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ON THE NEBENHÜLLE OF BOUNDED DOMAINS

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Introduction

In [1] Behnke and Thullen studied the convergence problem for a sequence of domains and the sequence of their envelopes of holomorphy. One half of their investigation was devoted to the study on the "Nebenhülle" and several interesting results were obtained. This paper was written to give light to them from the stand point of convexity.

As recent works of Diederich and Fornaess [2], [3] and other works suggest, a domain without Nebenhülle in the sense of [1] may be understood by a kind of convexity called in this article *n-convex.*

Above authors have treated the pseudoconvex domains with smooth boundary and obtained fundamental results on Nebenhülle and related matters. Some of their results seem to be very closely related to ours. Non-trivial example of n-convex domains will be found there, for example:

Let Ω be a strongly pseudoconvex domain with smooth boundary in C^n. Then Ω has a fundamental system of pseudoconvex neighborhoods.

In this paper we give a sufficient condition for a domain to be n-convex in terms of the convexity with respect to a family of functions, Theorem 17, where no condition is assumed for the boundary.

As is shown a n-convex bounded domain Ω is convex with respect to the set of functions holomorphic on $\overline{\Omega}$ and the maximal domain of continuation of which is schlicht, Proposition 5. The converse of this fact is very likely to hold, but in this paper we can only prove Theorem 17.

A survey on n-covex domains is given in **1** and **2.** In **3** a convexity properly observed for a n-convex domain is given. And in **4** a sufficient condition for a domain to be n-convex is proved. For the sake of simplicity we confine ourselves to the domains in C^n. For the standard knowledge of the theory of functions of several complex variables we refer to [4].

1. n-convexity

Let Ω be a bounded domain in C^n, which we fix in the sequel. We denote by

$\mathscr{N}$ the set of the domains of holomorphy in C^n which contain Ω as a relatively compact subset and by $N(\Omega)$ the open kernel of intersection of the domains belonging to $\mathscr{N}$:

$$N(\Omega) := (\bigcap_{\Omega' \in \mathscr{N}} \Omega')^0.$$

We call $N(\Omega)$ the *Nebenhülle* of the domain Ω. It is obvious that $N(\Omega)$ is connected and is a domain of holomorphy. When $\Omega \equiv N(\Omega)$ holds, we say that Ω is *n-convex*. As was already known in [1] Ω does not necessarily coincide with its Nebenhülle $N(\Omega)$, even if Ω is a domain of holomorphy. In other words "n-convex" is a stronger condition than the convexity observed for a domain of holomorphy. By definition we see

$$\Omega \subset H(\Omega) \subset N(\Omega)$$

if the envelope of holomorphy H(Ω) of Ω is schlicht.

Let K be a compact subset of Ω and $\mathscr{F}$ a subset of the set $\mathscr{A}(\Omega)$ of functions holomorphic in Ω. $\hat{K}_{\mathscr{F}} := \{x \in \Omega: |f(x)| \leq sup|f(K)|, f \in \mathscr{F}\}$. If $\mathscr{F} \equiv \mathscr{A}(\Omega)$, we write $\hat{K}_{\Omega}$ in stead of $\hat{K}_{\mathscr{F}}$, In the following by $\Omega \subset\subset \Omega_1$ we mean that Ω is relatively compact in Ω_1. From the definition of n-convexity we have

PROPOSITION 1. *The following statements are equivalent.*

(i) Ω *is n-convex*: $\Omega \equiv N(\Omega)$.

(ii) $(\overline{\Omega})^0 \equiv \Omega$ *and for any positive* ε *there exists an* $\Omega' \in \mathscr{N}$ *such that* $\Omega \subset\subset \Omega' \subset\subset \Omega_\varepsilon$, *where* Ω_ε *is the* ε*-neighborhood of* Ω.

(iii) $(\overline{\Omega})^0 \equiv \Omega$ *and* $\overline{\Omega} = \bigcap_{\Omega' \in \mathscr{N}} \Omega'$.

(iv) $(\overline{\Omega})^0 \equiv \Omega$ *and if* $\mathscr{N}'$ *is the set of analytic polyhedrons which contain* Ω *as relatively compact subset, then* $\overline{\Omega} = \bigcap_{P \in \mathscr{N}'} P$.

