Kumamoto J. Sci. (Math.)
Vol. 14, 76~80 March (1980)

# PSEUDODISTANCE DEFINED BY A SET OF HOLOMORPHIC FUNCTIOS AND n-CONVEX DOMAINS

Shawich SATO

(Received October 29, 1979)

## Introduction

In [3] the author gave a sufficient condition for a bounded domain to be n-convex. The conditions assumed there were somewhat complicated and restrictive.

If a bounded domain $\Omega$ in $C^n$ is convex with respect to a set $\mathscr{F}$ of functions holomorphic in some neighborhood of $\overline{\Omega}$, then, as is easily shown, $\Omega$ is convex also with res pect to $\mathscr{F}_1:=\{f/\|f\|_{\bar{\Omega}}: f\in\mathscr{F}, \|f\|_{\bar{\Omega}}\neq 0\}$, $\|f\|_{\bar{\Omega}}= sup\ |f(\Omega)|$. And $\mathscr{F}_1$ is equicontinuous in $\Omega$. Under several conditions a distance $d$ is defined on$\Omega$ from such equicontinuous set and by the aid of $d$ a kind of convexity is considered for $\Omega$, see **1**. It is shown that this convexity is very natural and implies n-convexity of $\Omega$.

A bounded domain $\Omega$ is said to be *n-convex* if $\overline{\Omega}$ possesses a fundamental system of neighborhoods consisting of the domains of holomorphy and the open kernel of $\overline{\Omega}$ is equal to $\Omega$, see [3].

The requirement (v) in Theorem 5 means that $\Omega$ is complete in some geometrical sense, and this circumstances are often treated in the theory of hyperbolic spaces, see [1].

## 1. Pseudodistance $\mathbf{d}_{\mathscr{F}}$

Let $\Omega$ be a bounded domain in $C^n$ and $\mathscr{F}$ a subset of $\mathscr{A}(\Omega)$ the family of functions holomorphic in $\Omega$. $\mathscr{A}(\Omega)$ is endowed with compact-open topology. We define a real-valued function $d_{\mathscr{F}}(p, q)$ in $\Omega\times\Omega$ by

$$d_{\mathscr{F}}(p, q) = \sup_{f\in\mathscr{F}} |f(p)|-f(q)|.$$

LEMMA 1. *If $\mathscr{F}$ is equicontinuous in $\Omega$, $d_{\mathscr{F}}(p, q)<+\infty$ for any $p$ and $p$ in $\Omega$.*

PROOF. $\mathscr{F}$ is equicontinuous if and only if $n$ families $\mathscr{F}_i=\left\{\dfrac{\partial f}{\partial x_i}: f\in\mathscr{F}\right\}$, $i=1,2,\cdots, n$ are all uniformly bounded on every compact subset of $\Omega$, see [2]. Applying this to the formula $f(p)-f(q) = \int_c df$ where $c$ is a path from $p$ to $q$, we obtain the required stimate. ◇

It is easily verified that the function $d_{\mathscr{F}}$ is a pseudodistance for a equicontinuous set $\mathscr{F}$. If $\mathscr{F}$ separates the points of $\Omega$, then obviously $d_{\mathscr{F}}$ defines a distance in $\Omega$. In the sequel the domain $\Omega_1$ is assumed to satisfy $\Omega \subset\subset \Omega_1$, that is, $\Omega_1$ contains $\Omega$ as its relatively compact subdomain, and $\mathscr{F}$ is assumed to be a subset of $\mathscr{A}(\Omega_1)$ and to be equicontinuous in $\Omega_1$. The restriction of $\mathscr{F}$ to $\Omega$ is denoted by the same notation $\mathscr{F}$ if there is no fear of confusion. It is convenient to see under what conditions the usual topology, that is, the topology induced by the Euclidian distance $d_e$, coincides with the topology in $\Omega$ induced by $d_{\mathscr{F}}$.

