \lambda \leq 1)$ by using the spectral analysis of the modular automorphism groups, and Takesaki showed the structure theorem by which a factor of type III was determined by the pair of a semi-finite von Neumann algebra and the dual action of the modular automorphism group. On the other hand, as well known, the theories of operator algebras and ergodic transformation groups are closely related. Many important examples of von Neumann algebras were given by virtue of the group measure space construction and conversely the study of general von Neumann algebras gave some methods to the study of ergodic non-singular transformation groups. For example, Krieger gave the weak equivalence classification of ergodic non-singular transformations without σ-finite invariant equivalent measures and proved that the weak equivalence between ergodic non-singular transformations is equivalent to the isomorphism between the correspond-

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ing von Neumann algebras by the group measure space construction ([10]), and Krieger's classification was again obtained by the associated flows ([7]). They are closely related to the classification of factors of type III by Connes, and the dual actions of the modular automorphism groups on factors, respectively. Thus the study of group actions, their dual actions and their spectral analysis is indispensable and useful in the theory of operator algebras.

In this paper we discuss the tensor products of continuous actions on von Neumann algebras and the type classification of the tensor products of von Neumann algebras. The latter is well known about the classification of the tensor product von Neumann algebras in the frame of the von Neumann's classification to the classes of types I, II or III. One of our purposes is to classify the tensor product von Neumann algebras in the frame of the Connes' classification to the classes of types III_λ $(0\leq\lambda\leq 1)$. The Connes' classification, which is due to the Tomita-Takesaki's theory, is finer than the von Neumann's one for the class of type III. Firstly it is necessary to discuss the tensor products of continuous group actions, their dual actions and their spectral analysis. More precisely, let α^1 and α^2 be continuous actions of a locally compact abelian group G on von Neumann algebras M_1 and M_2, respectively, and $\alpha^1\otimes\alpha^2$ the tensor product of α^1 and α^2 which is a continuous action of G on the tensor product $M_1\otimes M_2$ of M_1 and M_2 defined by $(\alpha^1\otimes\alpha^2)_t=\alpha^1_t\otimes\alpha^2_t$ for each t in G. Let $\hat{\alpha}^1$ and $\hat{\alpha}^2$ be the dual actions of α^1 and α^2, which are continuous actions of the dual group $\hat{G}$ of G on the crossed products $M_1\times_{\alpha^1}G$ and $M_2\times_{\alpha^2}G$, respectively. The joint action $(\hat{\alpha}^1,\hat{\alpha}^2)$ is a continuous action of $\hat{G}$ on $(N_1\otimes N_2)^{\hat{\alpha}}$ defined by $(\hat{\alpha}^1,\hat{\alpha}^2)_p=\hat{\alpha}^1_p\otimes\iota$, where $N_j=M_j\times_{\alpha j}G$ and $\hat{\alpha}_p=\hat{\alpha}^1_p\otimes\hat{\alpha}^2_{-p}$. Then we show that the dual action of $\alpha^1\otimes\alpha^2$ is given by the joint action of $\hat{\alpha}^1$ and $\hat{\alpha}^2$. Also, we show that if we assume the relative commutant property for α^1 and α^2, then the Connes spectrum $\Gamma(\alpha^1\otimes\alpha^2)$ of $\alpha^1\otimes\alpha^2$ is given as the kernel of the joint action of the dual actions $\hat{\alpha}^1$ and $\hat{\alpha}^2$ on the center of $(N_1\otimes N_2)^{\hat{\alpha}}$. From this follows that we can classify the tensor products of von Neumann algebras by the joint actions of the dual actions of the modular automorphism groups, since in particular the S-set of a von Neumann algebra is the Connes spectrum for the modular automorphism group associated with a faithful normal semi-finite weight. The same result is obtained for a smooth flow of weights as follows: the virtual spectrum $S_v(M_1\otimes M_2)$ is the closure of the product of $S_v(M_1)$ and $S_v(M_2)$ ([4]). Now, it is known that if M is a factor of type $III_\lambda(0<\lambda<1)$, or of type III_1, respectively, then the dual action of the modular automorphism group σ^φ has pure point spectrum on the center of the crossed product of M by the modular automorphism group, and on the other hand that there exists a factor of type III_0 such that the dual action of the modular automorphism group has pure point spectrum on the center of the crossed product of M by the modular automorphism group. We discuss the tensor products of such more general covariant systems and as its application examine the type classification of the tensor products of von Neumann algebras. For this purpose we introduce the notion of the joint action of continuous group actions and define a new type of von Neumann algebras of type III. This plays the important roles in our discussion.

In section 2 we recall fundamental terminologies and notations about a continuous group action, the crossed product von Neumann algebra, the dual action, and the Connes spectrum. In section 3 we use the notion of the joint action to show that the dual action of the tensor product of continuous actions is given as the joint action of their dual actions. In section 4 we discuss the joint action with pure point spectrum and define a new type of von Neumann algebras of type III. And then we apply the results in the previous sections to the type classification of the tensor products of von Neumann algebras. In the last section we discuss the joint flows directly in the case of ergodic non-singular transformation groups, which are closely related to our joint actions in the case of the corresponding von Neumann algebras given by the group measure space construction. By our discussion we see again that it is important to study ergodic non-singular transformation groups themselves for the development of the theory of operator algebras.

2. Preliminaries.

Let $B(\mathscr{H})$ be the algebra of all bounded linear operators on a complex Hilbert space $\mathscr{H}$. By a von Neumann algebra M we mean a σ-weakly closed self-adjoint subalgbera containing the identity operator $I_{\mathscr{H}}$ of the algebra $B(\mathscr{H})$ and by the predual M_* of a von Neumann algebra M the Banach space of all σ-weakly continuous linear functionals on M. Let $\mathrm{Aut}\,M$ be the group of all $*$-automorphisms of a von Neumann algebra M and G a locally compact abelian group. In this paper we assume that von Neumann algebras are σ-finite and locally compact abelian groups are separable. Then a homomorphism α of G into $\mathrm{Aut}\,M$ is called a continuous action of G on M if the map: $t \in G \rightarrow \alpha_t(x) \in M$ is σ-weakly continuous for each x in M. The pair $\{M, \alpha\}$ is sometimes called a covariant system on G. We denote by M^{α} the fixed point subalgebra of M under the action α, that is, $M^{\alpha}=\{x \in M;\ \alpha_t(x)=x$ for all $t \in G\}$. In the vector space

$K(\mathscr{H}; G)$ of all continuous $\mathscr{H}$-valued functions on G with compact support, we define the inner product by

$$(\xi | \eta) = \int_G (\xi(t) | \eta(t)) dt,$$

for ξ, η in $K(\mathscr{H}; G)$, where dt is the Haar measure of G. We denote by $L^2(\mathscr{H}; G)$ a Hilbert space which is the completion of the pre-Hilbert space $K(\mathscr{H}; G)$ with respect to this inner product. On the Hilbert space $L^2(\mathscr{H}; G)$, we define representations π_α of M and λ of G as follows

$$(\pi_\alpha(x)\xi)(s) = \alpha_s^{-1}(x)\xi(s),$$
$$(\lambda(t)\xi)(s) = \xi(s-t),$$

for x in M, s, t in G, and ξ in $L^2(\mathscr{H}; G)$. Then π_α is a faithful normal representation and

$$\lambda(t)\pi_\alpha(x)\lambda(t)^* = \pi_\alpha(\alpha_t(x)),\ x \in M,\ t \in G.$$

We denote by $M \times_\alpha G$ the von Neumann algebra on $L^2(\mathscr{H}; G)$ generated by $\pi_\alpha(M)$ and $\lambda(G)$, and call it the crossed product of M by G with respect to the action α, or simply the crossed product of M by the action α of G ([12]). Apparently, the crossed product $M \times_\alpha G$ depends also on the underlying Hilbert space $\mathscr{H}$. However, it is known that the algebraic structure of $M \times_\alpha G$ is independent of the Hilbert space.

Let $\hat{G}$ be the dual group of G, and we define a unitary representation μ of $\hat{G}$ on $L^2(\mathscr{H}; G)$ by

$$(\mu(p)\xi)(s) = \overline{\langle s, p \rangle}\xi(s),\ \xi \in L^2(\mathscr{H}; G),\ s \in G,\ p \in \hat{G},$$

where $\langle s, p \rangle$ denotes the value of p in $\hat{G}$ at s in G. Then the map: $p \to Ad\mu(p)$ induces a continuous action of $\hat{G}$ on $M \times_\alpha G$ which is called the dual action of $\hat{G}$ on $M \times_\alpha G$ and sometimes denoted by $\hat{\alpha}$, where $Ad\mu(p) = \mu(p) \cdot \mu(p)^*$. By Takesaki's duality theorem, the crossed product $(M \times_\alpha G) \times_{\hat{\alpha}} \hat{G}$ of $M \times_\alpha G$ by the dual action $\hat{\alpha}$ of the dual group $\hat{G}$ of G is isomorphic to the tensor product of M and the factor $B(L^2(G))$ of type I of all bounded operators on $L^2(G)$, and under this isomorphism the second dual action $\hat{\hat{\alpha}}$ of G on $(M \times_\alpha G) \times_{\hat{\alpha}} \hat{G}$ is equivalent to the action $\tilde{\alpha}$ of G on $M \otimes B(L^2(G))$, where $\tilde{\alpha}_t = \alpha_t \otimes Adv(t)^*$ and $(v(t)\xi)(s) = \xi(s-t)$, $(t, s \in G, \xi \in L^2(G))$.