(v) $(\overline{\Omega})^0 \equiv \Omega$ *and* $\overline{\Omega} = \bigcap_{\Omega_\lambda \in \mathscr{N}} (\overline{\Omega})_{\hat{\Omega}_\lambda}$.

PROOF. (i) $\Rightarrow$ (ii) Since in general $\Omega \subset (\overline{\Omega})^0$, we put $S = (\overline{\Omega})^0 - \Omega$ and assume $S \neq \phi$. Obviously $S \subset \partial\Omega$. Take any point $p \in S$, then from $p \in (\overline{\Omega})^0$ there exists a neighborhood U of p such that $U \subset (\overline{\Omega})^0$. For any $\Omega' \in \mathscr{N}$ we have $U \subset\subset \Omega'$ and hence $U \subset N(\Omega)$. Since Ω is n-convex, that is, $\Omega = N(\Omega)$, we have $U \subset\subset \Omega$. Consequently $p \in \Omega$. Thus $S \subset \Omega$, which implies $S \subset \Omega \cap \partial\Omega$. This means $S = \phi$, a contradiction.

We shall show the latter half of (ii). Since $\overline{\Omega}$ is compact, $\overline{\Omega}_\varepsilon$ is also compact. So $\partial\Omega_\varepsilon$ is compact. For any $p \in \partial\Omega_\varepsilon$ we may choose an $\Omega_p \in \mathscr{N}$ which does not contain p. We may assume that p is an exterior point of Ω_p. For if $p \in \partial\Omega_p$, considering $\Omega \subset\subset \Omega_p$ we construct an analytic polyhedron P defined by the functions of $\mathscr{A}(\Omega_p)$ such that $\Omega \subset\subset P \subset\subset \Omega_p$. Then we may replace Ω_p by P which satisfies the condition. We associate p with an Ω_p chosen as above. Then $\{(\overline{\Omega}_p)^c\}_{p\in\partial\Omega_\varepsilon}$ is an open covering of $\partial\Omega_\varepsilon$, where $(\overline{\Omega}_p)^c$ is the exterior of Ω_p. Since $\partial\Omega_\varepsilon$ is compact, some finite subset $(\overline{\Omega}_{p_1})^c, (\overline{\Omega}_{p_2})^c, \cdots, (\overline{\Omega}_{p_k})^c$ gives an covering of $\partial\Omega_\varepsilon$. Then the connected component Ω' of $\bigcap_{i=1}^{k}\Omega_{p_i}$ that contains Ω satisfies $\Omega \subset\subset \Omega'$ $\subset\subset \Omega_\varepsilon$. Ω is, as a finite intersection of the domains of holomorphy, also a domain of holomorphy.

(ii) $\Rightarrow$ (iii) $\overline{\Omega} \subset \bigcap_{\Omega' \in \mathscr{N}} \Omega'$ is obvious and therefore we shall show $\overline{\Omega} \supset \bigcap_{\Omega' \in \mathscr{N}} \Omega'$. Suppose $x \notin \overline{\Omega}$. Since the distance $d(x,\overline{\Omega}) = \rho > 0$, we can choose ε in (ii) so as $\varepsilon < \rho$. Then there exists an $\Omega'' \in \mathscr{N}$ satisfying $\Omega \subset\subset \Omega'' \subset\subset \Omega_\varepsilon$. Hence $x \notin \bigcap_{\Omega' \in \mathscr{N}} \Omega'$.

(iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) For any $\Omega' \in \mathscr{N}$ there exists an analytic polyhedron P such that $\Omega \subset\subset P \subset\subset \Omega'$. Then we have

$$\overline{\Omega} \subset \bigcap_{\Omega_\lambda \in \mathscr{N}} (\overline{\Omega})_{\hat{\Omega}_\lambda} \subset \bigcap_{p \in \mathscr{N}'} P \subset \bigcup_{\Omega' \in \mathscr{N}'} \Omega'.$$

Thus, if $\overline{\Omega} = \bigcap_{\Omega' \in \mathscr{N}} \Omega'$, then $\overline{\Omega} = \bigcap_{\Omega_\lambda \in \mathscr{N}} (\overline{\Omega})_{\hat{\Omega}_\lambda} = \bigcap_{p \in \mathscr{N}'} P$

The implication (iv) $\Rightarrow$ (iii) is obvious. By the same argument as in the proof of (i) $\Rightarrow$ (ii) we can show (v) $\Rightarrow$ (iv).

