Kumamoto J. Sci. (Math.)
Vol. 14, 81～84 March (1980)

SOME REMARKS ON GÖDEL'S MEMORANDUM FOR THE CARDINALITY OF THE CONTINUUM

Hiromitsu OKAZAKI

(Received Nobember 2, 1979)

In this paper, we show some remarks on Gödel's (unpublished) memorandum for the cardinality of the continuum. Let f, g and h be functions from ω_n to ω_n (n is a non-negative integer) and $\alpha, \beta, \ldots$ be ordinal numbers.

DEFINITION 1.

$$f < g \overset{\text{DF}}{\Longleftrightarrow} \exists\alpha<\omega_n \ \forall\beta(\alpha<\beta<\omega_n \rightarrow f(\beta)<g(\beta)).$$
$$f << g \overset{\text{DF}}{\Longleftrightarrow} \forall\alpha<\omega_n(f(\alpha)<g(\alpha)).$$

DEFINITION 2.

$A(\aleph_n, \aleph_n) \overset{\text{DF}}{\Longleftrightarrow} \exists F\subseteq\omega_n^{\omega_n} \ \exists M\subseteq \omega_n^{\omega_n}$(*F and M satisfy the conditions from* 1_n *to* 6_n).

1_n) *F is wellodered by* $<$ *and* $\bar{F}=\omega_{n+1}$ ($\bar{F}$ *means the order type of F*).

2_n) $\forall f\in\omega_n^{\omega_n} \ \exists g\in F(f<g)$.

3_n) $\overline{\overline{M}}=\aleph_{n+1}$.

4_n) $\forall f\in\omega_n^{\omega_n} \ \exists g\in M(f<<g)$.

5_n) $\overline{\overline{\{f\upharpoonright\alpha \mid \alpha<\omega_n \ \& \ f\in F\}}}=\aleph_n$.

6_n) $\overline{\overline{\{f\upharpoonright\alpha \mid \alpha<\omega_n \ \& \ f\in M\}}}=\aleph_n$.

Gödel's axiom is $\forall nA(\aleph_n, \aleph_n)$ plus Hausdorff's axiom.[1,2] From these definitions, we have the following propositions.

PROPOSITION 1. *If there exists an* $F\subseteq\omega_n^{\omega_n}$ *satisfying the conditions* 1_n, 2_n *and* 5_n, *then there exists an* $M\subseteq\omega_n^{\omega_n}$ *satisfying the conditions* 3_n, 4_n *and* 6_n.

PROOF. For $g\in F$,

$$g_{\beta,\gamma}(\mu)=\begin{cases}\gamma & \text{if} \quad \mu<\beta,\\ g(\mu) & \text{if} \quad \beta<\mu<\omega_n.\end{cases}$$
$$F_{\beta,\gamma} = \{ g_{\beta,\gamma} \mid g\in F\}.$$
$$M=\bigcup_{\beta,\gamma<\omega_n} F_{\beta,\gamma}.$$

Since $\overline{\overline{F}}_{\beta,\gamma}=\aleph_{n+1}$, we have $\overline{\overline{M}}=\aleph_{n+1}$. For $f\in\omega_n^{\omega_n}$, there exist $g\in F$ and β_0 such that

$$\forall\mu(\beta_0<\mu<\omega_n\rightarrow f(\mu)<g(\mu)).$$

Set

$$\gamma_0=\sup\{f(\mu)|\mu<\beta_0\}+1.$$

Then $\gamma_0<\omega_n$, since ω_n is a regular ordinal number. By the definition of g_{β_0,γ_0}, we have $f<<g_{\beta_0,\gamma_0}$. Now we show that M satisfies the condition 6_n. Let $\beta,\gamma<\omega_n$. For each $f\in F_{\beta,\gamma}$, take $g\in F$ such that $f=g_{\beta,\gamma}$, and denote such a g by g_f. If f, $f'\in F_{\beta,\gamma}$ and $f\neq f'$, then $g_f\neq g_{f'}$. So

$$\overline{\overline{\{f\upharpoonright\alpha|\alpha<\omega_n\ \&\ f\in F_{\beta,\gamma}\}}}\leqq\overline{\overline{\{g_f\upharpoonright\alpha|\alpha<\omega_n\ \&\ f\in F_{\beta,\gamma}\}}}$$
$$\leqq\overline{\overline{\{g\upharpoonright\alpha|\alpha<\omega_n\ \&\ g\in F\}}}\leqq\aleph_n.$$

Hence

$$\overline{\overline{\{f\upharpoonright\alpha|\alpha<\omega_n\ \&\ f\in M\}}}=\overline{\overline{\bigcup_{\beta,\gamma<\omega_n}\{f\upharpoonright\alpha|\alpha<\omega_n\ \&\ f\in F_{\beta,\gamma}\}}}\leqq\aleph_n.$$

Therefore M satisfies the condition 6_n. Q. E. D.

PROPOSITION 2. *If* $2^{\aleph_n}=\aleph_{n+1}$, *then there exists an* F *satisfying the conditions* 1_n *and* 2_n.