If α is a continuous action of G on M, then the spectrum $sp(\alpha)$ is the intersection of all kernels $\{p \in \hat{G}; \hat{f}(p) = 0\}$ of the Fourier transform $\hat{f}$ of f in $L^1(G)$

with $\alpha_f=\int_G f(t)\alpha_t dt=0$. For each projection e in M^α we denote by α^e the restriction of α to the reduced von Neumann algebra M_e. The Connes spectrum $\Gamma(\alpha)$ of α is defined by

$$\Gamma(\alpha)=\bigcap\{sp(\alpha^e);\ e\in M^\alpha,\ e\neq 0\}.$$

Then the Connes spectrum $\Gamma(\alpha)$ of α coincides with the kernel of the action $\hat{\alpha}|C_{M\times_\alpha G}$ which is the restriction of the dual action $\hat{\alpha}$ of α to the center $C_{M\times_\alpha G}$ of $M\times_\alpha G$, that is,

$$\begin{aligned}\Gamma(\alpha)&=Ker\ \hat{\alpha}|C_{M\times_\alpha G}\\&=\{p\in\hat{G};\hat{\alpha}_p=\iota \text{ on } C_{M\times_\alpha G}\}.\end{aligned}$$

The Connes spectrum for the modular automorphism group is the S-set $S(M)$ of M, more precisely, $\Gamma(\sigma^\varphi)=\{\log\lambda;\ \lambda\in S(M),\ \lambda\neq 0\}$ for a faithful normal semi-finite weight φ on M.

If α^1 and α^2 are continuous actions of the same locally compact abelian group G on von Neumann algebras M_1 and M_2, respectively, we denote

$$(\alpha^1\otimes\alpha^2)_t=\alpha_t^1\otimes\alpha_t^2 \text{ and } (\alpha^1\times\alpha^2)_{(s,t)}=\alpha_s^1\otimes\alpha_t^2.$$

The former $\alpha^1\otimes\alpha^2$ is a continuous action of G on $M_1\otimes M_2$ which is called the tensor product of α^1 and α^2, and the latter $\alpha^1\times\alpha^2$ is a continuous action of $G\times G$ on $M_1\otimes M_2$. Two covariant systems $\{M_1,\alpha^1\}$ and $\{M_2,\alpha^2\}$ on G are said to be equivalent if there exists an isomorphism γ of M_1 onto M_2 such that $\gamma\circ\alpha_t^1\circ\gamma^{-1}=\alpha_t^2$ for all t in G.

3. Dual action of the tensor product of continuous actions.

Let α^1 and α^2 be continuous actions of a locally compact abelian group G on von Neumann algebras M_1 and M_2, respectively. In this section, we shall give the dual action of the tensor product $\alpha^1\otimes\alpha^2$ of α^1 and α^2 by virtue of the dual actions $\hat{\alpha}^1$ and $\hat{\alpha}^2$ of α^1 and α^2, respectively, to study the tensor product $M_1\otimes M_2$ of von Neumann algebras M_1 and M_2.

Connes ([3]) introduced the S-set $S(M)$ of a von Neumann algebra M which is the Connes spectrum $\Gamma(\sigma^\varphi)$ for the modular automorphism group σ^φ associated with a faithful normal semi-finite weight φ on M and classified factors of type III into those of type $\mathrm{III}_\lambda(0\leq\lambda\leq 1)$. On the other hand, the Connes spectrum $\Gamma(\alpha)$ of a continuous action α of a locally compact abelian group G on M is given as the kernel $Ker\ \hat{\alpha}|C_{M\times_\alpha G}$ of the action $\hat{\alpha}|C_{M\times_\alpha G}$ which is the restriction of the dual action $\hat{\alpha}$ of α to the center $C_{M\times_\alpha G}$ of the crossed product $M\times_\alpha G$ of M by the action α of G ([4]). Since, in particular, the S-set $S(M)$ of M is given by the kernel of the action $\tilde{\theta}$ of the additive group $\boldsymbol{R}$ of real numbers on the center $C_{M\times_{\sigma\varphi}R}$ which is the restriction of the dual action θ of σ^φ to the center $C_{M\times_{\sigma\varphi}R}$ of the crossed product $M\times_{\sigma\varphi}\boldsymbol{R}$ of M by the action σ^φ of $\boldsymbol{R}$ if the action α is the modular automorphism group σ^φ for a faithful normal semi-finite weight φ, we can classify von Neumann algebras by the dual actions of the modular automorphism groups.

Let β^1 and β^2 be continuous actions of a locally compact abelian group Γ on von Neumann algebras N_1 and N_2, respectively, and β a continuous action of Γ on the tensor product $N_1\otimes N_2$ defined by $\beta_p=\beta^1_p\otimes\beta^2_{-p}$ for p in Γ. Since the action: $p\rightarrow\beta^1_p\otimes\iota$ commutes with the action β on $N_1\otimes N_2$, the action: $p\rightarrow\beta^1_p\otimes\iota$ induces a continuous action of Γ on $(N_1\otimes N_2)^\beta$, where ι is the trivial action. We note that $\beta^1_p\otimes\iota=\iota\otimes\beta^2_p$ on $(N_1\otimes N_2)^\beta$.

DEFINITION 1. We denote by (β^1,β^2) the continuous action of Γ on $(N_1\otimes N_2)^\beta$ induced by the action: $p\rightarrow\beta^1_p\otimes\iota$ and call it *the joint action of* β^1 *and* β^2.

Firstly we show that the dual action $(\alpha^1\otimes\alpha^2)^\wedge$ of the tensor product $\alpha^1\otimes\alpha^2$ of continuous actions α^1 and α^2 is given as the joint action $(\hat{\alpha}^1,\hat{\alpha}^2)$ of the dual actions $\hat{\alpha}^1$ and $\hat{\alpha}^2$ of α^1 and α^2, respectively.

Let α be a continuous action of a locally compact abelian group G on a von Neumann algebra M on a Hilbert space $\mathscr{H}$ and H a closed subgroup of G. We denote by α^H the restriction of α to H and by $M\times_\alpha H$ the von Neumann subalgebra of $M\times_\alpha G$ generated by $\pi_\alpha(M)$ and $\lambda(H)$. Then $M\times_{\alpha H}H$ is isomorphic to $M\times_\alpha H$ by the correspondence:

$$\begin{cases}\pi_{\alpha H}(x)\longrightarrow\pi_\alpha(x)\\ \lambda^H(t)\longrightarrow\lambda(t),\end{cases}$$

where λ^H is the unitary representation of H on $L^2(\mathscr{H};H)$ defined by $(\lambda^H(t)\xi)(s)=\xi(s-t)$ for ξ in $L^2(\mathscr{H};H)$ ([14]). Moreover,

$$M\times_\alpha H=\{y\in M\times_\alpha G;\ \hat{\alpha}_p(y)=y \text{ for all } p\in H^\perp\},$$

where $H^\perp=\{p\in\hat{G};\ \langle t,p\rangle=1 \text{ for all } t\in H\}$ ([16]).

THEOREM 1. *Let* α^1 *and* α^2 *be continuous actions of a locally compact abelian group* G *on von Neumann algebras* M_1 *and* M_2, *respectively. If* $N_j=M_j\times_{\alpha j}G$ *and*

$\hat{\alpha}_p=\hat{\alpha}^1_p\otimes\hat{\alpha}^2_{-p}$, *then covariant systems* $\{(M_1\otimes M_2)\times_{\alpha^1\otimes\alpha^2}G,\ (\alpha^1\otimes\alpha^2)^\wedge\}$ *and* $\{(N_1\otimes N_2)^{\hat{\alpha}},\ (\hat{\alpha}^1,\hat{\alpha}^2)\}$ *are equivalent, that is,*

$$\{(M_1\otimes M_2)\times_{\alpha^1\otimes\alpha^2}G,\ (\alpha^1\otimes\alpha^2)^\wedge\}\cong\{(N_1\otimes N_2)^{\hat{\alpha}},\ (\hat{\alpha}^1,\hat{\alpha}^2)\}.$$

PROOF. Let M_j be a von Neumann algebra acting on a Hilbert space $\mathscr{H}_j$ and $U^{(j)}$ a continuous unitary representation of G on $\mathscr{H}_j$ such that $\alpha^j_t=AdU^{(j)}_t$ for all t in G. Put $M=M_1\otimes M_2$ and $H=\{(s,s);\ s\in G\}$. Let V be the isomorphism which maps naturally $L^2(\mathscr{H}_1;\ G)\otimes L^2(\mathscr{H}_2;\ G)$ onto $L^2(\mathscr{H}_1\otimes\mathscr{H}_2;\ G\times G)$. Then AdV gives an equivalence of covariant systems:

$$\{N_1\otimes N_2,\ \hat{\alpha}^1\times\hat{\alpha}^2)\}\cong\{M\times_{\alpha^1\times\alpha^2}(G\times G),\ (\alpha^1\times\alpha^2)^\wedge\}.$$

Since $H^\perp=\{(p,-p);\ p\in\hat{G}\}$ and

$$M\times_{\alpha^1\times\alpha^2}H$$
$$=\{y\in M\times_{\alpha^1\times\alpha^2}(G\times G);\ (\alpha^1\times\alpha^2)^\wedge_{(p,q)}(y)=y \text{ for all } (p,q)\in H^\perp\},$$

by the above equivalence it follows that $Ad\ V((N_1\otimes N_2)^{\hat{\alpha}})=M\times_{\alpha^1\times\alpha^2}H$ and $AdV\circ(\hat{\alpha}^1_p\otimes\iota)\circ AdV^{-1}=(\alpha^1\times\alpha^2)^\wedge_{(p,0)}$, that is, covariant systems $\{(N_1\otimes N_2)^{\hat{\alpha}},\ (\hat{\alpha}^1,\hat{\alpha}^2)\}$ and $\{M\times_{\alpha^1\times\alpha^2}H,\ p\to(\alpha^1\times\alpha^2)^\wedge_{(p,0)}\}$ are equivalent.