(iii) $\Rightarrow$ (i) Taking the open kernel of the both sides of $\overline{\Omega} = \bigcap_{\Omega' \in \mathscr{N}} \Omega'$, we have

$$\Omega = (\overline{\Omega})^0 = (\bigcap_{\Omega' \in \mathscr{N}} \Omega')^0 = N(\Omega).$$

Thus (i) is proved. ◇

COROLLARY 2. *An analytic polyhedron in C^n is n-convex.*

COROLLARY 3 [1]. *Let Ω be a bounded domain in C^n and starlike with respect to a point P. If the distance $d(P, Q)$ is continuous as Q varies in $\partial\Omega$, then Ω is n-convex.*

PROOF. It is easily verified that the conditions in (ii) of Proposition 1 are satisfied. ◇

COROLLARY 4 [1]. *Let Ω be a bounded domain in C^n and $f \in \mathscr{A}(\Omega)$. If $V = \{x \in \Omega : f(x) = 0\}$ is not empty, then $\Omega - V$ can not be n-convex.*

PROOF. Since $\phi \neq V \subset (\overline{\Omega - V})^{\wedge}_{\mathscr{F}} - (\Omega - V)$, by (ii) of Proposition 1 $\Omega - V$ can not be n-convex. ◇

From the definition of n-convex domain the following elementary properties are easily shown:

1) Let Ω and Ω_1 be the domains in C^n and Φ a biholomorphic mapping from a neighborhood of $\overline{\Omega}$ to a neighborhood of $\overline{\Omega}_1$. Then Ω_1 is n-convex if and only if Ω is n-convex.

2) If $\Omega \subset \Omega_1$, Then $N(\Omega) \subset N(\Omega_1)$.

3) The product and the intersection of finite number of n-convex domains are both n-convex.

4) $N(\Omega \times \Omega_1) = N(\Omega) \times N(\Omega_1)$.

REMARK. As in the case of domain of holomorphy the latter half of (iii) is generalized: *Let* $\{\Omega_\lambda\}_{\lambda \in \Lambda}$ *be a family of n-convex domains. Then* $(\bigcap_{\lambda \in \Lambda} \Omega_\lambda)^0$ *is n-convex.*

PROOF. Put $\Omega = (\bigcap_{\lambda \in \Lambda} \Omega_\lambda)^0$. Since $\Omega \subset \bigcap_{\lambda \in \Lambda} \Omega_\lambda \subset \Omega_\mu$ for any $\mu \in \Lambda$, by 2) above we have $N(\Omega) \subset N(\Omega_\mu)$, $\mu \in \Lambda$. Hence we have $N(\Omega) \subset \bigcap_{\lambda \in \Lambda} N(\Omega_\lambda)$. By assumption $N(\Omega_\lambda) = \Omega_\lambda$. Consequently $N(\Omega) \subset \bigcap_{\lambda \in \Lambda} \Omega_\lambda$. Since $N(\Omega)$ is open and $\Omega = (\bigcap_{\lambda \in \Lambda} \Omega_\lambda)^0$, we have $N(\Omega) \subset \Omega$, which implies $N(\Omega) = \Omega$. ◇

Let $\mathscr{A}(\overline{\Omega})$ be the set of functions holomorphic in a neighborhood of $\overline{\Omega}$. For $f \in \mathscr{A}(\overline{\Omega})$ let us denote by H_f the maximal domain of continuation of f, see [5]. Then H_f is, as is well known, not necessarily schlicht. We denote by $\mathscr{F}(\overline{\Omega})$ or by $\mathscr{F}$ the set of such element f of $\mathscr{A}(\overline{\Omega})$ that H_f is schlicht.

Now we are able to state the following

PROPOSITION 5. *An n-convex domain in* C^n *is* $\mathscr{F}$*-convex.*

PROOF. we shall proceed by the aid of well known technique due to Cartan-Thullen, see [1] and [4]. Suppose Ω is not $\mathscr{F}$-convex. Then we may find a compact subset K whose $\mathscr{F}$-hull $\hat{K}_{\mathscr{F}}$ is not compact in Ω. The distance $\rho = d(K, \partial\Omega)$ is positive. Choose positive number r so as $r < \rho$. For any $f \in \mathscr{F}$ and for any differential operator D^α we have $D^\alpha f \in \mathscr{F}$ where

$$D^\alpha = \frac{\partial^{\alpha_1 + \cdots + \alpha_n}}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}}, \quad \alpha = (\alpha_1, \cdots, \alpha_n).$$