We denote the ball defined by $d_{\mathscr{F}}$ by $B_{\mathscr{F}}(p, r)$ and the ball defined by $d_e$ by $B_e(p, r)$.

Then we have

LEMMA 2. *Let $\mathscr{F}$ satisfy the following conditions*:

(i) *$\mathscr{F}$ is equicontinuous in $\Omega_1$.*

(ii) *$\mathscr{F}$ separates the points of $\Omega$.*

*Then the topology $T_{\mathscr{F}}$ induced by $d_{\mathscr{F}}$ coincides with the topology $T_e$ induced by $d_e$.*

PROOF. We can easily show that $d_{\mathscr{F}}$ is continuous in $\Omega \times \Omega$ with respect to $T_e$. First we show that $T_e$ is stronger than $T_{\mathscr{F}}$. Let $p$ be any point of $\Omega$. Since $\mathscr{F}$ is equicontinuous in $\Omega_1$ and hence in $\Omega$, for any positive number $\varepsilon$ there exists a positive number $\delta$ such that $|f(x)-f(p)|<\varepsilon/2$ if $d_e(p, x)<\delta$ for every $f\in\mathscr{F}$. Hence $d_{\mathscr{F}}(p, x)<\varepsilon$. This shows that $B_e(p, \delta)\subset B_{\mathscr{F}}(p, \varepsilon)$. So we have $T_e \geqq T_{\mathscr{F}}$.

Next we have to show $T_e \leqq T_{\mathscr{F}}$. So assume that there exist a point $p_0$ of $\Omega$ and a positive number $\varepsilon_0$ such that for any positive number $\delta$ holds $B_{\mathscr{F}}(p_0, \delta) \not\subset B_e(p_0, \varepsilon_0)$. If there exists such $\varepsilon_0$, then arbitrary $\varepsilon$ satisfying $\varepsilon<\varepsilon_0$ possesses the same property, that is, $B_{\mathscr{F}}(p_0, \delta)\not\subset B_e(p_0, \varepsilon)$. So we may assume without loss of generality that $B_e(p_0, \varepsilon_0)$ is relatively compact in $\Omega$. By the fact proved above that $T_e \geqq T_{\mathscr{F}}$ the ball $B_{\mathscr{F}}(p_0, \varepsilon)$ is open. We may assume that $B_{\mathscr{F}}(p_0, \varepsilon)$ is connected. For, if for any $p$ and positive number $\varepsilon'$ there can be chosen a positive number $\delta'$ so that the connected component of $B_{\mathscr{F}}(p, \delta')$ containing $p$ is included in $B_e(p', \varepsilon')$, obviously it holds that $T_e \leqq T_{\mathscr{F}}$. Therefore we assume that $B_{\mathscr{F}}(p_0, \delta)$ is connected. Since $B_e(p_0, \varepsilon_0)-B_e(p_0, \varepsilon)$ is relatively compact and $B_{\mathscr{F}}(p_0, 1/n)$ can be assumed connected for every $n$, and since $B_{\mathscr{F}}(p_0, 1/n) \not\subset B_e(p_0, \varepsilon)$, we have $B_{\mathscr{F}}(p_0, 1/n) \cap \{B_e(p_0, \varepsilon_0)-B_e(p_0, \varepsilon)\} \neq \phi$ for every $n$, from which we choose arbitrarily a point $x_n$. The set $\{x_n\}$ is relatively compact in $\Omega$ and therefore contains a subsequence

$\{y_m\}$ converging to a point $y_0$ of $\Omega$. From $y_m \notin B_e(p_0, \varepsilon)$ we have $d_e(y_m, p_0) \geqq \varepsilon$. Consequently we have $d_e(y_0, p_0) \geqq \varepsilon$. Because of $y_m \in B_{\mathscr{F}}(p_0, 1/n_m)$ we see that $|f(y_m) - f(p_0)| < 1/n_m$ for every $f \in \mathscr{F}$, which implies $f(y_0) - f(p_0) = 0$. This contradicts to the requirement (ii) such that there exists an $F \in \mathscr{F}$ satisfying $F(y_0) \neq F(p_0)$. Thus we proved $T_e \leqq T_{\mathscr{F}}$. ◇

We need some more preparations. Let $\Omega$, $\Omega_1$ and $\mathscr{F}$ be as before.