Let F be an isomorphism of $L^2(\mathscr{H}_1\otimes\mathscr{H}_2;\ G\times\hat{G})$ onto $L^2(\mathscr{H}_1\otimes\mathscr{H}_2;\ G\times G)$ defined by

$$(F\xi)(s,t)=\int_{\hat{G}}\overline{\langle t,p\rangle}\ \xi(s,p)\ dp$$

and W a unitary operator on $L^2(\mathscr{H}_1\otimes\mathscr{H}_2;\ G\times G)$ defined by

$$(W\eta)(s,t)=(I\otimes U^{(2)}_{s-t})\ \eta(s,t).$$

Put $\Phi=Ad(W\circ F)\circ\pi_{(\alpha^1\otimes\alpha^2)^\wedge}$. Then Φ gives an isomorphism of $M\times_{\alpha^1\otimes\alpha^2}G$ onto $M\times_{\alpha^1\times\alpha^2}H$ such that

$$\Phi:\ \begin{cases}\pi_{\alpha^1\otimes\alpha^2}(x_1\otimes x_2)\longrightarrow\pi_{\alpha^1\times\alpha^2}(x_1\otimes x_2)\\ \lambda(s)\longrightarrow\lambda(s,s).\end{cases}$$

Moreover, Φ gives an equivalence of covariant systems:

$$\{M\times_{\alpha^1\otimes\alpha^2}G,\ (\alpha^1\otimes\alpha^2)^\wedge\}\cong\{M\times_{\alpha^1\times\alpha^2}H,\ p\to(\alpha^1\times\alpha^2)^\wedge_{(p,0)}\}.$$

Really, it suffices to check this on the generators. Since $\hat{\alpha}^j_p(\pi_{\alpha^j}(x))=\pi_{\alpha^j}(x)$ and $\hat{\alpha}^j_p(\lambda(s))=\overline{\langle s,p\rangle}\ \lambda(s)$, it follows that

$$\Phi\circ(\alpha^1\otimes\alpha^2)^\wedge_p(\pi_{\alpha^1\otimes\alpha^2}(x_1\otimes x_2))=\Phi\circ\pi_{\alpha^1\otimes\alpha^2}(x_1\otimes x_2)$$
$$=(\alpha^1\times\alpha^2)^\wedge_{(p,0)}\circ\Phi(\pi_{\alpha^1\otimes\alpha^2}(x_1\otimes x_2))$$

and

$$\Phi\circ(\alpha^1\otimes\alpha^2)^\wedge_p(\lambda(s))=\overline{\langle s,p\rangle}\lambda(s,s)$$
$$=(\alpha^1\times\alpha^2)^\wedge_{(p,0)}\Phi(\lambda(s)).$$

Thus $AdV^{-1}\circ\Phi$ gives an equivalence of covariant systems:

$$\{(M_1\otimes M_2)\times_{\alpha^1\otimes\alpha^2}G,\ (\alpha^1\otimes\alpha^2)^\wedge\}\cong\{(N_1\otimes N_2)^{\hat{\alpha}},\ (\hat{\alpha}^1,\hat{\alpha}^2)\}.$$

This completes the proof.

The following proposition is a dual version of Theorem 1.

PROPOSITION 1. *Let $\beta^j(j=1,2)$ be a continuous action of a locally compact abelian group Γ on a properly infinite von Neumann algebra N_j and $\hat{\beta}_j$ its dual action of the dual group $\hat{\Gamma}$ of Γ on $N_j\times_{\beta^j}\Gamma$. Then covariant systems $\{(N_1\times_{\beta^1}\Gamma)\otimes(N_2\times_{\beta^2}\Gamma),\ \hat{\beta}^1\otimes\hat{\beta}^2\}$ and $\{(N_1\otimes N_2)^\beta\times_{(\beta^1,\beta^2)}\Gamma,\ (\beta^1,\beta^2)^\wedge\}$ are equivalent, where $\beta_p=\beta^1_p\otimes\beta^2_{-p}$.*

If M_j and α^j are as in Theorem 1 and $N_j=M_j\times_{\alpha^j}G$, by Takesaki's duality theorem, covariant systems $\{N_j\times_{\hat{\alpha}^j}\hat{G},\hat{\alpha}^j\}$ and $\{M_j\otimes B(L^2(G)),\ \alpha^j\otimes Ad\,v^*\}$ are equivalent and furthermore if M_j is properly infinite, the latter is equivalent to $\{M_j,\alpha^j\}$. Therefore if M_j is properly infinite, Theorem 1 is again concluded by Proposition 1.

Here we recall recent results due to Paschke ([12]). If α is a continuous action of a locally compact abelian group G on a von Neumann algebra M, it is said that α has the relative commutant property if the relative commutant of M in the crossed product $M\times_\alpha G$ is the center of $M\times_\alpha G$. According to [12], if $(M^\alpha)'\cap M\subseteq M^\alpha$, then α has the relative commutant property. This implies from the relative commutant theorem in [4] that all modular actions, which are continuous actions by modular automorphism groups, have the relative commutant property. Conversely, if α is integrable and has the relative commutant property, then $(M^\alpha)'\cap M\subseteq M^\alpha$. We recall that α is said to be integrable if the set of all x in M for which the integral $\int\alpha_t(x^*x)dt$ exists in M is a σ-weakly dense left ideal in M. Also, if the relative commutant of M in $M\times_\alpha G$ is the center of M, then α is said to have the dual relative commutant property. If α is a continuous action of G dual to some action of $\hat{G}$, then $\pi_\alpha(M)'\cap(M\times_\alpha G)=\pi_\alpha(C_M)$ is equivalent to

$(M^{\alpha})' \cap M = C_M$ ([12]).

PROPOSITION 2. *Let β^1 and β^2 be continuous actions of a locally compact abelian group Γ of von Neumann algebras N_1 and N_2, respectively. Let β be a continuous action of Γ on $N_1 \otimes N_2$ defined by $\beta_p = \beta^1_p \otimes \beta^2_{-p}$ for p in Γ. If β^j is integrable and has the dual relative commutant property: $\pi_{\beta^j}(N_j)' \cap (N_j \times_{\beta^j} \Gamma) = \pi_{\beta^j}(C_{N_j})$, then*

$$C_{(N_1 \otimes N_2)^\beta} = (C_{N_1 \otimes N_2})^\beta.$$

PROOF. Put $N = N_1 \otimes N_2$ and let G be the dual group of Γ. Then β is a continuous action of Γ on N. Since $H = \{(t, t);\ t \in G\} = \{(p, -p);\ p \in \Gamma\}^{\perp}$ in $G \times G$, it follows that the crossed product $N \times_\beta \Gamma$ is isomorphic to the fixed point subalgebra $(N \times_{\beta^1 \times \beta^2} (\Gamma \times \Gamma))^{\hat{\beta}^1 \otimes \hat{\beta}^2}$ and the generators $\pi_\beta(y)$ and $\lambda^\Gamma(p)$ correspond to $\pi_{\beta^1 \times \beta^2}(y)$ and $\lambda^{\Gamma \times \Gamma}(p, -p)$, respectively, ([16]). Therefore, the inclusion relation

$$\pi_\beta(N)' \cap (N \times_\beta \Gamma) \subseteq \pi_\beta(N)$$

is equivalent to the inclusion relation

$$\pi_{\beta^1 \times \beta^2}(N)' \cap (N \times_{\beta^1 \times \beta^2} (\Gamma \times \Gamma))^{\hat{\beta}^1 \otimes \hat{\beta}^2} \subseteq \pi_{\beta^1 \times \beta^2}(N).$$

On the other hand, as we assume the dual relative commutant property for β^j, we have

$$\pi_{\beta^j}(N_j)' \cap (N_j \times_{\beta^j} \Gamma) \subseteq \pi_{\beta^j}(N_j).$$

Hence we have

$$\pi_{\beta^1 \times \beta^2}(N)' \cap (N \times_{\beta^1 \times \beta^2} (\Gamma \times \Gamma)) \subseteq \pi_{\beta^1 \times \beta^2}(N).$$

This implies the inclusion

$$\pi_\beta(N)' \cap (N \times_\beta \Gamma) \subseteq \pi_\beta(N),$$

and hence

$$(N^\beta)' \cap N = C_N$$

by [12]. Therefore, $C_{N^\beta} \subseteq C_N$ and hence $C_{N^\beta} \subseteq (C_N)^\beta$. The converse inclusion is clear.