PROOF. Since $2^{\aleph_n}=\aleph_{n+1}$, there is an enumeration $h_0, h_1, \cdots, h_\nu, \cdots$ $(\nu<\omega_{n+1})$ of the elements of $\omega_n^{\omega_n}$. We define inductively the sequence $f_0, f_1, \cdots, f_\nu, \cdots$ $(\nu<\omega_{n+1})$ of functions from ω_n to ω_n as follows:

i) $f_0(\gamma)=h_0(\gamma)+1$ for $\gamma<\omega_n$.

ii) If ν is a successor ordinal number, let $\nu=\mu+1$ and define f_ν by

$$f_\nu(\gamma)=f_\mu(\gamma)+h_\nu(\gamma)+1\ (\gamma<\omega_n).$$

iii) If ν is a limit ordinal number and $\mathrm{cf}(\nu)<\omega_n$, let g_ν be a function from $\mathrm{cf}(\nu)$ to ν such that

$$\nu=\sup_{\xi<\mathrm{cf}(\nu)} g_\nu(\xi)$$

and define f_ν by

$$f_\nu(\gamma) = \sup_{\xi < \mathrm{cf}(\nu)} (f_{g_\nu(\xi)}(\gamma)) + h_\nu(\gamma) + 1.$$

iv) If ν is a limit ordinal number and $\mathrm{cf}(\nu) = \omega_n$, let g_ν: $\omega_n \to \nu$ be a bijection. For $f, g \in \omega_n^{\omega_n}$, set

$$\lambda(f, g) = \begin{cases} \mu\xi\,(\forall\sigma \geqq \xi\, f(\sigma) < g(\sigma)) & \text{if } f < g, \\ \mu\xi\,(\forall\sigma \geqq \xi\, f(\sigma) > g(\sigma)) & \text{if } g < f, \\ 0 & \text{otherwise.} \end{cases}$$

Define the function σ_ν: $\omega_n \to \omega_n$ inductively:

$$\sigma_\nu(0) = 0,$$

$$\sigma_\nu(\xi) = \max\{\sup_{\eta<\xi} (\lambda(f_{g_\nu(\eta)}, f_{g_\nu(\xi)})), \sup_{\eta<\xi} (\sigma_\nu(\eta))\} + 1, \text{ if } \xi > 0.$$

For $\gamma < \omega_n$, let ξ be the ordinal number such that

$$\sigma_\nu(\xi) \leqq \gamma < \sigma_\nu(\xi + 1).$$

We define

$$f_\nu(\gamma) = \sup_{\eta \leqq \xi} (f_{g_\nu(\eta)}(\gamma)) + h_\nu(\gamma) + 1.$$

$$F = \{f_\nu \mid \nu < \omega_{n+1}\}.$$

By induction on $\nu < \omega_{n+1}$, we can prove that $h_\nu << f_\nu$ and that $\mu < \nu$ implies $f_\mu < f_\nu$. Therefore F satisfies 1_n and 2_n. Q. E. D.

PROPOSITION 3. *If $\aleph_n^\beta = \aleph_n$ for all $\beta < \aleph_n$ and $2^{\aleph_n} = \aleph_{n+1}$, then the condition $A(\aleph_n, \aleph_n)$ is satisfied.*

PROOF. By the proposition 2, there exists an F satisfying the conditions 1_n and 2_n.

$$\begin{aligned} \aleph_n &\leqq \overline{\overline{\{f \restriction \alpha \mid \alpha < \omega_n \,\&\, f \in F\}}} \\ &= \overline{\overline{\bigcup_{\alpha<\omega_n} \{f \restriction \alpha \mid f \in F\}}} \\ &\leqq \sum_{\beta < \aleph_n} \aleph_n^\beta = \aleph_n \cdot \aleph_n = \aleph_n. \end{aligned}$$

Then F satisfies the condition 5_n. By the proposition 1, there exists an M satisfying the conditions 3_n, 4_n and 6_n. Q. E. D.

PROPOSITION 4. *The following three conditions are equivalent.*

(i) $A(\aleph_0, \aleph_0)$.

(ii) *There exists an* F *satisfying* 1_0, 2_0 *and* 5_0.

(iii) *There exists an* M *satisfying* 3_0, 4_0 *and* 6_0.

PROOF. It is trivial that (i) implies (ii) and (iii). By the proposition 1, we have that (ii) implies (i). Now we show that (iii) implies (ii). Let the sequence $h_0, h_1, \cdots, h_\nu, \cdots$ $(\nu < \omega_1)$ be an enumeration of the elements of M. By the similar construction of the proof of the proposition 2, we have the set

$$F = \{f_\nu \mid \nu < \omega_1\}.$$

By the construction of F, F satisfies the conditions 1_0 and 5_0. Let g be a function of $\omega_0^{\omega_0}$, there exists h_ν such that $g \ll h_\nu$. There exists f_ν such that $h_\nu < f_\nu$ by the construction of F. Then $g < f_\nu$. This means F satisfies the condition 2_0. Therefore we have that (iii) imples (ii). Q. E. D.

References

[1] Gödel, K.: Some consideration leading to the probable conclusion that the true power is $\aleph_2$, unpublished memorandum, 1970.

[2] Takeuti, G.: Sugaku kisoron no sekai, Nihon-hyoron-sha, Japan, 1972, 106–107.

Department of Mathematics
Faculty of Education
Kumamoto University