PROPOSOTION 3. *Let $\mathscr{F}$ satisfy the following conditions:*

(i) *$\mathscr{F}$ is equicontinuous in $\Omega_1$.*
(iii) *$\mathscr{F}(p) := \{f(p) : f \in \mathscr{F}\}$ is bounded for some point $p$.*
(iv) *$\mathscr{F}$ is closed in $\mathscr{A}(\Omega_1)$.*

*Then for any different two points $x_0, y_0$ of $\Omega_1$ there exists a function $f \in \mathscr{F}$ for which holds $d_{\mathscr{F}}(x_0, y_0) = |f(x_0) - f(y_0)|$.*

PROOF. We can choose a sequence $\{f_n\}$ in $\mathscr{F}$ so that

$$(*) \qquad |f_n(x_0) - f_n(y_0)| > d_{\mathscr{F}}(x_0, y_0) - 1/n.$$

The requirements (i) and (ii) imply that $\mathscr{F}$ is normal in $\Omega_1$ in the strong sense, that is, any sequence of $\mathscr{F}$ contains a subsequence which converges compact-uniformly in $\Omega_1$. Let $\{f_m\}$ be the subsequence, the limit of which we denote by $f$. Then (*) implies $|f(x_0) - f(y_0)| = d_{\mathscr{F}}(x_0, y_0)$. By (iv) we know that $f$ belongs to $\mathscr{F}$. ◇

## 2. Distance $d_{\mathscr{F}}$ and n-convex domain

Let $\Omega$, $\Omega_1$ and $\mathscr{F}$ be as in **1**. In this section we assume (i), (ii), (iii) and (iv) for $\mathscr{F}$. In application the following requirement, that is stronger than (iii), is convenient:

(iii)′ *There exists a point $o$ such that $f(o) = 0$* for every $f \in \mathscr{F}$.

We prove

LEMMA 4. *Let $\Omega$, $\Omega_1$ and $\mathscr{F}$ satisfy* (i) *and* (iii)′. *Then for any compact subset $K$ of $\Omega$ we have $\hat{K}_{\mathscr{F}} \subset \overline{B_{\mathscr{F}}(o, \rho)}$, where $\rho = \sup_{x \in K} d_{\mathscr{F}}(o, x)$ and $\hat{K}_{\mathscr{F}} = \{x \in \Omega : |f(x)| \leqq \sup |f(K)|, f \in \mathscr{F}\}$.*

PROOF. Since $d_{\mathscr{F}}$ is continuous in $\Omega \times \Omega$, $d_{\mathscr{F}}(o, x)$ is bounded on $K$. As $|f(x)| \leqq \sup |f(K)|$ for any point $x \in \hat{K}_{\mathscr{F}}$, (iii)′ implies

$$|f(x)| = |f(x) - f(o)| \leqq \sup_{y \in K} |f(y) - f(o)|$$
$$\leqq \sup_{y \in K} d_{\mathscr{F}}(y, o) = \rho.$$

Hence we obtain $d_{\mathscr{F}}(x, o) \leqq \rho$ which means that $\hat{K}_{\mathscr{F}} \subset \overline{B_{\mathscr{F}}(o, \rho)}$. ◇

Now we are ready to state our result:

THEOREM 5. *Let $\Omega$, $\Omega_1$ and $\mathscr{F}$ satisfy* (i), (ii), (iii)′, (iv) *and* (v) $\overline{B_{\mathscr{F}}(o,\rho)}$, $\rho = d_{\mathscr{F}}(o, q)$ *is compact for any point* $q \in \Omega$.
*Then $\Omega$ is n-convex.*

PROOF. Let $K$ be any compact subset of $\Omega$. Since $d_{\mathscr{F}}$ is continuous, $\rho = \sup_{x \in K} d_{\mathscr{F}}(o, x)$ is finite and for some point $p \in K$ we have $\rho = d_{\mathscr{F}}(p, o)$. By assumption $\overline{B_{\mathscr{F}}(o, \rho)}$ being compact, $\hat{K}_{\mathscr{F}}$ is compact and hence $\Omega$ is $\mathscr{F}$-convex.