This completes the proof.

COROLLARY. *Let α^1 and α^2 be continuous actions of a locally compact abelian group G on von Neumann algebras M_1 and M_2, respectively. If α^1 and α^2 satisfy the relative commutant property: $\pi_{\alpha^j}(M_j)' \cap (M_j \times_{\alpha^j} G) = C_{M_j \times_{\alpha^j} G}$, then*

$$C_{(N_1 \otimes N_2)^{\hat{\alpha}}} = (C_{N_1 \otimes N_2})^{\hat{\alpha}},$$

where $N_j = M_j \times_{\alpha^j} G$ *and* $\hat{\alpha}_p = \hat{\alpha}^1_p \otimes \hat{\alpha}^2_{-p}$.

The proof is evident from Proposition 2 since $\hat{\alpha}^j$ is integrable and has the dual relative commutant property.

Now we can prove the following theorem which permits us as its application to classify the tensor products of von Neumann algebras.

THEOREM 2. *If* $\alpha^j (j=1,2)$ *is a continuous action of a locally compact abelian group* G *on a von Neumann algebra* M_j *and has the relative commutant property, then the Connes spectrum* $\Gamma(\alpha^1 \otimes \alpha^2)$ *of the tensor product* $\alpha^1 \otimes \alpha^2$ *of* α^1 *and* α^2 *coincides with the kernel of the joint action* $(\hat{\alpha}^1, \hat{\alpha}^2)$ *of the restrictions* $\hat{\alpha}^1 | C_{N_1}$ *and* $\hat{\alpha}^2 | C_{N_2}$ *of dual actions* $\hat{\alpha}^1$ *and* $\hat{\alpha}^2$ *to the centers* C_{N_1} *and* C_{N_2} *of* N_1 *and* N_2:

$$\Gamma(\alpha^1 \otimes \alpha^2) = \{p \in \hat{G};\ (\hat{\alpha}^1, \hat{\alpha}^2)_p = \iota \text{ on } (C_{N_1} \otimes C_{N_2})^{\hat{\alpha}}\},$$

where $N_j = M_j \times_{\alpha^j} G$ *and* $\hat{\alpha}_p = \hat{\alpha}^1_p \otimes \hat{\alpha}^2_{-p}$.

PROOF. By virtue of [4], we have

$$\Gamma(\alpha^1 \otimes \alpha^2) = \{p \in \hat{G};\ (\alpha^1 \otimes \alpha^2)^\wedge_p = \iota \text{ on } C_{M \times_{\alpha^1 \otimes \alpha^2} G}\},$$

where $M = M_1 \otimes M_2$. By Theorem 1, $(\alpha^1 \otimes \alpha^2)^\wedge_p = \iota$ on the center of $M \times_{\alpha^1 \otimes \alpha^2} G$ if and only if $(\hat{\alpha}^1, \hat{\alpha}^2)_p = \iota$ on the center of $(N_1 \otimes N_2)^{\hat{\alpha}}$. Since α^1 and α^2 have the relative commutant property, $C_{(N_1 \otimes N_2)^{\hat{\alpha}}} = (C_{N_1} \otimes C_{N_2})^{\hat{\alpha}}$ by Corollary of Proposition 2.

This completes the proof.

4. Types of the tensor products of von Neumann algebras.

In this section we shall discuss the joint action with pure point spectrum and apply it to the results obtained in the previous sections.

Let β be a continuous action of a locally compact abelian group Γ on a von Neumann algebra N and φ a faithful normal state on N. Let $L^2(N, \varphi)$ be a Hilbert space which is the completion of a pre-Hilbert space N with respect to the inner product defined by $(x|y)_\varphi = \varphi(y^* x)$. Then β is said to *have pure point spectrum for* φ *on* N if the set of unitary elements u in N such that there exists a t in $\hat{\Gamma}$ satisfying $\beta_p(u) = \overline{\langle t, p \rangle}\, u$ for all p in Γ is total in $L^2(N, \varphi)$, where $\hat{\Gamma}$ is the dual group of Γ. By $\sigma_P(\beta)$ we denote the set of all t in $\hat{\Gamma}$ such that there exists a unitary u in N such as $\beta_p(u) = \overline{\langle t, p \rangle} u$ for all p in Γ and we call it *the point spectrum* of β.

Firstly we shall state our main result in this section.

THEOREM 3. *Let β^1 and β^2 be integrable continuous actions of a locally compact abelian group Γ on von Neumann algebras N_1 and N_2, respectively. Suppose that β^j is ergodic on the center C_{N_j} of N_j and has the dual relative commutant property: $\pi_{\beta j}(N_j)' \cap (N_j \times_{\beta j} \Gamma) = \pi_{\beta j}(C_{N_j})$, for $j=1,2$. If either*

(i) β^j has an invariant faithful normal state on C_{N_j} for $j=1,2$; or

(ii) β^1 or β^2 has pure point spectrum for some invariant faithful normal state on the center, then the restriction $(\beta^1, \beta^2)|C_{(N_1 \otimes N_2)^\beta}$ of the joint action to the center $C_{(N_1 \otimes N_2)^\beta}$ is ergodic, admits an invariant faithful normal state φ and has pure point spectrum for φ, where $\beta_p = \beta_p^1 \otimes \beta_{-p}^2$.

We shall prepare the following lemmas to prove our Theorem.

LEMMA 1. *Let $\beta^j (j=1,2)$ be an ergodic continuous action of Γ on an abelian von Neumann algebrea A_j and φ_j a faithful normal state on A_j. If φ_j is β^j-invariant for $j=1,2$, then the joint action (β^1, β^2) of β^1 and β^2 is ergodic, admits an invariant faithful normal state $\varphi_1 \otimes \varphi_2$, and has pure point spectrum $\sigma_P(\beta^1) \cap \sigma_P(\beta^2)$ for $\varphi_1 \otimes \varphi_2$.*

PROOF. It is obvious that the joint action (β^1, β^2) is ergodic and $\varphi_1 \otimes \varphi_2$ is (β^1, β^2)-invariant. Since φ_j is β^j-invariant, we may assume that A_j acts canonically on the Hilbert space $L^2(A_j, \varphi_j)$ and β^j agrees on A_j with a unitary representation $U^{(j)}$ of Γ on $L^2(A_j, \varphi_j)$:

$$(U_p^{(j)} x | y)_{\varphi_j} = (\beta_p^j(x) | y)_{\varphi_j}, \quad x, y \in A_j.$$

Let $\{E^{(j)}(t);\ t \in \hat{\Gamma}\}$ be the spectral resolution of $U^{(j)}$. Let E be the projection of $L^2(A_1 \otimes A_2, \varphi_1 \otimes \varphi_2)$ onto the closure of $(A_1 \otimes A_2)^\beta$ in $L^2(A_1 \otimes A_2, \varphi_1 \otimes \varphi_2)$, and P the projection onto the closed linear span of $\{u_t^{(1)} \otimes u_t^{(2)};\ t \in \sigma_P(\beta^1) \cap \sigma_P(\beta^2)\}$, where $\beta_p = \beta_p^1 \otimes \beta_{-p}^2$ and $u_t^{(j)}$ are the normalized eigenvectors of β^j belonging to $t \in \sigma_P(\beta^j)$. Then $P \leq E$.

We shall show that $E(I-P)=0$. First we notice that each pair of E, P, $U_p^{(1)} \otimes I$ and $I \otimes U_p^{(2)}$ mutually commutes. Put $F=E(I-P)$. Since $(U_p^{(1)} \otimes I)E = (U_{p-q}^{(1)} \otimes U_q^{(2)})E$, we have

$$\int \overline{\langle s, p \rangle}\, (dE^{(1)}(s) \otimes I)F = \iint \overline{\langle s, p \rangle}\, \overline{\langle t-s, q \rangle} (dE^{(1)}(s) \otimes dE^{(2)}(t))F.$$

If f, $g \in L^1(\Gamma)$ and $\hat{g}(0)=1$, then the Fubini theorem implies

$$(*) \quad \int \hat{f}(s)\, (dE^{(1)}(s) \otimes I)F = \iint \hat{f}(s)\, \hat{g}(t-s)\, (dE^{(1)}(s) \otimes dE^{(2)}(t))F.$$

We may assume that $\hat{f}$ and $\hat{g}$ have compact supports and $0 \leq \hat{f} \leq 1$, $0 \leq \hat{g} \leq 1$. The spectral measure $dE^{(1)}(s)$ or $dE^{(2)}(t)$ is continuous at every point s in $\hat{\Gamma}$ on $F(L^2(A_1 \otimes A_2, \varphi_1 \otimes \varphi_2))$. Therefore, if the support of $\hat{g}$ converges to the unit of $\hat{\Gamma}$, then the right hand side of (*) converges to 0. Because, the right hand side of (*) is the integration by a product measure. Hence the left hand side of (*) vanishes for any $\hat{f}$ with $0 \leq \hat{f} \leq 1$. If $\hat{f} \uparrow 1$, then $F = 0$.