Then for arbitrary point $q \in K$ and r-neighborhood $S(q, r)$ of q we have

$$|D^{\alpha}f(q)| \leqq \frac{\alpha!}{r^{|\alpha|}} \; sup \; |f(S(q, r))|,$$

where $\alpha! = \alpha_1! \cdots \alpha_n!$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$. Hence we obtain for r-neighborhood K_r of K and $p \in \hat{K}_{\mathscr{F}}$

$$|D^{\alpha}f(p)| = \frac{\alpha!}{r^{|\alpha|}} \; sup \; |f(K_r)|.$$

From this we see that the Taylor expansion of $f \in \mathscr{F}$ at p converges in $S(p, r)$ for any r, $r < \rho$. Thus every element of $\mathscr{F}$ is analytically continued to $S(p, r)$. Choosing ε as $\varepsilon < \rho$, by (ii) of Proposition 1 we conclude that the exterior $(\Omega')^c$ for some domain of holomorphy does contain a non-empty subset of $S(p, \rho)$. The function g whose maximal domain of continuation H_g coincides with Ω' belongs to $\mathscr{F}$ and thus continued to $S(p, \rho)$, a contradiction. We proved that $K_{\mathscr{F}}$ is compact in Ω. ◇

If the converse of Proposition 5 holds, we shall have a nice characterization of n-convex domain. But by this time we do not have affirmative proof. In 4 we shall show the converse of Proposition 5 under some stronger condition.

Here we give a version of Proposition 1 concerning the boundery point. Let us say a poin $p \in \partial\Omega$ possesses the property (B) if for any open neighborhood U of p there exist a point $q \in U$ and a function f holomorphic in a domain of holomorphy containing $\{q\} \cup \bar{\Omega}$ such that $|f(q)| > sup|f(\Omega)|$.

THEOREM 6. *Ω is n-convex if and only if every point of $\partial\Omega$ possesses the property* (B).

Proof. ($\Leftarrow$) Suppose Ω is not n-convex. Then $N(\Omega) - \Omega \neq \phi$. Take a point $q \in N(\Omega) \cap \partial\Omega$ and a neighborhood U of q as $V \subset\subset N(\Omega)$. By assumption there is a point r of V and a function f holomorphic in a domain of holomorphy Ω_r such that $\bar{\Omega} \cup \{r\} \subset \Omega_r$ and $|f(r)| >$ sup $|f(\Omega)|$. Since $\bar{\Omega}$ is contained in Ω_r, there can be constructed an analytic polyhedron P defined by the functions in $\mathscr{A}(\Omega_r)$ so that $\Omega \subset\subset P \subset\subset \Omega_r$. Put $Q = P \cap \{x \in \Omega_r: \; |f(x)| < |f(r)|\}$. Then clearly $r \notin Q$ and $\Omega \subset\subset Q$. This a contradiction, because Q is an open set of holomorphy and hence contains $N(\Omega)$ to which r belongs. Thus we have $N(\Omega) - \Omega = \phi$.

($\Rightarrow$) Assume the contrary. Then there exists a point $p_0 \in \partial\Omega$ which does not possesses the property (B): there can be found an open neighborhood V of p_0 such that for any point q of V, any domain of holomorphy Ω_q containing $\{q\} \cup \bar{\Omega}$ and any holomorphic function f in Ω_q holds the inequality $|f(q)| \leqq sup \; |f(\Omega)|$.

We may assume that f is non-constant. Then by replacing V by smaller V' as $V' \subset\subset V$ we may assume without loss of generality that $|f(q)| < \sup |f(\Omega)|$, $q \in V'$ holds. Further we may take the ball $B(p_0, \rho)$ centered at p_0 with radius ρ in place of V'. Choose positive number ε so as $\varepsilon < \rho$. Then, since Ω is n-convex, there exists a domain of holomorphy Ω_1 such that $\Omega \subset\subset \Omega_1 \subset\subset \Omega_\varepsilon$ where Ω_ε is the ε-neighborhood of $\overline{\Omega}$. By standard arguments we construct an analytic polyhedron P such that $\Omega \subset\subset P \subset\subset \Omega_1$. Then since $V' \cap P^c = B(p_0, \rho) \cap P^c \neq \phi$, for any $q' \in V' \cap P^c$ there is a function f holomorphic in Ω, such that $|f(q')| > \sup |f(\Omega)|$. Now q' can be taken as $\overline{\Omega} \cup \{q'\} \subset \Omega_1$, contradicting the choice of V'. Thus every point of $\partial\Omega$ possesses the property (B). ◇

REMARK. The inequality $|f(q)| > \sup |f(\Omega)|$ can be replaced by the property that $f(q) = 0$ and $f(x) \neq 0$ in $\overline{\Omega}$.