Let $\{K_\lambda\}_{\lambda \in \Lambda}$ be an exhaustion of $\Omega$ by compact sets. $\rho_\lambda = \sup_{x \in K_\lambda} d_{\mathscr{F}}(x, o)$. Then $\{\overline{B_{\mathscr{F}}(o, \rho_\lambda)}\}_{\lambda \in \Lambda}$ is also an exhaustion of $\Omega$. We take an arbitrary point $p_0 \in \partial\Omega$ and fix, and choose a sequence $\{x_\lambda\}$ of $\Omega$ that converges to $p_0$. We may choose $x_\lambda$ so that $x_\lambda \notin \overline{B_{\mathscr{F}}(o, \rho_\lambda)}$. By Proposition 3 we can find an $f_\lambda \in \mathscr{F}$ such that $|f_\lambda(o) - f_\lambda(x_\lambda)| = d_{\mathscr{F}}(o, x_\lambda)$ holds. The requirement (iii)′ implies $f_\lambda(o) = 0$. Hence we have

$$|f_\lambda(x_\lambda)| = d_{\mathscr{F}}(o, x_\lambda) > \rho_\lambda = \sup |f_\lambda(K_\lambda)|.$$

Since by (i) and (ii) $\mathscr{F}$ is normal in the strong sense, $\{f_\lambda\}$ contains a subsequence which converges compact-uniformly in $\Omega_1$. We assume that $\{f_\lambda\}$ itself is convergent and converges to $f$ holomorphic in $\Omega_1$. Then (iv) and (iii)′ imply $f \in \mathscr{F}$ and $f(o) = 0$.

Let $K$ be a compact subset of $\Omega$. We can find an integer $\lambda_0$ such tha $K_\lambda \supset K$, $\lambda \geqq \lambda_0$. We have then

$$|f_\lambda(x_\lambda)| \geqq \rho_\lambda = \sup |f_\lambda(K)|.$$

Letting $\lambda \to +\infty$ we obtain $|f(p_0)| \geqq \sup |f(K)|$. $K$ being arbitrary we conclude that $|f(p_0)| \geqq \sup |f(\Omega)|$ and hence that $|f(p_0)| = \sup |f(\Omega)|$. By the following we see $|f(p_0)| > 0$:

$$|f_\lambda(x_\lambda)| > \rho_\lambda = \sup_{x \in K_\lambda} d_{\mathscr{F}}(x,o) \geqq \sup_{x \in K} d_{\mathscr{F}}(x, o) > 0,\ \lambda \geqq \lambda_0.$$

Combining this with $f(o) = 0$ we see that $f$ is non-constant. Since $p_0 \in \partial\Omega$ is arbitrary, Theorem 14 in [3] implies that $\Omega$ is n-convex. ◇

An example of the domain considered in Theorem 5 is simply given by the polydisk in $C^n$, where we may take as $\mathscr{F}$ the set of coordinate functions. Thus, the domain satisfying the conditions from (i) to (v) in Theorem 5 seems to be a generalization of polydisk.

## Referencens

[1] Kobayashi, S.: Hyperbolic Manfolds and Holomorphic Mappings. Marcel Dekker, New York, (1970).

[2] Sato, S.: On equicontinuous families of holomorphic functions, Kumamoto J. Sci. (Math.), 9, (1973), 75-86.

[3] Sato, S.: On the Nebenhülle of bounded domains, Kumamoto J. Sci. (Math.), (1980), 64-75.

Department of Mathematics,
Faculty of Science,
Kumamoto University