This completes the proof.

LEMMA 2. *Let $\beta^j (j=1,2)$ be an ergodic continuous action of Γ on an abelian von Neumann algebra A_j and φ_j a faithful normal state on A_j. If φ_1 is β^1-invariant and β^1 has pure point spectrum for φ_1 on A_1, then the joint action (β^1, β^2) of β^1 and β^2 is ergodic, admits an invariant faithful normal state $\varphi_1 \otimes \varphi_2$, and has pure point spectrum $\sigma_P(\beta^1) \cap \sigma_P(\beta^2)$ for $\varphi_1 \otimes \varphi_2$.*

PROOF. It is obvious that the joint action (β^1, β^2) is ergodic and $\varphi_1 \otimes \varphi_2$ is (β^1, β^2)-invariant. Put $\beta_p = \beta^1_p \otimes \beta^2_{-p}$ on $A_1 \otimes A_2$. Since β^j is ergodic, we can choose a unitary $u_t^{(j)} \in A_j$ with $\beta^j_p(u_t^{(j)}) = \overline{<t, p>}\, u_t^{(j)}$ for $t \in \sigma_P(\beta^j)$. Since $\beta_p(u_t^{(1)} \otimes u_t^{(2)}) = u_t^{(1)} \otimes u_t^{(2)}$, it suffices to show that the set of all $u_t^{(1)} \otimes u_t^{(2)}$ with $t \in \sigma_P(\beta^1) \cap \sigma_P(\beta^2)$ is total in $L^2((A_1 \otimes A_2)^\beta, \varphi_1 \otimes \varphi_2)$.

For this we suppose that $x \in (A_1 \otimes A_2)^\beta$ and

$$(**) \qquad <(u_t^{(1)} \otimes u_t^{(2)})^* x, \varphi_1 \otimes \varphi_2> = 0, \quad t \in \sigma_P(\beta^1) \cap \sigma_P(\beta^2).$$

Let R and L be the right and the left slice mappings on $A_1 \otimes A_2$:

$$<R_\varphi(x), \psi> = <x, \varphi \otimes \psi> = <L_\psi(x), \varphi>, \quad \varphi \in A_{1,*}, \ \psi \in A_{2,*}.$$

Put $x_t = R_{\varphi_1 u_t^{(1)*}}(x)$ for $t \in \sigma_P(\beta^1)$. Then $x_t \in A_2$.
Since $\beta_p(x) = x$ and φ_1 is β^1-invariant, it follows that

$$\begin{aligned} <\beta^2_p(x_t), \psi> &= <(u_t^{(1)} \otimes 1)^* \beta_p(x), (\varphi_1 \otimes \psi) \circ \beta_{-p}> \\ &= <\overline{<t, p>}\, x_t, \psi>, \quad \psi \in A_{2,*}. \end{aligned}$$

Therefore $x_t = \lambda u_t^{(2)}$ for some $\lambda \in C$. If $t \notin \sigma_P(\beta^2)$, then $x_t = 0$. If $t \in \sigma_P(\beta^1) \cap \sigma_P(\beta^2)$, then (**) implies that

$$\lambda = <x_t, \varphi_2 u_t^{(2)*}> = <x, \varphi_1 u_t^{(1)*} \otimes \varphi_2 u_t^{(2)*}> = 0.$$

Consequently, $x_t = 0$ for all t in $\sigma_P(\beta^1)$. Therefore

$$<L_\psi(x) u_t^{(1)*}, \varphi_1> = <x_t, \psi> = 0, \quad t \in \sigma_P(\beta^1).$$

Since β^1 has pure point spectrum for φ_1 by assumption, $L_\psi(x) = 0$ for all $\psi \in A_{2,*}$ and hence $x = 0$.

This completes the proof.

PROOF OF THEOREM 3. By Proposition 2, the following covariant systems on Γ are equivalent:

$$\{C_{(N_1\otimes N_2)^\beta}, (\beta^1,\beta^2)|C_{(N_1\otimes N_2)^\beta}\}\cong\{(C_{N_1}\otimes C_{N_2})^\beta, (\beta^1|C_{N_1}, \beta^2|C_{N_2})\},$$

where $\beta_p=\beta_p^1\otimes\beta_{-p}^2$. Moreover, since the covariant system $\{C_{N_j}, \beta^j|C_{N_j}\}$ on Γ satisfies the assumptions of Lemma 1, or Lemma 2, the proof is evident from these lemmas.

This completes the proof.

The following theorem is a consequence of Theorem 3, which gives some applications.

THEOREM 4. *Let α^j be a continuous action of a locally compact abelian group G on a factor M_j having the relative commutant property*: $\pi_{\alpha^j}(M_j)'\cap(M_j\times_{\alpha^j}G)=C_{M_j\times_{\alpha^j}G}$. *If either*

(i) $\hat{\alpha}^j$ has an invariant faithful normal state on the center of $M_j\times_{\alpha^j}G$ for $j=1,2$; or

(ii) $\hat{\alpha}^1$ or $\hat{\alpha}^2$ has pure point spectrum for some invariant faithful normal state on the center, then the action $(\alpha^1\otimes\alpha^2)^\wedge|C_N$ of G has pure point spectrum for some invariant faithful normal state, where $N=(M_1\otimes M_2)\times_{\alpha^1\otimes\alpha^2}G$.

We note that $\hat{\alpha}^j$ is integrable and has the dual relative commutant property, and if M_j is a factor, $\hat{\alpha}_j$ is ergodic on the center $C_{M_j\times_{\alpha^j}G}$.

Now we define a new type of von Neumann algebras and apply the results in the previous sections to the type classification of the tensor products of von Neumann algebras.

DEFINITION 2. Let α be a continuous action of a locally compact abelian group G on a factor M. For a subgroup Λ of the dual group $\hat{G}$ of G, a covariant system $\{M,\alpha\}$ on G is *of type Λ*, if $\hat{\alpha}|C_{M\times_\alpha G}$ admits an invariant faithful normal state φ and has pure point spectrum Λ for φ on the center $C_{M\times_\alpha G}$ of $M\times_\alpha G$. In particular, if α is the modular automorphism group and $\{M,\alpha\}$ is of type Λ, we say that M is *of type III^Λ*.

We note that $\mathrm{III}^{\frac{2\pi}{\log\lambda}Z}=\mathrm{III}_\lambda$ $(0<\lambda<1)$ and $\mathrm{III}^{\{0\}}=\mathrm{III}_1$.

THEOREM 5. (1) *Let M_1 be a factor of type III^Λ and M_2 be any factor. Then $M_1\otimes M_2$ is of type $III^{\Lambda\cap T(M_2)}$, where $T(M_2)$ is the T-set of M_2.*

(2) *Let M_j $(j=1,2)$ be a factor with the modular automorphism group σ^j. If $\{C_{M_j \times_{\sigma^j} R}, \hat{\sigma}^j\}$ $(j=1,2)$ admits an invariant faithful normal state, then $M_1 \otimes M_2$ is of type $III^{T(M_1) \cap T(M_2)}$.*

The proof is evident from Theorem 4, since σ^j $(j=1,2)$ has the relative commutant property ([4]).

COROLLARY ([7]). *Let M be a factor and $T(M)$ the T-set of M.*
(1) *If the tensor product of M and a factor of type III_λ $(0<\lambda<1)$ is isomorphic to M, then M is of type $III_{\lambda^{1/k}}$ for some integer k, or of type III_1.*
(2) *The tensor product of M and a factor of type III_λ $(0<\lambda<1)$ is of type III_λ if and only if $2\pi/\log\lambda \in T(M)$.*

Furthermore, as its application for the tensor products of von Neumann algebras, this implies the following.
(1) $III_\lambda \otimes$(semi-finite)$=III_\lambda$, (2) $III_1 \otimes III_\lambda = III_1$,
(3) $III_\lambda \otimes III_{\lambda'} = III_{<\lambda,\lambda'>}$ $(0<\lambda, \lambda'<1)$, where $-\log<\lambda, \lambda'>$ is the positive greatest common divisor of $-\log\lambda$ and $-\log\lambda'$, or $<\lambda, \lambda'>=1$ if $\log\lambda/\log\lambda'$ is an irrational number, (4) $III_\lambda \otimes III_0 = III_{\lambda'}$, where $\lambda'=\lambda^{1/k}$, or 1 $(0<\lambda<1)$. ([1]).