2. A continuation property for $\mathscr{F}$

From the difinition of $N(\Omega)$ we have

PROPOSITION 7. *Every function in $\mathscr{F}$ is analytically continued to a neighborhood of $\overline{N(\Omega)}$.*

PROOF. Since $\overline{N(\Omega)} = \bigcap_{\Omega' \in \mathscr{N}} \Omega'$, we have $\overline{N(\Omega)} \subset \Omega'$ for every $\Omega' \in \mathscr{N}$. On the other hand every domain of $\mathscr{N}$ is the domain of existence for some function f of $\mathscr{F}$, denote it by H_f. Then we have $\overline{N(\Omega)} = \bigcap_{f \in F} H_f$ and $\overline{N(\Omega)} \subset H_f$ for every $f \in \mathscr{F}$. Since $\overline{N(\Omega)}$ is compact, H_f is a neighborhood of $\overline{N(\Omega)}$. This means that every $f \in \mathscr{F}$ is analytically continued to some neighborhood of $\overline{N(\Omega)}$. ◇

Propsition 7 asserts that any schlicht domain of holomorphy, that contains Ω as relatively compact subset, contains also $N(\Omega)$ as relatively compact subset. As a direct consequence we obtain

COROLLARY 8. $N(N(\Omega)) = N(\Omega)$.

More precisely we can prove the following

THEOREM 9. *For any $f \in \mathscr{F}$ holds*

$$d(\Omega, H_f^c) = d(N(\Omega), H_f^c).$$

PROOF. The proof is analogous to that of Proposition 5. Take any $f \in \mathscr{F}$.

Since $\overline{\Omega}$ is compact in H_f the domain of maximal continuation of f, the distance $\rho = d(\Omega, H_f^c)$ is positive. First we show that for any $q \in N(\Omega)$ the inequality $|f(q)| \leqq sup\ |f(\bar{\Omega})|$ holds. If q $\in \overline{\Omega}$ there is nothing to prove. So we assume $q \in N(\Omega) - \overline{\Omega}$. It suffices to show $f(q) \in f(\overline{\Omega})$. Suppose $f(q) \notin f(\overline{\Omega})$ Then since $f(\overline{\Omega})$ is compact, there exists an open neighborhood ω of $f(\overline{\Omega})$ such that $f(q) \notin \omega$. The function $g(x) = (f(x) - f(q))^{-1}$ has $H_g := H_f - \{x \in H_f: f(x) - \mathrm{f(q)}\}$ as its domain of existence. Since $g(x)$ is holomorphic in ω, we have $g \in \mathscr{F}$. g is singular at q and $q \in N(\Omega) \subset H_g$. This is a contradiction. Hence $f(q) \in f(\overline{\Omega})$ and therefore $|f(q)| \leqq sup\ |f(\overline{\Omega})|$ for every $q \in N(\Omega)$. Since f is arbitrary in $\mathscr{F}$, we have for any differential operator D^α the same inequality: $|D^\alpha f(q)| \leqq sup\ |D^\alpha f(\overline{\Omega})|$. In general $H_f \subset H_{D^\alpha f}$ for $f \in \mathscr{F}$. Taking positive number r so that $r < \rho$ we have $(\overline{\Omega})_r \subset H_f$ and $(\overline{\Omega})_r \subset H_{D^\alpha f}$, and hence

$$sup\ |D^\alpha f(\overline{\Omega})| \leqq \frac{\alpha!}{r^{|\alpha|}}\ sup\ |f((\overline{\Omega})_r)|.$$

From this we can show that the Taylor expansion of f at q converges in the ball $S(q, r)$ of radius r centered at q. Since we may take r arbitrarily near to ρ, the Taylor expansion converges in $S(q, \rho)$. The point q being taken arbitrarily in $N(\Omega)$ we have $(N(\Omega))_\rho \subset H_f$. This implies $d(N(\Omega), H_f^c) \geqq \rho = d(\Omega, H_f^c)$. On the other hand from $\Omega \subset N(\Omega)$ we have converse inequality $d(N(\Omega), H_f^c) \leqq d(\Omega, H_f^c)$. Thus $d(N(\Omega), H_f^c) = d(\Omega, H_f^c)$. ◇