5. Examples.

As well known, the theories of operator algebras and ergodic transfomation groups are closely related. For an ergodic non-singular tranformation group on a measure space, there corresponds a von Neumann algebra constructed as the crossed product of the abelian von Neumann algebra of all essentially bounded functions by the automorphism group induced by the given transformation group, which was firstly studied by von Neumann in the case of freely acting transformation group and generalized by Krieger ([8]). The corresponding algebra is called as the von Neumann algebra given by the group measure space construction. Dye introduced the notion of the weak equivalence for countable non-singular transformation groups which identifies non-singular transformation groups preserving the orbits, and proved that any countable ergodic measure-preserving transformation groups on Lebesgue measure spaces are weakly equivalent. Moreover, Krieger studied the non-singular case, and gave the classification of ergodic non-singular transformations and proved that the weak equivalence between ergodic non-singular transformations is equivalent to the isomorphism between the corresponding von Neumann algebras by the group measure space construction. Also many important examples of von Neumann algebras were given by the group measure space construction and conversely the study of general von Neumann algebras gave some methods to the study of ergodic non-singular transformation groups. Thus it is interesting and important to study ergodic non-singular transformation groups for the development of the theory of ergodic transformation groups itself but also of the theory of operator algebras ([1]).

In this section we discuss the joint flows directly in the case of ergodic non-singular transformation groups, which correspond to the joint actions in the case of the corresponding von Neumann algebras by the group measure space construction.

Let $(\Omega, \mathscr{F}, P)$ be a Lebesgue measure space. Two measures μ and υ on the measurable space $(\Omega, \mathscr{F})$ are mutually equivalent $\mu \sim \upsilon$, when $\mu(A)=0$ if and only if $\upsilon(A)=0$, $A \in \mathscr{F}$. A bijective mapping g from Ω onto itself is a non-singular transformation if it is bimeasurable (i. e. $g^{-1}\mathscr{F} \subseteq \mathscr{F}$ and $g\mathscr{F} \subseteq \mathscr{F}$) and $Pg \sim P$, where $Pg(A)=P(gA)$, $A \in \mathscr{F}$. Let G be a countable group of non-singular transformations of $(\Omega, \mathscr{F}, P)$. A measure μ defined on $(\Omega, \mathscr{F})$ is G-invariant if $\mu g=\mu$, $g \in G$ and a measurable function $f(\omega)$ is G-invariant if $f(g\omega)=f(\omega)$, $g \in G$, a. e. ω. G is ergodic if every G-invariant function on $(\Omega, \mathscr{F}, P)$ is a constant a.e.. We denote by $[G]$ the group of all non-singular transformations g of $(\Omega, \mathscr{F}, P)$ satisfying that there exist measurable sets $A_n \in \mathscr{F}$, $n=1,2, \ldots$ and non-singular transformations $g_n \in G$, $n=1,2, \ldots$ such that $\Omega=\bigcup_{n=1}^{\infty} A_n$ (disjoint) and $g\omega=g_n\omega$, a. e. $\omega \in A_n$, $n=1,2, \ldots$. The group $[G]$ is said to be the full group of G. Two countable non-singular transformation groups G and G' of $(\Omega, \mathscr{F}, P)$ and $(\Omega', \mathscr{F}', P')$, respectively, are called weakly equivalent if there exists a bimeasurable bijective mapping φ from Ω onto Ω' such that $\varphi[G]\varphi^{-1}=[G']$ and $P \sim P'\varphi$.

Let us now define the ratio set $r(G)$ and the T-set $T(G)$ of a countable non-singular transformation group G of $(\Omega, \mathscr{F}, P)$. The ratio set $r(G)$ is the set of all non-negative numbers r satisfying that for any $\varepsilon>0$ and any measurable set A with $P(A)>0$ there exist a measurable subset B of A with $P(B)>0$ and $g \in G$ such that $gB \subseteq A$ and $\left|\frac{dPg}{dP}(\omega)-r\right|<\varepsilon$, $\omega \in B$([9]), and the T-set $T(G)$ is the set of all real numbers t satisfying that there exists a measurable function $\exp i\,\xi(\omega)$ such that $\exp i\,\{\xi(g\omega)-\xi(\omega)\} = \exp it \cdot \log \frac{dPg}{dP}(\omega)$, $g \in G$, a. e. ω ([6]). The set $r(G)\backslash\{0\}$ is a multiplicative subgroup of positive numbers and $T(G)$ is an additive subgroup of $\boldsymbol{R}$. These two sets are invariant for the weak equivalence. Moreover, the ratio set $r(G)$ has the following properties: (1) $r(G)$ does not depend on the choice of a measure P among equivalent measures, (2) $r(G)\backslash\{0\}$

is a closed subset of $\boldsymbol{R}_+=(0,+\infty)$ and hence a closed subgroup of $\boldsymbol{R}_+$, (3) if G admits no equivalent σ-finite invariant measures, $r(G)$ contains 0, and (4) G admits an equivalent σ-finite invariant measure if and only if $r(G)=\{1\}$. Therefore, for a countable ergodic non-singular transformation group G of $(\Omega,\mathscr{F},P)$, the ratio set $r(G)$ is one of the following sets: $\{1\}$, $\{\lambda^n;\ n\in\boldsymbol{Z}\}\cup\{0\}$ $(0<\lambda<1)$, $[0,+\infty)$ and $\{0,1\}$.

Let G be a countable ergodic non-singular transformation group of $(\Omega,\mathscr{F},P)$. Then we say that G is of type III if G admits no equivalent σ-finite invariant measures. And also we say that G is of type III_λ $(0<\lambda<1)$, III_1 or III_0 accordingly as the case: $r(G)=\{\lambda^n;\ n\in\boldsymbol{Z}\}\dot{\cup}\{0\}$ $(0<\lambda<1)$, $[0,+\infty)$ or $\{0,1\}$.

Let $\{U_s\}_{-\infty<s<+\infty}$ be a one-parameter group of non-singular transformatons of a measure space $(X,\mathscr{B}_X,\mu_X)$ which we call simply a non-singular flow. We say $\{U_s\}_{-\infty<s<+\infty}$ is measurable if the mapping: $(s,x)\to U_sx$ of $\boldsymbol{R}\times X$ into X is measurable. Let G be a countable non-singular transformation group acting on a Lebesgue measure space $(\Omega,\mathscr{F},P)$. For each $g\in G$, we define a non-singular transformation $\tilde{g}$ on the product measure space $(\Omega\times\boldsymbol{R},\mathscr{F}\times\mathscr{B}(\boldsymbol{R}),dP\times du)$ by

$$\tilde{g}(\omega,u)=(g\omega,\ u+\log\frac{dPg}{dP}(\omega)),$$

and put $\tilde{G}=\{\tilde{g};\ g\in G\}$. Let $\zeta(\tilde{G})$ be the measurable partition ([13]) generated by all $\tilde{G}$-invariant measurable subsets. For $-\infty<s<+\infty$, put $T_s(\omega,u)=(\omega,u+s)$, $(\omega,u)\in\Omega\times\boldsymbol{R}$. Since $\{T_s\}_{-\infty<s<+\infty}$ commutes with $\tilde{G}$, we can define the factor flow $\{\tilde{T}_s\}_{-\infty<s<+\infty}$ of $\{T_s\}_{-\infty<s<+\infty}$ on the quotient space $\Omega\times\boldsymbol{R}/\zeta(\tilde{G})$. For each $s(-\infty<s<+\infty)$, $\tilde{T}_s$ is a non-singular transformation with respect to any σ-finite measure equivalent to the image measure of $dP\times du$ and $\{\tilde{T}_s\}_{-\infty<s<+\infty}$ is a measurable flow.

We call the factor flow $\{\tilde{T}_s\}_{-\infty<s<+\infty}$ the non-singular flow associated with the non-singular transformation group G or simply the associated flow of G.

We note that the associated flow $\{\tilde{T}_s\}_{-\infty<s<+\infty}$ of G is ergodic if and only if G is ergodic.

Let $(X,\mathscr{B}_X,\mu_X;\ \{U_s\}_{-\infty<s<+\infty})$ and $(Y,\mathscr{B}_Y,\mu_Y;\ \{V_s\}_{-\infty<s<+\infty})$ be measurable non-singular flows. These measurable flows $\{U_s\}_{-\infty<s<+\infty}$ and $\{V_s\}_{-\infty<s<+\infty}$ are mutually strongly equivalent if there exists a bimeasurable bijective mapping ψ from X onto Y such that $\mu_X\sim\mu_Y\psi$ and for all $-\infty<s<+\infty$, $\psi U_sx=V_s\psi x$, a.e.x. We note that the strong equivalence among ergodic non-singular flows is the same as the metrically isomorphic equivalence if they admit finite equivalent invariant measures. The strong equivalence of associated flows is an invariant for the weak equivalence of ergodic non-singular transformation groups. Namely, if ergodic countable non-singular transformation groups $(\Omega, \mathscr{F}, P; G)$ and $(\Omega', \mathscr{F}', P'; G')$ are mutually weakly equivalent, then their associated flows are mutually strongly equivalent.

Let $(X, \mathscr{B}_X, \mu_X; \{U_s\}_{-\infty<s<+\infty})$ be a measurable non-singular flow. A real number t belongs to the set $\sigma(\{U_s\})$, which is called the point spectrum of $\{U_s\}_{-\infty<s<+\infty}$, if there exists a measurable function $\exp i\xi(x)$ such that for all $-\infty<s<+\infty$

$$\exp i\xi(U_s x)=\exp its\cdot\exp i\xi(x), \text{ a.e. } x.$$

We note that if G is a countable non-singular transformation group and $\{\tilde{T}_s\}_{-\infty<s<+\infty}$ is its associated flow, then the T-set $T(G)$ of G is equal to the point spectrum $\sigma(\{\tilde{T}_s\})$: $T(G)=\sigma(\{\tilde{T}_s\})$.