COROLLARY 10 [1]. *Let $f(x)$ be meromorphic in a neighborhood ω of $\overline{\Omega}$ and let there exist a complex number a such that $f(x) \neq a$ for any $x \in \omega$. Then $f(x)$ is meromorphically continued to some neighborhood of $N(\Omega)$ and $f(x) \neq a$ there.*

PROOF. Consider the function $g(x) = (f(x) - a)^{-1}$ in ω. ◇

COROLLARY 11. *Let f be a holomorphic mapping from a neighborhood of $\overline{\Omega}$ to C^n. If the rank of f is constant and equal to k, then the rank of the extension of f to the neighborhood of $\overline{N(\Omega)}$ is also constant and equal to k.*

PROOF. Apply Corollary 10 to the functional matrix of f. ◇

COROLLARY 12. *Let f be a holomorphic mapping from a neighborhood of $\overline{N(\Omega)}$ to a domain D in C^k. If there exists a function F holomorphic in D such that $f(\Omega) \cap \{y \in D: F(y) = 0\} = \phi$, then $f(N(\Omega)) \cap \{y \in D: F(y) = 0\} = \phi$.*

PROOF Consider the function $F \circ f$. ◇

REMARK. If D is domain of holomorphy, we may replace the set $\{y \in D: F(y)=0\}$ by arbitrary analytic subset of D. For the case: $k=1$ we obtain again $f(\Omega)=f(N(\Omega))$.

The standard arguments imply

COROLLARY 13 [1]. *Let f be a holomorphic mapping from a neighborhood of $\overline{\Omega}$ to a neighborhood of $\overline{\Omega}$ which induces an automorphism of Ω. Then f is extended to an automorphism of $N(\Omega)$.*

From Theorem 9 we are able to deduce a sufficient condition for a domain to be n-convex. It is also a direct consequence of Theorem 6.

THEOREM 14. *If for any $p \in \partial\Omega$ there exists a non-constant f in $\mathscr{F}$ such that $|f(p)| = \sup |f(\Omega)|$, then Ω is n-convex.*

PROOF. Though this is trivially obtained from Theorem 6, we give here a proof along the line of Theorem 9. Let us assume $\Omega \subsetneqq N(\Omega)$. Then $\partial\Omega \cap N(\Omega) \neq \phi$. Take any point p of $\partial\Omega \cap N(\Omega)$. By assumption of the theorem there exists a non-constant f of $\mathscr{F}$ such that $|f(p)| = \sup |f(\Omega)|$. Since f is non-constant, for any positive number ε we may choose a point q of $S(p, \varepsilon)$ so that the inequality $|f(q)| > |f(p)|$ holds. Obviously $q \in S(p, \varepsilon) \cup (\overline{\Omega})^c$. Putting $g(x)=(f(x)-f(q))^{-1}$, $g(x)$ is holomorphic in a neighborhood of $\overline{\Omega}$. Hence $N(\Omega) \subset H_g$. Taking ε so that $S(p, \varepsilon) \subset\subset N(\Omega)$ we may assume that q is a point of $N(\Omega)$. Since g is singular at q, this is a contradiction. Thus $\Omega = N(\Omega)$, that is, Ω is n-convex. ◇

3. Another convexity for n-convex domains.

In this section we shall define a strong type of convexity that properly holds for n-convex domains.

Let $y=(y_3, \cdots, y_n)$ be the coordinates of C^n and consider the domain $S=\{y: |y_1|<|y_2|<1, |y_3|<1, \cdots, |y_n|<1\}$ and $D=\{y: |y_1|<1, |y_2|<1, \cdots, |y_n|<1\}$. Let Φ be a biholomorphic mapping of a neighborhood of D to C^n. $\sum=\Phi(S)$ and $\Delta=\Phi(D)$ are called a generalized half disk or simply half disk, and a generalized disk or simply disk, respectively. A domain Ω of C^n is said to be N-convex if it possesses the property that if Ω contains $\sum$, then Ω contains Δ.