Moreover, the metrical properties of the associated flows give us much more informations about non-singular transformation groups. For example, the T-set is the point spectrum of the associated flow and the S-set is given by the periodic motion of the associated flow. More precisely, let $\{\tilde{T}_s\}_{-\infty<s<+\infty}$ be the non-singular flow associated with an ergodic non-singular transformation group G of a Lebesgue measure space $(\Omega, \mathscr{F}, P)$. Then (1) G admits an equivalent σ-finite invariant measure if and only if $\{\tilde{T}_s\}_{-\infty<s<+\infty}$ is strongly equivalent to the translation: $\mathbf{R}\ni u \to u+s$, $-\infty<s<+\infty$, (2) G is of type III_λ $(0<\lambda<1)$ if and only if $\{\tilde{T}_s\}_{-\infty<s<+\infty}$ is strongly equivalent to the periodic flow: $[0, -\log\lambda)\ni u\to u+s$ (mod. $-\log\lambda$), $-\infty<s<+\infty$, (3) G is of type III_1 if and only if $\{\tilde{T}_s\}_{-\infty<s<+\infty}$ is th trivial flow, and (4) G is of type III_0 if and only if $\{\tilde{T}_s\}_{-\infty<s<+\infty}$ is an ergodic, aperiodic and conservative flow. Also, it is shown by Krieger's skew-product method that any ergodic measurable non-singular flow is realized as the associated flow of an ergodic non-singular transformation group. In fact, let $(\Omega, \mathscr{F}, P; G)$ be a countable ergodic transfomation group of type III_1 and $(X, \mathscr{B}, \mu; \{U_s\}_{-\infty<s<+\infty})$ be an ergodic measurable non-singular flow and $\mathscr{G}=\{<g, g'>; g, g'\in G\}$ be acting on $(\Omega\times\Omega\times X\times\mathbf{R}, P\times P\times\mu\times m)$ as follows:

$$<g, g'>(\omega, \omega', x, u)$$
$$=(g\omega, g'\omega', U_{a(g,\omega)}x, u-a(g', \omega')-\log\frac{d\mu U_{a(g,\omega)}}{d\mu}(x)),$$

where $dm(u)=e^u du$ and $a(g, \omega)=\log\frac{dPg}{dP}(\omega)$. Then the associated flow $\{\tilde{T}_s\}_{-\infty<s<+\infty}$ of the ergodic non-singular transformation group $\mathscr{G}$ is strongly equivalent to $\{U_s\}_{-\infty<s<+\infty}$.

Now we define the joint flow of measurable non-singular flows acting on Lebesgue measure spaces and discuss the weakly equivalnet classes of the product

$G\times G'$ of countable ergodic non-singular transformation groups by using its associated flow and we introduce a new class (type III^{Γ}) of non-singular transformation groups of type III.

Let $\{U_s\}_{-\infty<s<+\infty}$ and $\{V_s\}_{-\infty<s<+\infty}$ be measurable non-singular flows acting on Lebesgue measure spaces $(X, \mathscr{B}_X, \mu_X)$ and $(Y, \mathscr{B}_Y, \mu_Y)$, respectively. Let $\{U_s\times I\}_{-\infty<s<+\infty}$ be a flow defined by $(U_s\times I)(x, y)=(U_s x, y)$. Since $\{U_s\times I\}_{-\infty<s<+\infty}$ commutes with $\{U_s\times V_{-s}\}_{-\infty<s<+\infty}$, we can define the factor flow of the flow $\{U_s\times I\}_{-\infty<s<+\infty}$ on the quotient space $X\times Y/\zeta(\{U_s\times V_{-s}\})$ and denote it by $\{(U, V)_s\}_{-\infty<s<+\infty}$. $\{(U, V)_s\}_{-\infty<s<+\infty}$ is a measurable non-singular flow with respect to the image measure of $\mu_X\times\mu_Y$ on $X\times Y/\zeta(\{U_s\times V_{-s}\})$.

DEFINITION 3. We call $\{(U, V)_s\}_{-\infty<s<+\infty}$ *the joint flow of* $\{U_s\}_{-\infty<s<+\infty}$ *and* $\{V_s\}_{-\infty<s<+\infty}$.

We note that $\{(U, V)_s\}_{-\infty<s<+\infty}$ is strongly equivalent to $\{(V, U)_s\}_{-\infty<s<+\infty}$ and that $\sigma(\{(U, V)_s\})=\sigma(\{(V, U)_s\})$.

THEOREM 6. *Let G and G' be countable non-singular transformation groups on Lebesgue spaces $(\Omega, \mathscr{F}, P)$ and $(\Omega', \mathscr{F}', P')$, respectively, and $G\times G'=\{g\times g';\ g\in G, g'\in G'\}$ be the product non-singular transformation group of G and G', where $(g\times g')(\omega, \omega')=(g\omega, g'\omega')$, $\omega\in\Omega, \omega'\in\Omega'$. Then the associated flow of $G\times G'$ is strongly equivalent to the joint flow of each associated flows.*

PROOF. We define a mapping ψ from $\Omega\times\boldsymbol{R}\times\Omega'\times\boldsymbol{R}$ onto $\Omega\times\Omega'\times\boldsymbol{R}$ as follows

$$\psi(\omega, u, \omega', u')=(\omega, \omega', u+u').$$

Since

$$(\tilde{g}\times\tilde{g}')(\omega, u, \omega', u')=\left(g\omega, u+\log\frac{dPg}{dP}(\omega), g'\omega', u'+\log\frac{dP'g'}{dP'}(\omega')\right)$$

and

$$\widetilde{g\times g'}(\omega, \omega', u)=\left(g\omega, g'\omega', u+\log\frac{dPg}{dP}(\omega)+\log\frac{dP'g'}{dP'}(\omega')\right),$$

we have

$$\psi(\tilde{g}\times\tilde{g}')=\widetilde{g\times g'}\,\psi, \quad g\in G,\ g'\in G'.$$

Hence ψ induces a mapping from the product space of the quotient spaces $(\Omega\times\boldsymbol{R}/\zeta(\tilde{G}))\times(\Omega'\times\boldsymbol{R}/\zeta(\tilde{G}'))$ onto the quotient space $\Omega\times\Omega'\times\boldsymbol{R}/\zeta(\widetilde{G\times G'})$. Since $\psi(\omega,$

$u+s,\omega',u'-s)=(\omega,\omega',u+u')$ and $\psi(\omega,u+s,\omega',u')=(\omega,\omega',u+u'+s)$, ψ induces a strongly equivalent mapping between the joint flow of the associated flows of G and G' and the associated flow of $G\times G'$.

This completes the proof.

DEFINITION 4. For a countable additive subgroup Γ of $\boldsymbol{R}$, a countable ergodic non-singular transformation group G is *of type* III^{Γ} if the associated flow $\{\tilde{T}_s\}_{-\infty<s<+\infty}$ of G is ergodic finite measure-preserving and if it has pure point spectrum Γ.

We note that $\mathrm{III}^{\frac{2\pi}{\log\lambda}Z}=\mathrm{III}_\lambda((0<\lambda<1)$ and $\mathrm{III}^{\{0\}}=\mathrm{III}_1$ since ergodic finite measure-preserving flows with the same pure point spectrum are mutually metrically isomorphic.

THEOREM 7. (1) *Let G be of type III^{Γ} and G' be any countable ergodic non-singular transformation group. Then $G\times G'$ is of type $III^{\Gamma\cap T(G')}$.*
(2) *Let G and G' be countable ergodic non-singular transformation groups whose associated flows have finite invariant measures. Then $G\times G'$ is of type $III^{T(G)\cap T(G')}$.*

PROOF. The proof follows from the next lemma.

LEMMA 3. (1) *Let $(X,\mathscr{B}_X,\mu_X;\ \{U_s\}_{-\infty<s<+\infty})$ be an ergodic finite measure-preserving flow which has pure point spectrum and $(Y,\mathscr{B}_Y,\mu_Y;\ \{V_s\}_{-\infty<s<+\infty})$ be an ergodic non-singular flow. Then the joint flow $\{(U,V)_s\}_{-\infty<s<+\infty}$ is ergodic finite measure-preserving and has pure point spectrum $\sigma(\{U_s\})\cap\sigma(\{V_s\})$.*
(2) *Let $(X,\mathscr{B}_X,\mu_X;\ \{U_s\}_{-\infty<s<+\infty})$ and $(Y,\mathscr{B}_Y,\mu_Y;\ \{V_s\}_{-\infty<s<+\infty})$ be ergodic finite measure-preserving flows. Then the joint flow $\{(U,V)_s\}_{-\infty<s<+\infty}$ is ergodic finite measure-preserving and has pure point spectrum $\sigma(\{U_s\})\cap\sigma(\{V_s\})$.*

PROOF. If μ_X is a $\{U_s\}_{-\infty<s<+\infty}$-invariant finite measure and if μ_Y is a finite measure, $\{(U,V)_s\}_{-\infty<s<+\infty}$ preserves the image measure of $\mu_X\times\mu_Y$ on the qoutient space $X\times Y/\zeta(\{U_s\times V_{-s}\})$. There exist measurable functions $\exp i\xi_t(x)$ for $t\in\sigma(\{U_s\})$ such that $\exp i\xi_t(U_sx)=\exp its\cdot\exp i\xi_t(x)$ and measurable functions $\exp i\eta_t(y)$ for $t\in\sigma(\{V_s\})$ such that $\exp i\eta_t(V_sy)=\exp its\cdot\exp i\eta_t(y)$. Since

$$\begin{aligned}\exp i\xi_t(U_sx)\cdot\exp i\eta_t(V_{-s}y)&=\exp its\cdot\exp i\xi_t(x)\times\exp\{-its\}\cdot\exp i\eta_t(y)\\&=\exp i\xi_t(x)\cdot\exp i\eta_t(y)\end{aligned}$$

for $t\in\sigma(\{U_s\})\cap\sigma(\{V_s\})$, $\exp i\xi_t(x)\cdot\exp i\eta_t(y)$ is a $\{U_s\times V_{-s}\}_{-\infty<s<+\infty}$-invariant function.