Now, we show

PROPOSITION 15. *Let Ω be a bounded domain of C^n. If Ω is n-convex, then it is N-convex.*

PROOF. Assume Ω is not N-convex. Then there exists a pair (Δ, Σ) of disk and half disk so that $\Sigma \subset \Omega$ and $\Delta \not\subset \Omega$. Then every function of $\mathscr{F}(\overline{\Omega})$ is analytically continued to $\Omega \cap \Delta$. By choosing a point p in $\Delta - \Omega$ we may assume that $d(p, \overline{\Omega}) > 0$. Since Ω is n-convex, choosing ε in (ii) in Proposition 1 so that $\varepsilon < d\ (p, \overline{\Omega})$ we arrive at a contradiction. Thus $\Delta \subset \Omega$. ◇

An example of a domain of holomorphy which is not N-convex is the half disk itself.

Is is natural to ask if the converse of Proposition 15 hold. We have no answer till now. In consideration of the convexity of Oka we can show the fact: *any domain of holomorphy in* C^n *contains* Δ *if it contains* Σ *as relatively compact subset.*

4. A condition for a domain to be n-convex

A domain of holomorphy in C^n is approximated by an increasing sequence of domains of holomorphy. This is a direct consequence of the fact that the domain of holomorphy is holomorphically convex. It is a remarkable contrast to this fact that a n-convex domain is approximated by a decreasing sequence of domains of holomorphy. Not as an accurate terminology we may call the former *the convexity from interior* and the latter the *convexity from exterior.* Then, in general for a domain convexity from interior does not induce convexity from exterior, but since a n-convex domain is holomorphically convex, convexity from exterior induces always convexity from interior. Our problem is under what condition the convexity from interior does imply the convexity from exterior, that is, under what condition the converse of Proposition 5 holds.

Now we begin with generalizing well known condition for a domain to be holomorphically convex to more general one which works also for a family of bounded functions.

PROPOSITION 16. *A necessary and sufficient condition for a domain Ω to be convex with respect to a subset $\mathscr{F}$ of $\mathscr{A}(\Omega)$ is that for any infinite discrete sequence $\{x_k\}$ in Ω and for any compact subset K of Ω these exists an $f \in \mathscr{F}$ such that the inequality* $\sup_k |f(x_k)| > \sup |f(K)|$ *holds.*

PROOF. ($\Rightarrow$) Assume that the conclusion of proposition is false. Then there exist a sequence $\{x_k\}$ of Ω and a compact subset K such that $\sup_k |f(x_k)| \leq \sup |f(K)|$ for all $f \in \mathscr{F}$. Hence $\{x_k\} \subset \hat{K}_{\mathscr{F}}$. Since $\{x_k\}$ is discrete in Ω, $\hat{K}_{\mathscr{F}}$ can not be compact in Ω. Thus Ω can not be $\mathscr{F}$-convex.

($\Leftarrow$) Assume that Ω is not $\mathscr{F}$-convex. Then there exists a compact subset K of Ω such that $\hat{K}_{\mathscr{F}}$ is not compact in Ω. So we can choose a infinite sequence $\{x_k\}$ in $\hat{K}_{\mathscr{F}}$ which is discrete in Ω. By the definition of $\hat{K}_{\mathscr{F}}$ we see that for every $f \in \mathscr{F}$ the inequality $|f(x_k)| \leqq sup\ |f(K)|$ holds for every k. Thus we have $\sup_k |f(x_k)| \leqq sup\ |f(K)|$ for every $f \in \mathscr{F}$. This is a contradiction. We proved that Ω is $\mathscr{F}$-convex. $\Diamond$

Our result is the following restricted converse of Proposition 5.

THEOREM 17. *Let Ω and Ω_1 be domains in C^n, $\Omega \subset\subset \Omega_1$ and $\mathscr{F}$ a subset of $\mathscr{A}(\Omega_1)$ satisfying the following conditions*:

(i) *$\mathscr{F}$ is equicontinuous in Ω_1.*

(ii) *There exists a point $p \in \Omega$ such that $\mathscr{F}(p)=\{f(p): f \in \mathscr{F}\}$ is bounded.*

(iii) *There exist a point $q \in \Omega$ and a positive number ρ such that $\sum_{i=1}^{n} \left| \frac{\partial f}{\partial x_i}(q) \right| \geqq \rho$ for every $f \in \mathscr{F}$.*

If Ω is $\mathscr{F}$-convex. then it is n-convex.