We will show that the set of all $\{U_s\times V_{-s}\}_{-\infty<s<+\infty}$-invariant square-integrable functions is generated by $\{\exp i\xi_t(x)\cdot\exp i\eta_t(y);\ t\in\sigma(\{U_s\})\cap\sigma(\{V_s\})\}$. Let $f(x,y)$ be a bounded $\{U_s\times V_{-s}\}_{-\infty<s<+\infty}$-invariant measurable function and assume

$$<f(\cdot,\cdot),\exp i\xi_t(\cdot)\ \exp\ i\eta_t(\cdot)>_{L^2(\mu_X\times\mu_Y)}=0$$

for $t\in\sigma(\{U_s\})\cap\sigma(\{V_s\})$. Define

$$\hat{f}_t(y)=<f(\cdot,y),\ \exp\ i\xi_t(\cdot)>_{L^2(\mu_X)},\ y\in Y,$$

for $t\in\sigma(\{U_s\})$. Then we have

$$\begin{aligned}\hat{f}_t(V_sy)&=<f(\cdot,V_sy),\ \exp\ i\xi_t(\cdot)>_{L^2(\mu_X)}\\&=<f(U_s\cdot,\ y),\ \exp\ i\xi_t(\cdot)>_{L^2(\mu_X)}\\&=<f(\cdot,\ y),\ \exp\ i\xi_t(U_{-s}\cdot)>_{L^2(\mu_X)}\\&=<f(\cdot,\ y),\ \exp\{-its\}\cdot\exp\ i\xi_t(\cdot)>_{L^2(\mu_X)}\\&=\exp\ its\cdot\hat{f}_t(y),\ \ t\in\sigma(\{U_s\}).\end{aligned}$$

Since $\{V_s\}_{-\infty<s<+\infty}$ is ergodic, we have

$$\hat{f}_t(y)=\begin{cases}c_t\exp\ i\eta_t(y) & \text{if } t\in\sigma(\{U_s\})\cap\sigma(\{V_s\})\\ 0 & \text{if } t\in\sigma(\{U_s\})\setminus\sigma(\{V_s\}),\end{cases}$$

where c_t is a constant. Hence we have

$$\hat{f}_t(y)=0,\quad \text{a.e. } y,$$

for any $t\in\sigma(\{U_s\})$ from the assumption on f.

Next consider the case (2) and take a measurable bounded function $\xi(x)$ which is orthogonal to every $\exp i\xi_t(x)$, $t\in\sigma(\{U_s\})$. Defining $\hat{f}_\xi(y)=<f(\cdot,y),\ \xi>_{L^2(\mu_X)}$, we will see $\hat{f}_\xi(y)=0$, a.e. y. Indeed

$$\begin{aligned}\hat{f}_\xi(V_sy)&=<f(\cdot,V_sy),\ \xi>_{L^2(\mu_X)}\\&=<f(U_s\cdot,\ y),\ \xi>_{L^2(\mu_X)}\\&=<f(\cdot,\ y),\ \xi(U_{-s}\cdot)>_{L^2(\mu_X)}.\end{aligned}$$

From Stone's spectral decomposition theorem,

$$\xi(U_s\cdot)=\int_{-\infty}^{\infty}\exp\ is\lambda\ dE(\lambda)\xi.$$

Then

$$\hat{f}_\xi(V_sy)=\int_{-\infty}^{\infty}\exp\ is\lambda\ d<f(\cdot,\ y),\ E(\lambda)\xi(\cdot)>_{L^2(\mu_X)}.$$

We put $dF(\lambda)=d<f(\cdot,y),\ E(\lambda)\ \xi(\cdot)>_{L^2(\mu_X)}$. This measure is non-atomic since $\xi(\cdot)$ is orthogonal to all eigenfuctions of $\{U_s\}_{-\infty<s<+\infty}$. Therefore

$$\frac{1}{s}\int_0^s |\hat{f}_\xi(V_s y)|^2\, ds = \iint \frac{\exp is(\lambda'-\lambda)}{s(\lambda'-\lambda)}\, dF(\lambda)dF(\lambda'),\ \text{a.e.}\, y.$$

The right term converges to 0 as $s\to\infty$ by Lebesgue's convergence theorem. Since $\{V_s\}_{-\infty<s<+\infty}$ is ergodic finite measure-preserving, from Birkhoff's pointwise ergodic theorem we have $\hat{f}_\xi(y)=0$, a.e. y.

Thus for almost all y, $f(x,y)$ is orthogonal to any $\mathscr{B}_X$-measurable bounded function and so $f(x,y)=0$, a.e. (x,y). This means that the subspace of all $\{U_s\times V_{-s}\}_{-\infty<s<+\infty}$-invariant square-integrable functions is generated by $\{\exp i\xi_t(x)\cdot \exp i\eta_t(y);\ t\in\sigma(\{U_s\})\cap\sigma(\{V_s\})\}$ and that $\{(U,V)_s\}_{-\infty<s<+\infty}$ has pure point spectrum $\sigma(\{U_s\})\cap\sigma(\{V_s\})$.

This completes the proof.

COROLLARY ([2]). *Let G_λ be a countable ergodic non-singular transformation group of type III_λ $(0<\lambda<1)$ and G be an ergodic non-singular transformation group. Then (1) G is of type $III_{\lambda^{1/k}}$ for some integer k or is of type III_1 if $G_\lambda\times G$ is weakly equivalent to G, and (2) $G_\lambda\times G$ is of type III_λ if and only if* $\frac{2\pi}{\log\lambda}\in T(G)$.

PROOF. The proof is clear from Theorem 7.

References

[1] Araki, H. and Nakagami, Y., Recent progress on operator algebras, Suugaku, **26** (1974), 330-345.

[2] Araki, H. and Woods, E. J., A classification of factors, Publ. RIMS, Kyoto Univ., **4** (1968), 51-130.

[3] Connes, A., Une classification des facteurs de type III, Ann. Ecole Norm. Sup., **6** (1973),133-252.

[4] Connes, A. and Takesaki, M., The flow of weights on factors of type III, Tohoku Math. J., **29** (1977), 473-575.

[5] Dixmier, J., Les algebres d'operateurs dans l'espace Hilbertien, Gauthier-Villars, Paris, 1969.

[6] Hamachi, T., Oka, Y. and Osikawa, M., A classification of ergodic non-singular transformation groups, Mem. Fac. Sci. Kyushu Univ., **28** (1974), 113-133.

[7] Hamachi, T., Oka, Y. and Osikawa, M., Flows associated with ergodic non-singular transformation groups, Publ. RIMS, Kyoto Univ., **11** (1975), 31-50.

[8] Krieger, W., On non-singular transformations of a measure space I, II, Z. Wahrscheinlichkeitstheorie verw. Geb., **11** (1969), 83-97, 98-119.

[9] Krieger, W., On the Araki-Woods' asymptotic ratio set and non-singular transformations of a measure space, Lecture Notes in Math., **160** (1970), 158-177.

[10] Krieger, W., On ergodic flows and the isomorphism of factors, Math. Ann., **223** (1976), 19-70.

[11] Nakagami, Y. and Oka, Y., On Connes spectrum Γ of a tensor product of actions on von Neumann algebras, Yokohama Math. J,. **26** (1978), 189-200.

[12] Paschke, W. L., Relative commutant of a von Neumann algebra in its crossed product by a group action, Math. Z., **163** (1978), 5-13.

[13] Rohlin, V. A., On the fundamental ideas of a measure space, Mat. Sb., **25** (1949), 107-150. =A. M. S. Transl., **71** (1962).

[14] Takesaki, M., Covariance representations of C*-algebras and their locally compact automorphism groups, Acta Math., **119** (1967), 273-303.

[15] Takesaki, M., Tomita's theory of modular Hilbert algebras and its applications, Lecture Notes in Math., **128** (1970), Springer-Verlag.

[16] Takesaki, M., Duality for crossed products and the structure of von Neumann algebras of type III, Acta Math., **131** (1973), 249-310.

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