PROOF. Let $\{K_\lambda\}$ be an exhaustion of Ω by compact subsets; $K_\lambda \subset K_{\lambda+1} \subset \cdots$ and $\bigcap_\lambda K_\lambda = \Omega$. Take an arbitrary point $p \in \partial\Omega$ and fix. Then we can choose a sequence $\{x_k\}$ which is discrete in Ω and converges to p. By Proposition 14 there is an $f_\lambda \in \mathscr{F}$ for every λ such that it holds

$$\sup_k |f_\lambda(x_k)| > sup\ |f_\lambda(K_\lambda)|.$$

Then we can find an integer k_λ satisfying

$$|f_\lambda(x_{k_\lambda})| > sup\ |f_\lambda(K_\lambda)|.$$

We determine $\{x_{k_\lambda}\}$ inductively so that $k_1 < k_2 < \cdots$. For this purpose assume that $f_1, f_2, \quad , f_{\lambda-1}$ and $x_{k_1}, x_{k_2} \quad , x_{k_{\lambda-1}}$ are already determined. Then for $\{x_k\}_{k>k_{\lambda-1}}$ and K_λ we apply Proposition 14 and determine f_λ.

(i) and (ii) imply that $\mathscr{F}$ is uniformly bounded on every compact subset of Ω. Choose a domain Ω_0 such that $\Omega \subset\subset \Omega_0 \subset\subset \Omega_1$. Then by the therem of Montel $\mathscr{F}$ is normal in $\overline{\Omega}_0$, in other words, any infinite sequence of $\mathscr{F}$ contains a subsequence which converges uniformly in $\overline{\Omega}_0$. Applying this to the sequence $\{f_\lambda\}$ we can find a subsequence which converges uniformly in $\overline{\Omega}_0$. We may assume that $\{f_\lambda\}$ itself is convergent. Let f be the limit of $\{f_\lambda\}$. Then $f \in \mathscr{A}(\Omega_0)$. Further, th esequences $\left\{\frac{\partial f}{\partial x_i}\right\}$, $i=1, 2, \cdots, n$ also converge uniformly to $\frac{\partial f}{\partial x_i}$ in Ω_0 respectively.

Clearly for arbitrary μ we have

$$|f_\lambda(x_{k_\lambda})| > sup\ |f_\lambda(K_\mu)|,\ {}^\forall\lambda \geqq \mu.$$

Since $\{K_{k\lambda}\} \cup \{p\}$ is compact in Ω_0 and f is continuous in Ω_0, we have, as $\lambda \to +\infty$,

$$|f(p)| \geqq sup\ |f(K_\mu)|,\ {}^\forall\mu.$$

Since sup $|f(K_\mu)| \to sup\ |f(\Omega)|$ as $\mu \to +\infty$, we obtain

$$|f(p)| \geqq sup\ |f(\Omega)|,$$

which implies

$$|f(p)| = sup\ |f(\Omega)|.$$

By (iii) f is non-constant in Ω_0. Since p is arbitrary in $\partial\Omega$, the assumption of Theorem 10 is satisfied. Thus Ω is n-convex. ◇

We may state Theorem 17 also in the following way.

THEOREM 17′. *Let Ω and Ω_1 be the domains in C^n, $\Omega \subset\subset \Omega_1$ and $\mathscr{F}$ a subet of $\mathscr{A}(\Omega_1)$ satisfying the following conditions*:

(i) *$\mathscr{F}$ is compact in $\mathscr{C}(\Omega_1)$.*
(ii) *$\mathscr{F} \cap (\mathscr{C} \cup \{\infty\}) = \phi$.*

Then, if Ω is $\mathscr{F}$-convex, it is n-convex. Here $\mathscr{C} \cup \{\infty\}$ is the set constant of functions possibly ∞-valued.

PROOF. The condition (i) is equivalent to (i) in Theorem 17. In general the necessary and sufficient condition for $\mathscr{F}$ to be equicontinuous is that any infinite sequeuce of $\mathscr{F}$ contains a compact-uniformly convergent subsequence or a subsequence which diverges compact-uniformly to constant ∞. By the condition (ii) any infinite sequence of $\mathscr{F}$ can not contain the subsequence diverging to ∞. Again by (ii) any converging sequence of $\mathscr{F}$ can not have constant limit. Remaining part of proof goes as in the proof of Theorem 17. ◇

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